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ON GENERALIZED TRIANGULAR MATRIX RINGS

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Abstract. For a generalized triangular matrix ring $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$, over rings R and S having only the idempotents 0 and 1 and over an (R, S)-bimodule M, we characterize all homomorphisms α's and all α-derivations of T. Some of the homomorphisms are compositions of an inner homomorphism and an extended or a twisted homomorphism.

1. Introduction

For R and S are rings with identity and M is an (R, S)-bimodule, we consider a generalized triangular matrix ring $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$. Automorphisms of T were characterized when R and S have only the trivial idempotents 0 and 1(see[5]). Moreover, in case R and S are strongly indecomposable all automorphisms of T are observed in [1]. In these cases, every automorphism is a composition of an extended automorphism and an inner automorphism. In [4], Ghosseiri determined the structure of (α, β)-derivations of T, where α and β are automorphisms of T. Moreover, Ghahramani and Moussavi characterized homomorphisms and derivations of T (see [2],[3]).

In this paper, we characterize all homomorphisms α's of T and all α-derivations of T, and get four types of homomorphisms of T, where one type is a composition of an inner homomorphism and an extended homomorphism and other one type is a composition of an inner homomorphism and a twisted homomorphism.

Throughout this paper, for a generalized triangular matrix ring T, R and S have only the trivial idempotents 0 and 1 and M is an (R, S)-bimodule. Every endomorphism α means a ring homomorphism preserving identity i.e., $\alpha(1) = 1$ and for a homomorphism α the additive map $\delta : T \to T$ is called an α-derivation if $\delta(tt^{'}) = \alpha(t)\delta(t^{'}) + \delta(t)t^{'}, (t, t^{'} \in T)$.

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2. Homomorphisms of T

In this section, we characterize that there exist only four types of homomorphisms of T. First we define four types of homomorphisms.

Type I. Let $\phi_1 : R \to R$ and $\psi_1 : S \to S$ be homomorphisms and $\theta_1 : M \to M$ is a ϕ_1, ψ_1-bimodule homomorphism. For a fixed element m of M, if we define $\alpha : T \to T$ by

$$\alpha\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} \phi_1(a) & \theta_1(b) \\ 0 & \psi_1(c) \end{bmatrix}, \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in T\right).$$

Then we can easily check that α is a homomorphism of T. Since $\begin{bmatrix} 1 & -m \\ 0 & 1 \end{bmatrix}$ is an inverse element of $\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$ induces an inner automorphism inn_{m_t}, where m_t stands for $\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$.

If we define $\alpha_1 = inn_{m_t} \cdot \alpha$, then α_1 is a homomorphism i.e.,

$$\alpha_1\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 1 & -m \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \phi_1(a) & \theta_1(b) \\ 0 & \psi_1(c) \end{bmatrix}\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in T\right).$$

This homomorphism α_1 is a composition of an inner automorphism and an extended homomorphism.

Type II. Let $\phi_2 : R \to S$ and $\psi_2 : S \to R$ be homomorphisms. For a fixed element m of M, if we define $\alpha_2 : T \to T$ by

$$\alpha_2\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \psi_2(c) & 0 \\ 0 & \phi_2(a) \end{bmatrix}\begin{bmatrix} 1 & -m \\ 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in T\right).$$

Then α_2 is a homomorphism, which is a composition of an inner automorphism and a twisted homomorphism.

Type III. Let $\phi_3 : R \to R$ and $\psi_3 : R \to S$ be homomorphisms and $\theta_3 : R \to M$ be an additive map such that $\theta_3(aa^{'}) = \phi_3(a)\theta_3(a^{'}) + \theta_3(a)\psi_3(a^{'})(a, a^{'} \in R)$. If we define $\alpha_3 : T \to T$ by

$$\alpha_3\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} \phi_3(a) & \theta_3(a) \\ 0 & \psi_3(a) \end{bmatrix}, \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in T\right).$$

Then α_3 is a homomorphism.

Type IV. Let $\phi_4 : S \to R$ and $\psi_4 : S \to S$ be homomorphisms and $\theta_4 : S \to M$ be an additive homomorphism such that $\theta_4(cc^{'}) = \phi_4(c)\theta_4(c^{'}) + \theta_4(c)\psi_4(c^{'}), (c, c^{'} \in S)$. If we define $\alpha_4 : T \to T$ by

$$\alpha_4\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} \phi_4(c) & \theta_4(c) \\ 0 & \psi_4(c) \end{bmatrix}, \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in T\right).$$

Then α_4 is a homomorphism.

For convenience, denote the idempotents $e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Also, $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Lemma 2.1. *If $\alpha : T \to T$ is a homomorphism such that $\alpha(e_{11}) = \begin{bmatrix} r_1 & m_1 \\ 0 & s_1 \end{bmatrix}$, $\alpha(e_{22}) = \begin{bmatrix} r_2 & m_2 \\ 0 & s_2 \end{bmatrix}$ and $\alpha\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} r_b & m_b \\ 0 & s_b \end{bmatrix}$ for $b \in M$. Then*

i) r_1, s_1, r_2, s_2 are 0 or 1 and $r_i m_i + m_i s_i = m_i (i = 1, 2)$.

ii) $r_1 + r_2 = 1, m_1 + m_2 = 0$ and $s_1 + s_2 = 1$.

iii) $r_b = s_b = 0, r_1 m_b = m_b$ and $m_b s_1 = 0$.

Proof.

i) Since $\alpha(e_{11}) = (\alpha(e_{11}))^2$, $\begin{bmatrix} r_1 & m_1 \\ 0 & s_1 \end{bmatrix} = \begin{bmatrix} r_1 & m_1 \\ 0 & s_1 \end{bmatrix}\begin{bmatrix} r_1 & m_1 \\ 0 & s_1 \end{bmatrix} = \begin{bmatrix} {r_1}^2 & r_1 m_1 + m_1 s_1 \\ 0 & {s_1}^2 \end{bmatrix}$. Then $r_1^2 = r_1$, $s_1^2 = s_1$ and $r_1 m_1 + m_1 s_1 = m_1$. Similarly, ${r_2}^2 = r_2, {s_2}^2 = s_2$ and $r_2 m_2 + m_2 s_2 = m_2$. Thus r_1, s_1, r_2, s_2 are 0 or 1.

ii) since $1 = \alpha(1) = \alpha(e_{11}) + \alpha(e_{22}) = \begin{bmatrix} r_1 + r_2 & m_1 + m_2 \\ 0 & s_1 + s_2 \end{bmatrix}$, $r_1 + r_2 = 1, m_1 + m_2 = 0$ and $s_1 + s_2 = 1$.

iii) By hypothesis, $\alpha\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} r_b & m_b \\ 0 & s_b \end{bmatrix}$ and $\alpha\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \alpha(e_{11})\alpha\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} r_1 & m_1 \\ 0 & s_1 \end{bmatrix}\begin{bmatrix} r_b & m_b \\ 0 & s_b \end{bmatrix} = \begin{bmatrix} r_1 r_b & r_1 m_b + m_1 s_b \\ 0 & s_1 s_b \end{bmatrix}$. So, $r_1 r_b = r_b, s_1 s_b = s_b$ and $r_1 m_b + m_1 s_b = m_b$.

On the other hand, $0 = \alpha(0) = \alpha\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right)\alpha(e_{11}) = \begin{bmatrix} r_b & m_b \\ 0 & s_b \end{bmatrix}\begin{bmatrix} r_1 & m_1 \\ 0 & s_1 \end{bmatrix} = \begin{bmatrix} r_b r_1 & r_b m_1 + m_b s_1 \\ 0 & s_b s_1 \end{bmatrix}$. So, $r_b r_1 = 0, s_b s_1 = 0$ and $r_b m_1 + m_b s_1 = 0$. Thus $r_b = s_b = 0, r_1 m_b = m_b$ and $m_b s_1 = 0$. □

Remark 2.2. From Lemma 2.1, we conclude that there are only four cases of homomorphisms α's which are depended on the values of r_i, s_i, m_i. The four cases are followings ;

Case I. $r_1 = 1, r_2 = 0, s_1 = 0, s_2 = 1$. This implies $m_1 = m, m_2 = -m$.

Case II. $r_1 = 0, r_2 = 1, s_1 = 1, s_2 = 0$. This implies $m_1 = m, m_2 = -m$ and $m_b = 0$.

Case III. $r_1 = 1, r_2 = 0, s_1 = 1, s_2 = 0$. This implies $m_1 = m_2 = m_b = 0$.

Case IV. $r_1 = 0, r_2 = 1, s_1 = 0, s_2 = 1$. This implies $m_1 = m_2 = m_b = 0$.

From Lemma 2.1 and Remark 2.2, we have the following theorems.

Theorem 2.3. *Case I homomorphisms are Type I homomorphisms.*

Proof. Since $r_1 = s_2 = 1, r_2 = s_1 = 0$, $\alpha(e_{11}) = \begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix}, \alpha(e_{22}) = \begin{bmatrix} 0 & -m \\ 0 & 1 \end{bmatrix}$ and $\alpha\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & m_b \\ 0 & 0 \end{bmatrix}$. Thus we can define $\theta_1 : M \to M$ by $\theta_1(b) = m_b$.

Let $\alpha(ae_{11}) = \left(\begin{bmatrix} r_a & m_a \\ 0 & s_a \end{bmatrix}\right)$. Then $\alpha(ae_{11}) = \alpha(ae_{11})\alpha(e_{11}) = \begin{bmatrix} r_a & m_a \\ 0 & s_a \end{bmatrix} \begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r_a & r_a m \\ 0 & 0 \end{bmatrix}$. So, $s_a = 0$ and $m_a = r_a m$. Thus we can define $\phi_1 : R \to R$ by $\phi_1(a) = r_a$.

Let $\alpha(ce_{22}) = \left(\begin{bmatrix} r_c & m_c \\ 0 & s_c \end{bmatrix}\right)$. Then $\alpha\left(ce_{22}\right) = \alpha(e_{22})\alpha(ce_{22})] = \begin{bmatrix} 0 & -m \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_c & m_c \\ 0 & s_c \end{bmatrix} = \begin{bmatrix} 0 & -ms_c \\ 0 & s_c \end{bmatrix}$. So, $r_c = 0$ and $m_c = -ms_c$. Thus we can define $\psi_1 : S \to S$ by $\psi_1(c) = s_c$.

Therefore, we conclude $\alpha\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} r_a & r_a m + m_b - ms_c \\ 0 & s_c \end{bmatrix} = \begin{bmatrix} \phi_1(a) & \phi_1(a)m + \theta_1(b) - m\psi_1(c) \\ 0 & \psi_1(c) \end{bmatrix} = \begin{bmatrix} 1 & -m \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_1(a) & \theta_1(b) \\ 0 & \psi_1(c) \end{bmatrix} \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$.

Moreover, for $a \in R$ and $b \in M$, $\alpha\left(\begin{bmatrix} 0 & ab \\ 0 & 0 \end{bmatrix}\right) = \alpha(ae_{11})\alpha\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} \phi_1(a) & \phi_1(a)m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \theta_1(b) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \phi_1(a)\theta_1(b) \\ 0 & 0 \end{bmatrix}$ and $\alpha \begin{bmatrix} 0 & ab \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \theta_1(ab) \\ 0 & 0 \end{bmatrix}$. So, $\theta_1(ab) = \phi_1(a)\theta_1(b)$.

Similarly, for $b \in M$ and $c \in S$, $\theta_1(bc) = \theta_1(b)\psi_1(c)$. □

Theorem 2.4. *Case II homomorphisms are Type II homomorphisms.*

Proof. Since $r_1 = s_2 = 0, r_2 = s_1 = 1$ and $r_b = s_b = m_b = 0$, $\alpha(e_{11}) = \begin{bmatrix} 0 & m \\ 0 & 1 \end{bmatrix}, \alpha(e_{22}) = \begin{bmatrix} 1 & -m \\ 0 & 0 \end{bmatrix}$ and $\alpha\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = 0$.

Let $\alpha(ae_{11}) = \begin{bmatrix} r_a & m_a \\ 0 & s_a \end{bmatrix}$. Then $\alpha(ae_{11}) = \alpha(e_{11})\alpha(ae_{11}) = \begin{bmatrix} 0 & m \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} r_a & m_a \\ 0 & s_a \end{bmatrix} = \begin{bmatrix} 0 & ms_a \\ 0 & s_a \end{bmatrix}$. So, $r_a = 0$. Thus we can define $\phi_2 : R \to S$ by $\phi_2(a) = s_a$.

Let $\alpha\left(ce_{22})\right) = \begin{bmatrix} r_c & m_c \\ 0 & s_c \end{bmatrix}$. Then $\alpha(ce_{22}) = \alpha(ce_{22})\alpha(e_{22}) = \begin{bmatrix} r_c & m_c \\ 0 & s_c \end{bmatrix}$ $\begin{bmatrix} 1 & -m \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r_c & -r_c m \\ 0 & 0 \end{bmatrix}$. So, $s_c = 0$. Thus we can define $\psi_2 : S \to R$ by $\psi_2(c) = r_c$.

Therefore, we conclude $\alpha\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} r_c & ms_a - r_c m \\ 0 & s_a \end{bmatrix}$
$= \begin{bmatrix} \psi_2(c) & m\phi_2(a) - \psi_2(c)m \\ 0 & \phi_2(a) \end{bmatrix} = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \psi_2(c) & 0 \\ 0 & \phi_2(a) \end{bmatrix} \begin{bmatrix} 1 & -m \\ 0 & 1 \end{bmatrix}$.
□

Theorem 2.5. *Case III homomorphisms are Type III homomorphisms.*

Proof. Since $r_1 = s_1 = 1$, $r_2 = s_2 = 0$ and $m_1 = m_2 = m_b = 0$, $\alpha(e_{11}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\alpha(e_{22}) = 0$ and $\alpha\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = 0$.

Let $\alpha(ae_{11}) = \begin{bmatrix} r_a & m_a \\ 0 & s_a \end{bmatrix}$, then we can define $\phi_3 : R \to R$ by $\phi_3(a) = r_a, \psi_3 : R \to S$ by $\psi_3(a) = s_a$ and $\theta_3 : R \to M$ by $\theta_3(a) = m_a$.

Now, for $a, a' \in R$, $\alpha(aa'e_{11}) = \alpha(ae_{11})\alpha(a'e_{11}) = \begin{bmatrix} \phi_3(a) & \theta_3(a) \\ 0 & \psi_3(a) \end{bmatrix}$ $\begin{bmatrix} \phi_3(a') & \theta_3(a') \\ 0 & \psi_3(a') \end{bmatrix} = \begin{bmatrix} \phi_3(a)\phi_3(a') & \phi_3(a)\theta_3(a') + \theta_3(a)\psi_3(a') \\ 0 & \psi_3(a)\psi_3(a') \end{bmatrix}$ and $\alpha(aa'e_{11})$ $= \begin{bmatrix} \phi_3(aa') & \theta_3(aa') \\ 0 & \psi_3(aa') \end{bmatrix}$. This means that ϕ_3, ψ_3 are homomorphisms and θ_3 is additive, which satisfies $\theta_3(aa') = \phi_3(a)\theta_3(a') + \theta_3(a)\psi_3(a')$. □

Theorem 2.6. *Case IV homomorphisms are Type IV homomorphisms.*

Proof. The proof is similar to the proof of Theorem 2.5. □

Remark 2.7. Case II, III and IV homomorphisms cannot be isomorphisms. So, every automorphism of T is a Type I automorphism which is a composition of an inner automorphism and an extended automorphism.

3. Derivations of T

In this section, we will observe all α-derivations where α is a homomorphism given in Section 2. Since there are four types of homomorphisms, we get four types of α-derivations.

Theorem 3.1. *Let $\alpha_1 : T \to T$ be a Type I homomorphism for some fixed element $m \in M$. Then $\delta : T \to T$ is an α_1- derivation if and only if there exist*

(i) $f : R \to R$ is a ϕ_1-derivation,

(ii) $g : S \to S$ is a ψ_1-derivation and

(iii) $h : M \to M$ is an additive map satisfying $h(ab) = \phi_1(a)h(b) + f(a)b$ and $h(bc) = \theta_1(b)g(c) + h(b)c$ such that for some $b_1 \in M$,

$$\delta\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} f(a) & \phi_1(a)b_1 - b_1c - mg(c) + h(b) \\ 0 & g(c) \end{bmatrix},$$

where ϕ_1, ψ_1, θ_1 are maps given in Type I homomorphisms in Section 2.

Proof. Since α_1 is a Type I homomorphism, assume $\alpha_1\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 1 & -m \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \phi_1(a) & \theta_1(b) \\ 0 & \psi_1(c) \end{bmatrix}\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \phi_1(a) & \phi_1(a)m + \theta_1(b) - m\psi_1(c) \\ 0 & \psi_1(c) \end{bmatrix}$ for some $m \in M$.

Let $\delta(e_{11}) = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}$. Then $\delta(e_{11}) = \alpha_1(e_{11})\delta(e_{11}) + \delta(e_{11})e_{11} = \begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}e_{11} = \begin{bmatrix} a_1 & b_1 + mc_1 \\ 0 & 0 \end{bmatrix} + a_1e_{11} = \begin{bmatrix} 2a_1 & b_1 + mc_1 \\ 0 & 0 \end{bmatrix}$. So, $c_1 = a_1 = 0$. Thus $\delta(e_{11}) = \begin{bmatrix} 0 & b_1 \\ 0 & 0 \end{bmatrix}$.

Since $0 = \delta(1) = \delta(e_{11}) + \delta(e_{22})$, $\delta(e_{22}) = \begin{bmatrix} 0 & -b_1 \\ 0 & 0 \end{bmatrix}$.

(i) Let $\delta(ae_{11}) = \begin{bmatrix} a_R & a_M \\ 0 & a_S \end{bmatrix}$. Then $\delta(ae_{11}) = \delta(e_{11}ae_{11}) = \alpha_1(e_{11})\delta(ae_{11}) + \delta(e_{11})(ae_{11}) = \begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_R & a_M \\ 0 & a_S \end{bmatrix} + \begin{bmatrix} 0 & b_1 \\ 0 & 0 \end{bmatrix}ae_{11} = \begin{bmatrix} a_R & a_M + ma_S \\ 0 & 0 \end{bmatrix}$. So, $a_S = 0$. Thus $\delta(ae_{11}) = \begin{bmatrix} a_R & a_M \\ 0 & 0 \end{bmatrix}$.

On the other hand, $\delta(ae_{11}) = \delta(ae_{11}e_{11}) = \alpha_1(ae_{11})\delta(e_{11}) + \delta(ae_{11})e_{11} = \begin{bmatrix} \phi_1(a) & \phi_1(a)m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & b_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_R & a_M \\ 0 & 0 \end{bmatrix}e_{11} = \begin{bmatrix} 0 & \phi_1(a)b_1 \\ 0 & 0 \end{bmatrix} + a_Re_{11} = \begin{bmatrix} a_R & \phi_1(a)b_1 \\ 0 & 0 \end{bmatrix}$. So, $a_M = \phi_1(a)b_1$. Thus $\delta(ae_{11}) = \begin{bmatrix} a_R & \phi_1(a)b_1 \\ 0 & 0 \end{bmatrix}$.

If we define $f : R \to R$ by $f(a) = a_R$, then we can easily check that f is a ϕ_1-derivation.

(ii) Let $\delta(ce_{22}) = \begin{bmatrix} c_R & c_M \\ 0 & c_S \end{bmatrix}$. Then $\delta(ce_{22}) = \delta(e_{22}ce_{22}) = \alpha_1(e_{22})\delta(ce_{22})+$ $\delta(e_{22})ce_{22} = \begin{bmatrix} 0 & -m \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_R & c_M \\ 0 & c_S \end{bmatrix} + \begin{bmatrix} 0 & -b_1 \\ 0 & 0 \end{bmatrix} ce_{22} = \begin{bmatrix} 0 & -mc_S \\ 0 & c_S \end{bmatrix} +$ $\begin{bmatrix} 0 & -b_1c \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -mc_S - b_1c \\ 0 & c_S \end{bmatrix}$. So, $c_R = 0$ and $-b_1c - mc_S = c_M$. Thus $\delta(ce_{22}) = \begin{bmatrix} 0 & -b_1c - mc_S \\ 0 & c_S \end{bmatrix}$.

If we define $g : S \to S$ by $g(c) = c_S$, then g is a ψ_1-derivation.

(iii) Let $\delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} b_R & b_M \\ 0 & b_S \end{bmatrix}$. Then $\delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \delta\left(e_{11}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right)$ $= \alpha_1(e_{11})\delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right)+\delta(e_{11})\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b_R & b_M \\ 0 & b_S \end{bmatrix}+\begin{bmatrix} 0 & b_1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} b_R & b_M + mb_S \\ 0 & 0 \end{bmatrix}$. So, $b_S = 0$.

On the other hand, $\delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} e_{22}\right) = \alpha_1\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right)\delta(e_{22})+$ $\delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right)e_{22} = \begin{bmatrix} 0 & \theta_1(b) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -b_1 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} b_R & b_M \\ 0 & b_S \end{bmatrix}e_{22} = \begin{bmatrix} 0 & b_M \\ 0 & b_S \end{bmatrix}$. So, $b_R = 0$. Thus $\delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & b_M \\ 0 & 0 \end{bmatrix}$.

If we define $h : M \to M$ by $h(b) = b_M$, then h satisfies the property of (iii).

This means $\delta(ae_{11}) = \begin{bmatrix} f(a) & \phi_1(a)b_1 \\ 0 & 0 \end{bmatrix}$, $\delta(ce_{22}) = \begin{bmatrix} 0 & -b_1c - mg(c) \\ 0 & g(c) \end{bmatrix}$, and $\delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & b_M \\ 0 & 0 \end{bmatrix}$. That is,

$$\delta\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} f(a) & \phi_1(a)b_1 - b_1c - mg(c) + h(b) \\ 0 & g(c) \end{bmatrix}.$$

Conversely, we define $\delta\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} f(a) & \phi_1(a)b_1 - b_1c - mg(c) + h(b) \\ 0 & g(c) \end{bmatrix}$. Then $\alpha_1\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)\delta\left(\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}\right) + \delta\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}$

$$= \begin{bmatrix} \phi_1(a) & \phi_1(a)m + \theta_1(b) - m\psi_1(c) \\ 0 & \psi_1(c) \end{bmatrix}\begin{bmatrix} f(a') & \phi_1(a')b_1 - b_1c' - mg(c') + h(b') \\ 0 & g(c') \end{bmatrix}$$
$$+ \begin{bmatrix} f(a) & \phi_1(a)b_1 - b_1c - mg(c) + h(b) \\ 0 & g(c) \end{bmatrix}\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix}$$

$= \begin{bmatrix} \phi_1(a)f(a') & A \\ 0 & \psi_1(c)g(c') \end{bmatrix} + \begin{bmatrix} f(a)a' & A' \\ 0 & g(c)c' \end{bmatrix}$, where $A = \phi_1(a)\phi_1(a')b_1-$ $\phi_1(a)b_1c' - \phi_1(a)mg(c') + \phi_1(a)h(b') + \phi_1(a)mg(c') + \theta_1(b)g(c') - m\psi_1(c)g(c')$ and $A' = f(a)b' + \phi_1(a)b_1c' - b_1cc' - mg(c)c' + h(b)c'$.

Here, $(1,1)$-component is $\phi_1(a)f(a') + f(a)a' = f(aa')$.

$(2,2)$- component is $\psi_1(c)g(c^{'}) + g(c)c^{'} = g(cc^{'})$.

$(1,2)$-component is $\phi_1(a)\phi_1(a^{'})b_1 - \phi_1(a)b_1c^{'} - \phi_1(a)mg(c^{'}) + \phi_1(a)h(b^{'}) + \phi_1(a)mg(c^{'}) + \theta_1(b)g(c^{'}) - m\psi_1(c)g(c^{'}) + f(a)b^{'} + \phi_1(a)b_1c^{'} - b_1cc^{'} - mg(c)c^{'} + h(b)c^{'} = \phi_1(aa^{'})b_1 - b_1cc^{'} - mg(cc^{'}) + h(ab^{'} + bc^{'})$.

On the other hand, $\delta\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} a^{'} & b^{'} \\ 0 & c^{'} \end{bmatrix}\right) = \delta\begin{bmatrix} aa^{'} & ab^{'}+bc^{'} \\ 0 & cc^{'} \end{bmatrix} = \begin{bmatrix} f(aa^{'}) & \phi_1(aa^{'})b_1 - b_1cc^{'} - mg(cc^{'}) + h(ab^{'}+bc^{'}) \\ 0 & g(cc^{'}) \end{bmatrix}$.

Therefore δ is an α_1-derivation. □

Corollary 3.2. *In Theorem 3.1, let* $\delta_1 : T \to T$ *by* $\delta_1\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 0 & \phi_1(a)b_1 - b_1c \\ 0 & 0 \end{bmatrix}$ *and* $\delta_2 : T \to T$ *by* $\delta_2\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} f(a) & -mg(c)+h(b) \\ 0 & g(c) \end{bmatrix}$. *Then* δ_1 *and* δ_2 *are* α_1*-derivations.*

Remark 3.3. In Corollary 3.2, $\delta_1\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 0 & \phi_1(a)b_1 - b_1c \\ 0 & 0 \end{bmatrix} = \alpha_1\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)\begin{bmatrix} 0 & b_1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & b_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$. So, δ_1 is an inner α_1-derivation for $\begin{bmatrix} 0 & b_1 \\ 0 & 0 \end{bmatrix}$.

Moreover, if $m = 0$, then α_1 is a trivial extension on T and every derivation is a sum of an inner α_1-derivation and an extended derivation. This satisfies [2, Proposition 2.6] and [3, Theorem 3.2].

Theorem 3.4. *Let* $\alpha_2 : T \to T$ *be a Type II homomorphism for some fixed element* $m \in M$. *Then* $\delta : T \to T$ *is an* α_2*-derivation if and only if there exist*

(i) $g : M \to S$ *is an additive map satisfying* $\forall a \in R, b \in M, c \in S, g(ab) = \phi_2(a)g(b)$ *and* $g(bc) = g(b)c$ *and*

(ii) $h : S \to M$ *is an additive map satisfying* $\forall c, s \in S, h(cs) = \psi_2(c)h(s) + h(c)s + \psi_2(c)mc_1s$ *for some* $c_1 \in S$ *such that for some* $a_1 \in R$,

$$\delta\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a_1a - \psi_2(c)a_1 & m\phi_2(a)c_1 + a_1b + mg(b) + h(c) \\ 0 & \phi_2(a)c_1 + g(b) - c_1c \end{bmatrix},$$

where ϕ_2, ψ_2, θ_2 *are maps given in Type II homomorphisms in Section 2.*

Proof. Assume δ is an α_2-derivation and α_2 be Type II homomorphism. So, $\alpha_2\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \psi_2(c) & 0 \\ 0 & \phi_2(a) \end{bmatrix}\begin{bmatrix} 1 & -m \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \psi_2(c) & m\phi_2(a) - \psi_2(c)m \\ 0 & \phi_2(a) \end{bmatrix}$ for some $m \in M$.

Let $\delta(e_{11}) = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}$. Then $\delta(e_{11}) = \delta(e_{11}e_{11}) = \alpha_2(e_{11})\delta(e_{11}) + \delta(e_{11})e_{11} = \begin{bmatrix} 0 & m \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} e_{11} = \begin{bmatrix} a_1 & mc_1 \\ 0 & c_1 \end{bmatrix}$. Thus $\delta(e_{11}) = \begin{bmatrix} a_1 & mc_1 \\ 0 & c_1 \end{bmatrix}$ and $\delta(e_{22}) = \begin{bmatrix} -a_1 & -mc_1 \\ 0 & -c_1 \end{bmatrix}$.

Let $\delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} b_R & b_M \\ 0 & b_S \end{bmatrix}$. Then $\delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} e_{22}\right)$
$= \alpha_2\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right)\delta(e_{22}) + \delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right)e_{22} = 0\begin{bmatrix} -a_1 & -mc_1 \\ 0 & -c_1 \end{bmatrix} + \begin{bmatrix} b_R & b_M \\ 0 & b_S \end{bmatrix} e_{22}$
$= \begin{bmatrix} 0 & b_M \\ 0 & b_S \end{bmatrix}$. So, $b_R = 0$.

On the other hand, $\delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \delta\left(e_{11}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \alpha_2(e_{11})\delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) + \delta(e_{11})\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & mb_S \\ 0 & b_S \end{bmatrix} + \begin{bmatrix} 0 & a_1 b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & mb_S + a_1 b \\ 0 & b_S \end{bmatrix}$. So, $b_M = a_1 b + mb_S$.

Therefore we can define $g : M \to S$ by $g(b) = b_S$, where $\delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a_1 b + mb_S \\ 0 & b_S \end{bmatrix}$.

Let $\delta(ae_{11}) = \begin{bmatrix} a_R & a_M \\ 0 & a_S \end{bmatrix}$. Then $\delta(ae_{11}) = \delta(ae_{11}e_{11})) = \alpha_2(ae_{11})\delta(e_{11}) + \delta(ae_{11})e_{11} = \begin{bmatrix} 0 & ms_a \\ 0 & s_a \end{bmatrix}\begin{bmatrix} a_1 & mc_1 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} a_R & a_M \\ 0 & a_S \end{bmatrix} e_{11} = \begin{bmatrix} 0 & ms_a c_1 \\ 0 & s_a c_1 \end{bmatrix} + a_R e_{11} = \begin{bmatrix} a_R & ms_a c_1 \\ 0 & s_a c_1 \end{bmatrix} = \begin{bmatrix} a_R & m\phi_2(a)c_1 \\ 0 & \phi_2(a)c_1 \end{bmatrix}$. Thus $a_S = \phi_2(a)c_1$ and $a_M = m\phi_2(a)c_1$.

On the other hand, $\delta(ae_{11}) = \delta(e_{11}ae_{11})$ implies $a_R = a_1 a$. Thus $\delta(ae_{11}) = \begin{bmatrix} a_1 a & m\phi_2(a)c_1 \\ 0 & \phi_2(a)c_1 \end{bmatrix}$.

Let $\delta(ce_{22}) = \begin{bmatrix} c_R & c_M \\ 0 & c_S \end{bmatrix}$. Then $\delta(ce_{22}) = \delta(ce_{22}e_{22}) = \alpha_2(ce_{22})\delta(e_{22}) + \delta(ce_{22})e_{22} = \begin{bmatrix} r_c & -r_c m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -a_1 & -mc_1 \\ 0 & -c_1 \end{bmatrix} + \begin{bmatrix} c_R & c_M \\ 0 & c_S \end{bmatrix} e_{22}$
$= \begin{bmatrix} -r_c a_1 & -r_c mc_1 + r_c mc_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c_M \\ 0 & c_S \end{bmatrix} = \begin{bmatrix} -r_c a_1 & c_M \\ 0 & c_S \end{bmatrix} = \begin{bmatrix} -\psi_2(c)a_1 & c_M \\ 0 & c_S \end{bmatrix}$.
So, $c_R = -\psi_2(c)a_1$.

On the other hand, $\delta(ce_{22}) = \delta(e_{22}ce_{22})$ implies that $c_S = -c_1 c$. Thus $\delta(ce_{22}) = \begin{bmatrix} -\psi_2(c)a_1 & c_M \\ 0 & -c_1 c \end{bmatrix}$.

Therefore we can define $h : S \to M$ by $h(c) = c_M$ where $\delta(ce_{22}) = \begin{bmatrix} -\psi_2(c)a_1 & c_M \\ 0 & -c_1c \end{bmatrix}$.

Now $\forall a \in R, b \in M, c \in S$, $\delta\left(\begin{bmatrix} 0 & ab \\ 0 & 0 \end{bmatrix}\right) = \delta\left(ae_{11}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \alpha_2(ae_{11})$ $\delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) + \delta(ae_{11})\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & m\phi_2(a) \\ 0 & \phi_2(a) \end{bmatrix}\begin{bmatrix} 0 & mg(b)+a_1b \\ 0 & g(b) \end{bmatrix} + \begin{bmatrix} a_1a & m\phi_2(a)c_1 \\ 0 & \phi_2(a)c_1 \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & m\phi_2(a)g(b) \\ 0 & \phi_2(a)g(b) \end{bmatrix} + \begin{bmatrix} 0 & a_1ab \\ 0 & 0 \end{bmatrix}$ $= \begin{bmatrix} 0 & m\phi_2(a)g(b)+a_1ab \\ 0 & \phi_2(a)g(b) \end{bmatrix}$ and $\delta\left(\begin{bmatrix} 0 & ab \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & mg(ab)+a_1ab \\ 0 & g(ab) \end{bmatrix}$. Thus $g(ab) = \phi_2(a)g(b)$

Similarly, $\delta\left(\begin{bmatrix} 0 & bc \\ 0 & 0 \end{bmatrix}\right) = \delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}ce_{22}\right)$ implies that $g(bc) = g(b)c$.

Moreover, $\delta(cse_{22}) = \delta(ce_{22}se_{22}) = \alpha_2(ce_{22})\delta(se_{22}) + \delta(ce_{22})se_{22}$

$= \begin{bmatrix} \psi_2(c) & -\psi_2(c)m \\ 0 & \end{bmatrix}\begin{bmatrix} -\psi_2(s)a_1 & h(s) \\ 0 & -c_1s \end{bmatrix} + \begin{bmatrix} -\psi_2(c)a_1 & h(c) \\ 0 & -c_1c \end{bmatrix}se_{22}$

$= \begin{bmatrix} -\psi_2(c)\psi_2(s)a_1 & \psi_2(c)h(s)+\psi_2(c)mc_1s \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & h(c)s \\ 0 & -c_1cs \end{bmatrix}$

$= \begin{bmatrix} -\psi_2(c)\psi_2(s)a_1 & \psi_2(c)h(s)+\psi_2(c)mc_1s+h(c)s \\ 0 & -c_1cs \end{bmatrix}$ and $\delta\begin{bmatrix} 0 & 0 \\ 0 & cs \end{bmatrix} = \begin{bmatrix} -\psi_2(cs)a_1 & h(cs) \\ 0 & -c_1cs \end{bmatrix}$. Thus $h(cs) = \psi_2(c)h(s) + h(c)s + \psi_2(c)mc_1s$.

Conversely, let $\delta\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a_1a - \psi_2(c)a_1 & B \\ 0 & \phi_2(a)c_1 + g(b) - c_1c \end{bmatrix}$, where $B = m\phi_2(a)c_1 + a_1b + mg(b) + h(c)$. Then we can check δ is an α_2-derivation by a similar proof of Theorem 3.1. □

Theorem 3.5. *Let $\alpha_3 : T \to T$ be a Type III homomorphism. Then $\delta : T \to T$ is an α_3-derivation if and only if there exist*

(i) $f : R \to R$ is a ϕ_3-derivation,

(ii) $g : M \to S$ is an additive map satisfying $\forall a \in R, b \in M, c \in S, g(ab) = \psi_3(a)g(b)$ and $g(bc) = g(b)c$ and

(iii) $h : M \to M$ is an additive map satisfying $\forall a \in R, b \in M, c \in S, h(ab) = \phi_3(a)h(b)+\theta_3(a)g(b)+f(a)b$ and $h(bc) = h(b)c$ such that for some $b_1 \in M, c_1 \in S$,

$$\delta\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} f(a) & \phi_3(a)b_1 + \theta_3(a)c_1 + h(b) - b_1c \\ 0 & \psi_3(a)c_1 + g(b) - c_1c \end{bmatrix},$$

where ϕ_3, ψ_3, θ_3 are maps given in Type III homomorphisms in Section 2.

Sketch of proof. Let $\delta(e_{11}) = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}$. Then $\delta(e_{11}) = \begin{bmatrix} 0 & b_1 \\ 0 & c_1 \end{bmatrix}$ and $\delta(e_{22}) = \begin{bmatrix} 0 & -b_1 \\ 0 & -c_1 \end{bmatrix}$.

Let $\delta(ae_{11}) = \begin{bmatrix} a_R & a_M \\ 0 & a_S \end{bmatrix}$. Then $\delta(ae_{11}) = \begin{bmatrix} a_R & \phi_3(a)b_1 + \theta_3(a)c_1 \\ 0 & \psi_3(a)c_1 \end{bmatrix}$. Thus $a_S = \psi_3(a)c_1$ and $a_M = \phi_3(a)b_1 + \theta_3(a)c_1$.

Define $f : R \to R$ by $f(a) = a_R$. Then f is an ϕ_3-derivation.

Let $\delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} b_R & b_M \\ 0 & b_S \end{bmatrix}$. Then $\delta\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & b_M \\ 0 & b_S \end{bmatrix}$. So, $b_R = 0$.

Define $g : M \to S$ by $g(b) = b_S$. Then g satisfies $\forall a \in R, b \in M, c \in S, g(ab) = \psi_3(a)g(b)$ and $g(bc) = g(b)c$.

Define $h : M \to M$ by $h(b) = b_M$. Then h satisfies $\forall a \in R, b \in M, c \in S, h(ab) = \phi_3(a)h(b) + \theta_3(a)g(b) + f(a)b$ and $h(bc) = h(b)c$.

Let $\delta(ce_{22}) = \begin{bmatrix} c_R & c_M \\ 0 & c_S \end{bmatrix}$. Then $\delta(ce_{22}) = \begin{bmatrix} 0 & -b_1c \\ 0 & -c_1c \end{bmatrix}$. This means

$$\delta\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} f(a) & \phi_3(a)b_1 + \theta_3(a)c_1 + h(b) - b_1c \\ 0 & \psi_3(a)c_1 + g(b) - c_1c \end{bmatrix}.$$

Conversely, let $\delta\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} f(a) & \phi_3(a)b_1 + \theta_3(a)c_1 + h(b) - b_1c \\ 0 & \psi_3(a)c_1 + g(b) - c_1c \end{bmatrix}$.

Then we can check δ is an α_3-derivation by a similar proof of Theorem 3.1. □

Theorem 3.6. *Let $\alpha_4 : T \to T$ be a Type IV homomorphism. Then $\delta : T \to T$ is an α_4-derivation if and only if there exist*

(i) $g : S \to S$ is a ψ_4-derivation and

(ii) $h : S \to M$ is an additive map satisfying $\forall c, s \in S, h(cs) = \phi_4(c)h(s) + \theta_4(c)g(s) + h(c)s$ such that for some $a_1 \in R$,

$$\delta\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a_1a - \phi_4(c)a_1 & a_1b + h(c) \\ 0 & g(c) \end{bmatrix}$$

for some $a_1 \in R$, where ϕ_4, ψ_4, θ_4 are maps given in Type IV homomorphisms in Section 2.

Proof. Since the proof is similar to above theorems, we omit. □

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