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NOTE ON THE NEGATIVE DECISION NUMBER IN DIGRAPHS

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Abstract. Let D be a finite digraph with the vertex set $V(D)$ and the arc set $A(D)$. A function $f : V(D) \to \{-1,\ 1\}$ defined on the vertices of a digraph D is called a bad function if $f(N^-(v)) \le 1$ for every v in D. The weight of a bad function is $f(V(D)) = \sum_{v \in V(D)} f(v)$. The maximum weight of a bad function of D is the the negative decision number $\beta_D(D)$ of D. Wang [4] studied several sharp upper bounds of this number for an undirected graph. In this paper, we study sharp upper bounds of the negative decision number $\beta_D(D)$ of for a digraph D.

1. Introduction

It has been studied that an interconnection network is modelled by a graph with vertices representing sites of the network and edges representing links between sites of the network. The motivation for studying this new parameter on a directed network system may be varied from a modelling perspective. For instance, in a social network (a network of people), if we give an arc uv when u influences v and assign the values -1 or 1 to the vertices of a digraph, we can model networks of people in which global decisions must be made(e.g. positive or negative responses). In certain circumstances, a positive decision can be made only if there are significantly more people voting for than those voting against. We assume that each individual has one vote, and each has an initial opinion. We assign 1 to vertices (individuals) which have a positive opinion and -1 vertices which have a negative opinion. A voter votes 'good' if there are two more vertices in its open neighborhood with positive opinion than with negative opinion, otherwise the vote is 'bad'. We seek an assignment of opinions that guarantee an unanimous decision; namely, for which every vertex votes 'bad'. Such an assignment of opinions is called a uniformly negative assignment. Among all uniformly negative assignments of opinions, we are

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particularly interested in the minimum number of vertices (individuals) which have a negative opinion. The negative decision number is the maximum possible sum of all opinions, 1 for a positive opinion and -1 for a negative opinion, in a uniformly negative assignment of opinions. The negative decision number corresponds the minimum number of individuals who can have negative opinions and in doing so force every individual to vote bad.

All digraphs considered in this paper are finite, without loops and multiple arcs. For a general reference on graph theory, the reader is directed to [1]. For a digraph D, we denote the vertex set of D and the arc set of D by $V(D)$ and $A(D)$, respectively. We say that u is an *in-neighbor* of v and v is an *out-neighbor* of u if uv is an arc of D. For a vertex $v \in V(D)$, the sets of in-neighbors and out-neighbors of v are called the *open in-neighborhood* and the *open out-neighborhood* of v are denoted by $N_D^-(v)$ and $N_D^+(v)$, respectively. The *closed in-neighborhood* of v is $N_D^-[v] = N_D^-(v) \cup \{v\}$. The numbers $d_D^-(v) = |N_D^-(v)|$ and $d_D^+(v) = |N_D^+(v)|$ are the *in-degree* and the *out-degree* of v, respectively. We use $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$, and $\Delta^+ = \Delta^+(D)$ to denote the *minimum in-degree*, the maximum in-degree, the minimum out-degree and the *maximum out-degree* of a vertex in D, respectively. For $S \subseteq V(D)$, $D[S]$ denotes the subdigraph induced by S. If $S \subseteq V(D)$ and $v \in V(D)$, then $E(S, v)$ is the set of arcs from S to v. If S and T are two disjoint vertex sets of a digraph D, then $E(S, T)$ is the set of arcs from S to T.

For a function $f : V(D) \to \{-1,\ 1\}$, the *weight* of f is defined $w(f) = \sum_{v \in V(D)} f(v)$, and for $S \subseteq V(D)$ we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V(D))$.

A function $f : V(D) \to \{-1,\ 1\}$ is called a *bad function* (or *signed total 2-independence function*) if $f(N^-(v)) \leq 1$ for every vertex $v \in V(D)$. The *negative decision number* (or *singed total 2-independence number*) of a digraph D, denoted by $\beta_D(D)$, is the maximum weight $w(f)$, taken over all bad functions f on D.

C. Wang [4] presented several sharp upper bounds of the negative decision number for undirected graphs. The study of signed 2-independence number of undirected graphs was initiated by Zelink [5] and continued in [2] and elsewhere. Recently, Volkmann [3] began to investigate some upper bounds of the singed 2-independence number $\alpha_s^2(D)$ for a digraph D. In this paper, we study some upper bounds of the negative decision number $\beta_D(D)$ for for a digraph D.

Throughout this paper, when f is a bad function of D, we let P and M denote the sets of those vertices in D which are assigned under f the value 1 and -1, respectively and let $p = |P|$ and $m = |M|$. Then $|V(D)| = p + m$ and $\beta_D(D) = n - 2m$.

2. Main results

Theorem 2.1. *Let D be a digraph of order n and n_e the number of vertices whose in-degree of $V(D)$ is even. Then*

$$\beta_D(D) \leq min\{\frac{n(1-\delta^+) - n_e + |A(D)|}{\delta^+}, \ \frac{n(1+\Delta^+) - n_e - |A(D)|}{\Delta^+}\}.$$

This bound is sharp.

Proof. Let f be a bad function on D for which $\beta_D(D) = f(V(D))$, and let V_e and $n_e = |V_e|$ be defined as in the proof of Theorem 1.
Following the proof of Theorem 1 and $\sum_{v\in V(D)} d^+(v) = |A(D)|$,

$$\begin{aligned} n - n_e &\geq \sum_{v\in V(D)} f(N^-(v)) = \sum_{v\in P} d^+(v) - \sum_{v\in M} d^+(v) \\ &= 2\sum_{v\in P} d^+(v) - |A(D)| = |A(D)| - 2\sum_{v\in M} d^+(v). \end{aligned}$$

Now, we obtain two inequities ; $n - n_e \geq 2p\delta^+ - |A(D)|$ and $n - n_e \geq |A(D)| - 2(n-p)\Delta^+$. These implies

$$2p \leq \frac{n - n_e + |A(D)|}{\delta^+}$$

and

$$2p \leq \frac{n(\Delta^+ + 1) - n_e - |A(D)|}{\Delta^+}.$$

So,

$$\beta_D(D) = 2p - n \leq \frac{n(1-\delta^+) - n_e + |A(D)|}{\delta^+}$$

and

$$\beta_D(D) = 2p - n \leq \frac{n(1+\Delta^+) - n_e - |A(D)|}{\Delta^+}.$$

Thus,

$$\beta_D(D) \leq min\{\frac{n(1-\delta^+) - n_e + |A(D)|}{\delta^+}, \ \frac{n(1+\Delta^+) - n_e - |A(D)|}{\Delta^+}\}.$$

It is easy to show that $\beta_D(\overrightarrow{C_n}) = n$ satisfies a sharp bound. □

Theorem 2.2. *Let D be a digraph of order n. Then*

$$\beta_D(D) \leq mim\{\frac{\Delta^+ - 2\lceil\frac{\delta^- - 1}{2}\rceil}{\Delta^+} \cdot n, \ \frac{2\lfloor\frac{\Delta^- + 1}{2}\rfloor - \delta^+}{\delta^+} \cdot n\}.$$

This bound is sharp.

Proof. Let f be a bad function on D for which $\beta_D(D) = f(V(D))$, and let P, M, p and m be defined as in the proof of Theorem 1. Since $|E(P,v)| \leq |E(M,v)| + 1$ for each $v \in V(D)$, $\delta^- \leq d^-(v) = |E(P,v)| + |E(M,v)| \leq 2|E(M,v)|+1$ for each $v \in V(D)$. We have $|E(M,v)| \geq \lceil \frac{\delta^- -1}{2} \rceil$ for each $v \in V(D)$. Therefore,

$$|E(M,P)| = \sum_{v \in P} |E(M,v)| \geq p\lceil \frac{\delta^- - 1}{2} \rceil = (n-m)\lceil \frac{\delta^-}{2} \rceil.$$

Since

$$|E(D[M])| = \sum_{v \in M} |E(M,v)| \geq m\lceil \frac{\delta^- - 1}{2} \rceil,$$

$$|E(M,P)| = \sum_{v \in M} d^+(v) - |E(D[M])| \leq m\Delta^+ - m\lceil \frac{\delta^- - 1}{2} \rceil.$$

Thus,

$$(n-m)\lceil \frac{\delta^- - 1}{2} \rceil \leq m\Delta^+ - m\lceil \frac{\delta^- - 1}{2} \rceil.$$

It implies that

$$m \geq \frac{\lceil \frac{\delta^- -1}{2} \rceil}{\Delta^+} \cdot n.$$

We have a bound

$$\beta_D(D) = n - 2m \leq \frac{\Delta^+ - 2\lceil \frac{\delta^- -1}{2} \rceil}{\Delta^+} \cdot n.$$

Since $|E(P,v)| \leq |E(M,v)| + 1$ for each $v \in V(D)$, $\Delta^- \geq d^-(v) = |E(P,v)| + |E(M,v)| \geq 2|E(P,v)| - 1$ for each $v \in V(D)$. We have $|E(P,v)| \leq \lfloor \frac{\Delta^- +1}{2} \rfloor$ for each $v \in V(D)$. Therefore,

$$|E(P,M)| = \sum_{v \in M} |E(P,v)| \leq m\lfloor \frac{\Delta^- + 1}{2} \rfloor.$$

Since

$$|E(D[P])| = \sum_{v \in P} |E(P,v)| \leq p\lfloor \frac{\Delta^- + 1}{2} \rfloor,$$

$$|E(P,M)| = \sum_{v \in P} d^+(v) - |E(D[P])| \geq p\delta^+ - p\lfloor \frac{\Delta^- + 1}{2} \rfloor.$$

Now, we get

$$p\delta^+ - p\lfloor \frac{\Delta^- + 1}{2} \rfloor \leq m\lfloor \frac{\Delta^- + 1}{2} \rfloor.$$

So,

$$(n-m)\delta^+ - n\lfloor \frac{\Delta^- + 1}{2} \rfloor \leq 0.$$

It follows that

$$m \geq \frac{\delta^+ - \lfloor \frac{\Delta^- + 1}{2} \rfloor}{\delta^+} \cdot n.$$

We obtain

$$\beta_D(D) = n - 2m \leq \frac{2\lfloor \frac{\Delta^- + 1}{2} \rfloor - \delta^+}{\delta^+} \cdot n.$$

It is easy to show that $\beta_D(\overrightarrow{C_n}) = n$ satisfies a sharp bound. □

C. Wang [4] studied the negative decision number of a graph G as a bad function $f : V(G) \to \{-1, 1\}$ such that $f(N^-(v)) \leq 1$ for every vertex $v \in G$. The sum $\sum_{v \in G} f(v)$ is the weight $w(f)$ of f. The maximum of weights $w(f)$, taken over all bad functions f on G is called the negative decision number of a graph G, denoted by $\beta_D(G)$.

The associated digraph $D(G)$ of a graph G is the digraph oriented when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}[v] = N_G[v]$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

Overservation : If $D(G)$ be the associated digraph of a graph G, then

$$\beta_D(D(G)) = \beta_D(G).$$

From Theorem 1 and Theorem 2, we get the following corollary.

Corollary 2.3. ([4] *Theorem 4) Let G be a r-regular of order n. Then*

$$\beta_D(G) \leq \begin{cases} 0 & \text{if } r \text{ is even,} \\ \frac{1}{r} \cdot n & \text{if } r \text{ is odd.} \end{cases}$$

Theorem 2.4. *Let D be a digraph of order n. If n_e is the number of vertices of even in-degree of $V(D)$, Then*

$$\beta_D(D) \leq \frac{n(\Delta^+ + 1) - n_e}{\delta^+ + \Delta^+}.$$

This bound is sharp.

Proof. Let f be a bad function on D for which $\beta_D(D) = f(V(D))$. Let V_e be the set of vertices of even indegree and $n_e = |V_e|$. Then it is clear that $f(x) \leq 0$ for each $x \in V_e$.
It follows that

$$\sum_{v \in V(D)} f(N^-(v)) = \sum_{v \in V_e} f(N^-(v)) + \sum_{v \in V(D) - V_e} f(N^-(v))$$
$$\leq |V(D) - V_e| = n - n_e.$$

Moreover, we have

$$\sum_{v\in V(D)} f(N^-(v)) = \sum_{v\in V(D)} d^+(v)f(v) = \sum_{v\in P} d^+(v) - \sum_{v\in M} d^+(v)$$
$$\geq p\delta^+ - m\Delta^+ = (n-m)\delta^+ - m\Delta^+ = n\delta^+ - m(\delta^+ + \Delta^+).$$

Thus,

$$n\delta^+ - m(\delta^+ + \Delta^+) \leq n - n_e.$$

It implies

$$\beta_D(D) = n - 2m \leq n - \frac{n(\delta^+ - 1) + n_e}{\delta^+ + \Delta^+}$$
$$\leq \frac{n(\Delta^+ + 1) - n_e}{\delta^+ + \Delta^+}.$$

It is easy to show that $\beta_D(\overrightarrow{C_n}) = n$ satisfies a sharp bound. □

Theorem 2.5. *Let D be a digraph of order n such that $\delta^-(v) \geq 1$ for every v in $V(D)$.*

Then

$$\beta_D(D) \leq n - 2\lceil \frac{\Delta^- - 1}{2} \rceil.$$

This bound is sharp.

Proof. Let $\omega \in V(D)$ be a vertex of maximum indegree $d^-(\omega) = \Delta^-$ and f be a bad function on D for which $f(V(D)) = \beta_D(D)$. Let P and M denote the sets of those vertices in D which are assigned under f the value 1 and -1, respectively. It is clear that $|E(P,\omega)| + |E(M,\omega)| = \Delta^-$. Since $f(\omega) \leq 1$, $|E(P,\omega)| - |E(M,\omega)| \leq 1$. Thus,

$$m \geq |E(M,\omega)| \geq \frac{2|E(M,\omega)|}{2} \geq \frac{|E(P,\omega)| + |E(M,\omega)| - 1}{2} = \frac{\Delta^- - 1}{2},$$

and so $m \geq \lceil \frac{\Delta^- - 1}{2} \rceil$. It follows that $\beta_D(D) = n - 2m \leq n - 2\lceil \frac{\Delta^- - 1}{2} \rceil$. Moreover, it is easy to show that $\beta_D(\overrightarrow{C_n}) = n = n - 2\lceil \frac{\Delta^- - 1}{2} \rceil, n \geq 2$.

□

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