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SOME REMARKS ON NON-SYMPLECTIC AUTOMORPHISMS OF K3 SURFACES OVER A FIELD OF ODD CHARACTERISTIC

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Abstract. In this paper, we present a simple proof of Corollary 3.3 in [5] using the fact that for a K3 surface of finite height over a field of odd characteristic, the height is a multiple of the non-symplectic order. Also we prove for a non-symplectic CM K3 surface defined over a number field, the Frobenius invariant of the reduction over a finite field is determined by the congruence class of residue characteristic modulo the non-symplectic order of the K3 surface.

1. Introduction

Let k be an algebraically closed field of odd characteristic p. Let W be the ring of Witt vectors of k and K be the fraction field of W. Assume X is a K3 surface over k. The second crystalline cohomology of X, $H^2_{cris}(X/W)$ is a free W-module of rank 22 equipped with a canonical Frobenius linear morphism

$$\mathbf{F} : H^2_{cris}(X/W) \to H^2_{cris}(X/W).$$

Let $H^2_{cris}(X/K) = H^2_{cris}(X/W) \otimes K$ be the rational crystalline cohomology. If all the Newton slopes of F-isocrystal $(H^2_{cris}(X/K), F)$ are 1, we say X is supersingular. If X is not supersingular, there exists an integer h between 1 and 10 such that the slopes of $H^2_{cris}(X/K)$ are $1-1/h,\ 1,\ 1+1/h$ of length $h, 22-h, h$ respectively. In this case, we say X is of height h.

If X is of finite height h, the Dieudonné module of the formal Brauer group of X is

$$\mathbb{D}(\widehat{Br}_X) = W[F,V]/(FV = p, F = V^{h-1}).$$

Here $\mathbb{D}(\widehat{Br}_X)$ is a free W-module of rank h equipped with a Frobenius linear operator F and a Frobenius-inverse linear operator V. The Dieudonné module

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$\mathbb{D}(\widehat{Br}_X)$ is isomorphic to $H^2(X, W\mathcal{O}_X)$ ([1]). $H^2_{cris}(X/W)$ has an F-crystal decomposition according to the Newton slopes

$$H^2_{cris}(X/W) = H^2_{cris}(X/W)_{[1-1/h]} \oplus H^2_{cris}(X/W)_{[1]} \oplus H^2_{cris}(X/W)_{[1+1/h]}.$$

Here

$$H^2_{cris}(X/W)_{[1-1/h]} = \mathbb{D}(\widehat{Br}_X)$$

and

$$H^2_{cris}(X/W)_{[1+1/h]} = \mathrm{Hom}(H^2(X, W\mathcal{O}_X), W(p^2)).$$

By the cup product pairing, $H^2_{cris}(X/W)_{[1-1/h]}$ and $H^2_{cris}(X/W)_{[1+1/h]}$ are isotropic and dual to each other. $H^2_{cris}(X/W)_{[1]}$ is unimodular. The discriminant of the $\mathbb{Z}_p$-lattice $H^2_{cris}(X/W)^{F=p}_{[1]}$ is $(-1)^{h+1}$ ([10]). The image of the cycle map

$$NS(X) \otimes W \hookrightarrow H^2_{cris}(X/W)$$

sits inside $H^2_{cris}(X/W)_{[1]}$. Therefore the Picard number of X, the rank of $NS(X)$ is at most $22 - 2h$. Let $T_{cris}(X)$ be the orthogonal complement of the embedding $NS(X) \hookrightarrow H^2_{cris}(X/W)$. We call $T_{cris}(X)$ the crystalline transcendental lattice of X. By the above observation, we can see $H^2_{cris}(X/W)_{[1-1/h]} \oplus H^2_{cris}(X/W)_{[1+1/h]}$ is a direct factor of $T_{cris}(X)$. From the exact sequence of sheaves on X

$$0 \to W\mathcal{O}_X \xrightarrow{V} W\mathcal{O}_X \to \mathcal{O}_X \to 0,$$

we have a canonical morphism

$$H^2_{cris}(X/W)_{[1-1/h]} = \mathbb{D}(\widehat{Br}_X) \simeq H^2(X, W\mathcal{O}_X) \twoheadrightarrow H^2(X, \mathcal{O}_X).$$

Let

$$\chi_{cris,X} : \mathrm{Aut}\, X \to O(T_{cris}(X))$$

and

$$\rho_X : \mathrm{Aut}\, X \to Gl(H^0(X, \Omega^2_{X/k}))$$

be the representation of Aut X on the crystalline transcendental lattice and on global 2 forms. The images of $\chi_{cris,X}$ and ρ_X are finite and there is a canonical projection ([5])

$$\mathrm{Im}\, \chi_{cris,X} \to \mathrm{Im}\, \rho_X.$$

If X is a supersingular K3 surface over k, the rank of $NS(X)$ is 22 ([2], [6]). The cycle map $NS(X) \otimes W \hookrightarrow H^2_{cris}(X)$ is an embedding of W-lattices of same rank and we have

$$NS(X) \otimes W \subset H^2_{cris}(X/W) \subset NS(X)^* \otimes W.$$

The quotient $H^2_{cris}(X/W)/(NS(X) \otimes W)$ is a σ dimensional k-space for an integer σ between 1 and 10. We say σ is the Artin-invariant of X. It is known that $NS(X)$ is completely determined by p and σ ([12]). We denote the Neron-Severi lattice of a supersingular K3 surface of Artin invariant σ over a field of characteristic p by $\Lambda_{p,\sigma}$. For a lattice L, let $d(L)$ be the discriminant of L. The discriminant $d(\Lambda_{p,\sigma})$ is $-p^{2\sigma}$. It is also known that there is a decomposition

$$\Lambda_{p,\sigma} \otimes \mathbb{Z}_p = E_0(p) \oplus E_1.$$

Here E_0 and E_1 are unimodular $\mathbb{Z}_p-$lattices of rank 2σ and $22-2\sigma$ respectively. And $d(E_0) = (-1)^\sigma \delta$ and $d(E_1) = (-1)^{\sigma+1}\delta$, where δ is a non-square unit of $\mathbb{Z}_p$. Note that a unimodular $\mathbb{Z}_p-$lattice is uniquely determined up to isomorphism by the rank and the discriminant, square or non-square.

We denote the characteristic polynomial over an indeterminate T of a linear endomorphism L by $\varphi(L)$. When $\alpha \in \operatorname{Aut} X$ is an automorphism of a K3 surface of X, $\varphi(\alpha^*|H^2_{cris}(X/W))$ is a polynomial with integer coefficients ([4], 3.7.3). If X is of finite height, $\varphi(\chi_{cris,X})$ is also an integral polynomial. An automorphism $\alpha \in \operatorname{Aut} X$ is symplectic if $\rho_X(\alpha) = 1$. An automorphism $\alpha \in \operatorname{Aut} X$ is purely non-symplectic if α is of finite order and the order of α is equal to the order of $\rho_X(\alpha)$. For $\alpha \in \operatorname{Aut} X$, we say the order of $\rho_X(\alpha)$ is the non-symplectic order of α. Also, we call the order of $\operatorname{Im} \rho_X$ the non-symplectic order of X. An automorphism $\alpha \in \operatorname{Aut} X$ is tame if α is of finite order and the order of α is not divisible by p. An automorphism of finite order which is divisible by p is called a wild automorphism. It is known that if $p > 11$, there is no wild automorphism ([3]). When X is of finite height, an automorphism $\alpha \in \operatorname{Aut} X$ is weakly tame if the order of $\chi_{cris,X}(\alpha)$ is not divisible by p. Every tame automorphism is weakly tame. Since $\chi_{cris,X}(\alpha)$ is of finite order, all roots of $\varphi(\chi_{cris,X}(\alpha))$ are roots of unity. Hence if a primitive n-th root of unity is an eigenvalue of $\chi_{cris,X}(\alpha)$, the rank of $T_{cris}(X)$ is at least $\phi(n)$ because $\varphi(\chi_{cris,X}(\alpha)) \in \mathbb{Z}[T]$. Therefore when X is of finite height and $p \geq 23$, every automorphism of X is weakly tame.

If X is of height h, α is a weakly tame automorphism of X and $\rho(\alpha) = \xi$ is of order n, then all the eigenvalues of $\alpha^*|H^2_{cris}(X/W)_{[1-1/h]} \oplus H^2_{cris}(X/W)_{[1+1/h]}$ are $\xi^{\pm 1}, \xi^{\pm p^{-1}}, \cdots, \xi^{\pm p^{1-h}}$ up to multiplicity ([5], Theorem 2.9). By this and an argument based on the Tate conjecture for K3 surfaces ([9], [6]), we also proved the following.

Proposition([5], Corollary 3.3) Let k be an algebraically closed filed of odd characteristic p. Let X be a K3 surface over k equipped with an automorphism $\alpha \in \operatorname{Aut}(X)$ such that the order of $\rho_X(\alpha)$ is $N > 2$. We assume the rank of the Neron-Severi group of X is at least $22 - \phi(N)$. If $p^m \equiv -1$ modulo N for some m, X is supersingular. If $p^m \not\equiv -1$ modulo N for any m and the order of p in $(\mathbb{Z}/N\mathbb{Z})^*$ is n, X is of height n.

In the next section, we present a simple proof of the above proposition. For that, we prove the following theorem.

Theorem 2.1. Let k be an algebraically closed field of odd characteristic p. Assume X is a K3 surface over k and $N > 2$ is the non-symplectic order of X. If $p^m \not\equiv -1$ modulo N for any m, X is of finite height. In this case, if n is the order of p in $(\mathbb{Z}/N)^*$, the height of X is a multiple of n.

When X is a complex algebraic K3 surface, the transcendental lattice of X is the orthogonal complement of the embedding

$$NS(X) \hookrightarrow H^2(X, \mathbb{Z}).$$

We denote the transcendental lattice of X by $T(X)$. When $\rho(X)$ is the Picard number of X, the signature of $T(X)$ is $(2, 20-\rho(X))$. If N is the non-symplectic order of X, the rank of $T(X)$ is a multiple of $\phi(N)$ ([7]). We say X is a non-symplectic CM K3 surface of order N if $\operatorname{rank} T(X)$ is equal to $\phi(N)$. A non-symplectic CM K3 surface gives a CM point in a moduli Shimura variety, so it has a model over a number field ([11]). It seems like that there are only few non-symplectic CM K3 surfaces. Also it is known that there is a unique non-symplectic CM K3 surface of order N for many N. In a previous work ([5]), if X is a non-symplectic CM K3 surface of order N with $\phi(N) > 10$ and $\operatorname{Im} \rho_X$ is generated by a purely non-symplectic automorphism, we proved that the height and the Artin invariant of a reduction of X over a field of odd characteristic p is determined by the congruence class of p modulo N. In the next section, we give a generalization of this result for an arbitrary non-symplectic CM K3 surface.

2. Results

Theorem 2.1. Let k be an algebraically closed field of odd characteristic p. Assume X is a K3 surface over k and $N > 2$ is the non-symplectic order of X. If $p^m \not\equiv -1$ modulo N for any m, X is of finite height. In this case, if n is the order of p in $(\mathbb{Z}/N)^*$, the height of X is a multiple of n.

Proof. If X is a supersingular K3 surface of Artin-invariant σ, the non-symplectic order of X divides $p^\sigma + 1$ ([8], Theorem 2.1). Hence under the assumption, X is not supersingular. Let us choose an automorphism $\alpha \in \operatorname{Aut} X$ such that $\operatorname{Im} \rho_X$ is generated by $\rho_X(\alpha)$. If the order of $\chi_{cris,X}(\alpha)$ is $p^r M$ for $r, M \in \mathbb{N}$ with $p \nmid M$, α^{p^r} also generates $\operatorname{Im} \rho_X$ and is weakly tame. After replacing α by α^{p^r}, we may assume α is weakly tame. Suppose the height of X is h. If $\xi = \rho_X(\alpha)$, by ([5], Theorem 2.9), all the eigenvalues of $\alpha^*|H^2_{cris}(X/W)_{[1-1/h]}$ are $\xi^{-1}, \xi^{-1/p}, \cdots, \xi^{-1/p^{h-1}}$ up to multiplicity. On the other hand, if $\alpha^* x = \lambda x$ for some $x \in H^2_{cris}(X/W)_{[1-1/h]}$,

$$\alpha^*(Vx) = V(\alpha^* x) = V(\lambda x) = \lambda^{1/p} V x.$$

Therefore for any $i \in \mathbb{Z}$, ξ^{-1/p^i} is an eigenvalue of $\alpha^*|H^2_{cris}(X/W)_{[1-1/h]}$ and the rank of eigenspace for the eigenvalue ξ^{-1/p^i} is equal to the rank of eigenspace for the eigenvalue ξ^{-1}. Hence the height h is divided by the order of p in $(\mathbb{Z}/N)^*$. □

Corollary 2.2. Let k be an algebraicly closed filed of odd characteristic p. X is a K3 surface over k equipped with an automorphism $\alpha \in \mathrm{Aut}(X)$ such that the order of $\rho_X(\alpha)$ is $N > 2$. We assume the rank of the Neron-Severi group of X is at least $22 - \phi(N)$. If $p^m \not\equiv -1$ modulo N for any m and the order of p in $(Z/NZ)^*$ is n, X is of height n.

Proof. By Theorem 2.1, X is of finite height and the height of X is a multiple of n. By the assumption, the rank of $T_{cris}(X)$ is $\phi(N)$. Since $\varphi(\chi_{cris,X}(\alpha))$ is an integral polynomial of degree $\phi(N)$ and a primitive N-th root of unity is an eigenvalue of $\chi_{cris,X}(\alpha)$, $\varphi(\chi_{cris,X}(\alpha))$ is the N-th cyclotomic polynomial. It follows that every eigenvalue of $\chi_{cris,X}(\alpha)$ occurs only one time. Considering ([5], Theorem 2.9), the height of X is at most n. Therefore the height of X is n. □

Let X be a non-symplectic CM K3 surface of order N. We may assume X is defined over a number field F. Let us fix a smooth projective integral model X_R of X over an integer ring R with $\mathrm{Spec}\, R \subset \mathrm{Spec}\, \mathfrak{o}_F$. For each place $v \in \mathrm{Spec}\, R$, let p_v be the residue characteristic of v. We assume $p_v \nmid N$ and p_v is unramified in F for any $v \in \mathrm{Spec}\, R$. We denote the reduction of X_R over the residue field $k(v)$ by X_v.

Theorem 2.3. If $p_v^m \not\equiv -1$ modulo N for any $m \in \mathbb{Z}$, X_v is of finite height and the height of X_v is the order of p_v in $(\mathbb{Z}/N)^*$. If $p_v^m \equiv -1$ for some $m \in \mathbb{Z}$, X_v is supersingular. Moreover if p_v does not divide $d(NS(X))$, the Artin-invariant of X_v is the half of the order of p_v in $(\mathbb{Z}/N)^*$.

Proof. There is an embedding

$$NS(X) \hookrightarrow NS(X_v),$$

so the rank of $NS(X_v)$ is at least $22 - \phi(N)$. Let N_v be the non-symplectic order of X_v. Then N_v is a multiple of N. If $p_v^m \not\equiv -1$ modulo N for any $m \in \mathbb{Z}$, $p_v^m \not\equiv -1$ modulo N_v, so X_v is of finite height. Since the rank of $T(X_v$ is at least $\phi(N_v)$, $\phi(N) = \phi(N_v)$ and $N_v = N$ or $N_v = 2N$. In any case, the order of p in $(\mathbb{Z}/N_v)^*$ is equal to the order of p in $(\mathbb{Z}/N)^*$. By Corollary 2.2, the height of X_v is the order of p_v in $(\mathbb{Z}/N)^*$. Assume the order of $\xi = \rho_X(\alpha)$ is N for some $\alpha \in \mathrm{Aut}\, X$. Let $T_{cris}(X)$ be the orthogonal complement of the embedding

$$NS(X) \otimes W \hookrightarrow NS(X_v) \otimes W \hookrightarrow H^2_{cris}(X_{upsilon}/W).$$

Since $H^2_{cris}(X/W)$ is canonically isomorphic to $H^2_{dr}(X_R/R) \otimes W$, $\alpha^*|T_{cris}(X)$ is of finite order. If X_v is of finite height, $T_{cris}(X_v) \subset T_{cris}(X)$ and all the eigenvalues of $\alpha^*|T_{cris}(X_v)$ are distinct. If $p^m \equiv -1$ modulo N, $\xi^{-1/p^m} = \xi$ occurs as an eigenvalue of $\alpha^*|H^2_{cris}(X/W)_{[1-1/h]}$. But ξ is also an eigenvalue of $\alpha^*|H^2_{cris}(X/W)_{[1+1/h]}$ and it is a contradiction. Therefore X_v is supersingular.

If p does not divide $d(NS(X))$, $NS(X) \otimes W$ is unimodular and the Artin-invariant of X_v is at most $\phi(N)/2$. Also when σ is the Artin-invariant of X_v, N divides $p^\sigma + 1$, so $p^\sigma \equiv -1$ modulo N. We have an isomorphism $NS(X_v)^*/NS(X_v) = T_{cris}(X)^*/T_{cris}(X)$ compatible with the action of $\mathrm{Aut}\, X$. By ([5], theorem 2.9), if σ is greater than the half of the order of p_v in $(\mathbb{Z}/N)^*$, ξ appears more than twice in the eigenvalues of $\alpha^*|T_{cris}(X)$. Therefore the Artin-invariant of X_v is the half of the order of p_v in $(\mathbb{Z}/N)^*$. □

Remark 2.4. If a non-symplectic CM K3 surface of order N, X has a reduction of height $\phi(N)/2$ or of Artin invariant $\phi(N)/2$, the Legendre symbol $\left(\frac{d(NS(X))}{p}\right)$ is constant for all primes p in a congruence class modulo N.

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