행렬 부등식 접근법을 이용한 비선형 시스템의 측정 피드백 제어

Measurement Feedback Control of a Class of Nonlinear Systems via Matrix Inequality Approach

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Abstract: We propose a measurement state feedback controller for a class of nonlinear systems that have uncertain nonlinearity and sensor noise. The new design method based on the matrix inequality approach solves the measurement feedback control problem of a class of nonlinear systems. As a result, the proposed methods using a matrix inequality approach has the flexibility to apply the controller. In addition, the sensor noise can be attenuated for more generalized systems containing uncertain nonlinearities.

Keywords: measurement feedback, sensor noise, matrix inequality approach, uncertain nonlinearity

I. INTRODUCTION AND PROBLEM FORMULATION

For practical systems, the measurement feedback controller with x and the sensor noise $s(t)$ in an additive form is applied to the systems in place of the nominal controller with state x. To dilute the effect of sensor noise, there have been various control results on measurement feedback control problems [1-4,7]. The presented controller in [1,2] via high gain observer can reduce the sensor noise. In [4], using a low-pass filter for feedforward nonlinear systems, the sensor noise is reduced for the unknown magnitude, frequency, and phase. However, the considered systems and control methods in [4] are limited to a class of feedforward systems and there are other limitation such that the condition in [4] is conservative because the condition is developed based on norm-bound. In order to relax over the norm-bound condition as [6] and [9], the matrix inequality condition is presented in [3].

The goal of this paper is to present a new design method which is motivated by [3]. In this paper, the matrix inequality approach is used to solve the measurement feedback control problem of a class of nonlinear systems.

We consider the nonlinear system given by

$$\dot{x} = Ax + Bu + \delta(t, x, u) \tag{1}$$

where $x \in R^n$ is the state, $u \in R$ is the input. The system matrices (A, B) are a Brunovsky canonical pair.

The nonlinearity $\delta(t, x, u) = [\delta_1(t, x, u), \cdots, \delta_n(t, x, u)]^T \in R^n$ is such that $\delta_i(t, x, u) : R \times R^n \times R \to R, \ i = 1, \cdots, n$ are C^1.

We show the conditions in [4] as follows.

Assumption 1: [4] For $i = 1, \cdots, n-2$, there exists a constant

$c \geq 0$ such that

$$| \delta_i(t, x, u) | \leq c(| x_{i+2} | + \cdots + | x_n | + | u |) \tag{2}$$

where $0 \leq a_i \leq a, \ w \leq w_i < \infty$, and $0 \leq \phi_i \leq 2\pi$ with the constants $a > 0$ and $w > 0$.

Assumption 2: [4] For $i = 1, \cdots, n$, there exists a constant $a_i \geq 0, \ w_i > 0$, and ϕ_i such that

$$s_i(t) = a_i \sin(w_i t + \phi_i) \tag{3}$$

where $0 \leq a_i \leq a, \ w \leq w_i < \infty$, and $0 \leq \phi_i \leq 2\pi$ with the constants $a > 0$ and $w > 0$.

In [4], the following controller has been introduced

$$u = K(\varepsilon)(x + s(t)) * e^{k_{n+1}/\varepsilon t} \tag{4}$$

$K(\varepsilon) = [k_1 / \varepsilon^{n+1}, \cdots, k_n / \varepsilon^2], \ *$ denotes the convolution operation, and $s(t) = [s_1(t), \cdots, s_n(t)]^T$. Thus, the controller (4) is equivalent to the following form.

$$
\begin{aligned}
u &= \sum_{i=1}^{n} \frac{k_i}{\varepsilon^{n+i}} \int_0^t (x_i(\tau) + s_i(\tau)) e^{k_{n+1}\varepsilon^{-1}(t-\tau)} d\tau \\
&= e^{k_{n+1}\varepsilon^{-1}t} \sum_{i=1}^{n} \frac{k_i}{\varepsilon^{n+i}} \int_0^t (x_i(\tau) + s_i(\tau)) e^{-k_{n+1}\varepsilon^{-1}\tau} d\tau
\end{aligned}
\tag{5}
$$

The restriction with Assumption 1 mainly comes from the fact the norm-bound condition is used. As a result, the design of the controller tends to overly conservative and less applicable to variety of nonlinearity. Next, we introduce a controller and a new matrix inequality condition on the nonlinearity, which leads to more flexibility in applying the controller.

II. MAIN RESULTS

First, we address some mathematical setups and notations.

Setups: Let $A_1 = [a_{ij}], \ i = 1, \cdots, n+1, \ j = 1, \cdots, n+1$, with

$a_{ij} = 1$ if $i = 1, \cdots, n$, $j = i + 1$ and $a_{ij} = 0$ $j \neq i + 1$ $B_1 = [0\cdots,$ $0,1]^T$ is a $1 \times (n+1)$ vector, and $K_1(\varepsilon) = [k_1/\varepsilon^{n+1}, \cdots, k_n/\varepsilon^2,$ $k_{n+1}/\varepsilon]$ is a $(n+1) \times 1$ vector. Then, $A_K(\varepsilon) = A_1 + B_1(K_1(\varepsilon))$. Then, we define $K_1 = K_1(1)$, $A_K = A_K(1)$. Also, we define a positive definite matrix $E_\varepsilon = diag[1, \cdots, \varepsilon^{i-1}, \cdots, \varepsilon^n]$, $i = 1, \cdots,$ $n+1$. If given that A_K is Hurwitz, we can obtain a Lyapunov equation of $A_K(\varepsilon)^T P(\varepsilon) + P(\varepsilon) A_K(\varepsilon) = -\varepsilon^{-1} E_\varepsilon^2$ with $P(\grave{o}) =$ $E_\varepsilon P E$ from $A_K^T P + P A_K = -I$ where I denotes an $(n+1) \times$ $(n+1)$ identity matrix.

Notations: Throughout the paper, X denotes a $1 \times (n+1)$ vector as $X = [x, x_{n+1}]^T$ and $|X|$ denotes $|X| = [|x_1|, \cdots, |x_i|,$ $\cdots, |x_{n+1}|]$ for $i = 1, \cdots, n+1$. Also, $\|X\|$ denotes the Euclidean norm.

From [4], set a virtual state as

$$x_{n+1} = e^{k_{n+1}\varepsilon^{-1}t} \sum_{i=1}^{n} \frac{k_i}{\varepsilon^{n+2-i}} \int_0^t x_i(\tau) e^{-k_{n+1}\varepsilon^{-1}\tau} d\tau \tag{6}$$

From (6), we have

$$\dot{x}_{n+1} = \sum_{i=1}^{n+1} \frac{k_i}{\varepsilon^{n+2-i}} x_i \tag{7}$$

Then, from the system (1)-(4), (6)-(7), we have

$$\dot{X} = A_K(\varepsilon)X + \delta_1(t,x,u) + \theta(t) \tag{8}$$

where $\delta_1(t,x,u) = [\delta(t,x,u),0]^T$ and

$$\theta(t) = [0, \cdots, 0, e^{\frac{k_{n+1}}{\varepsilon}t} \sum_{i=1}^{n} \frac{k_i}{\varepsilon^{n+2-i}} \int_0^t s_i(\tau) e^{-\frac{k_{n+1}}{\varepsilon}\tau} d\tau, 0]^T \tag{9}$$

Remark 1: The proposed controller (4) in [4] is composed of a scaling gain for robust control with uncertain nonlinearities and a low-pass filter for attenuating the high-frequency sensor noise. In order to analyze the steady-state behavior with the filtered measurement feedback information multi plied by a control gain, we use the virtual state (6) in [4].

Assumption 3: There exist a matrix $M(\varepsilon)$ such that

$$X^T E_\varepsilon P E_\varepsilon \delta_1(t,x,u) \leq |X|^T E_\varepsilon M(\varepsilon) E_\varepsilon |X| \tag{10}$$

which plays a key role in a matrix inequality condition.

Theorem 1: (i) Select K_1 such that A_K are Hurwitz. (ii) Obtain P of $A_K^T P + P A_K^T = -I$. (iii) Suppose that Assumptions 2 and 3 hold. (iv) Suppose that there exists ε such that the following matrix inequality condition holds

$$h\frac{xy}{z}I - 2M(\ ) > 0 \tag{11}$$

with $0 < h < 1$. Then, all states of the system (1) with the controller (4) are globally ultimately bounded.

Proof: Set $V(X) = X^T P(\varepsilon) X$. Then, along the trajectory of (8), we have

$$\dot{V}(X) \leq -\varepsilon^{-1} X^T E_\varepsilon E_\varepsilon X + 2 X^T P(\varepsilon) \delta_1(t,x,u) + 2 X^T P(\varepsilon) \theta(t) \tag{12}$$

By Assumption 3, we have

$$\dot{V}(X) \leq -X^T E_\varepsilon(\varepsilon^{-1}I - 2M(\varepsilon))E_\varepsilon X + 2 X^T P(\varepsilon)\theta(t) \tag{13}$$

By Assumption 2, from [4], we have

$$\|\theta(t)\|_1 \leq \sum_{i=1}^{n} \frac{|k_i|}{\varepsilon^{n+2-i}} \frac{a_i}{\sqrt{(k_{n+1}/\varepsilon)^2 + w_i^2}}(1 + e^{k_{n+1}\varepsilon^{-1}t}) \tag{14}$$

Using (14), we have

$$X^T P(\varepsilon)\theta(t) \leq \|P\|\|E_\varepsilon X\|\varepsilon^{n-1}\|\theta(t)\|_1$$
$$\leq \left(\sum_{i=1}^{n} \frac{|k_i|}{\varepsilon^{3-i}} \frac{a_i}{\sqrt{(k_{n+1}/\varepsilon)^2 + w_i^2}}(1 + e^{k_{n+1}\varepsilon^{-1}t})\right) \tag{15}$$
$$\times \|P\|\|E_\varepsilon X\|$$

From (13) and (15), we have

$$\dot{V}(X) \leq -X^T E_\varepsilon(h\varepsilon^{-1}I - 2M(\varepsilon))E_\varepsilon X$$
$$- \|E_\varepsilon X\|((1-h)\varepsilon^{-1}\|E_\varepsilon X\| - \sigma(\varepsilon)) \tag{16}$$

where $\sigma(\varepsilon) = \left(\sum_{i=1}^{n} \frac{|k_i|}{\varepsilon^{3-i}} \frac{a_i}{\sqrt{(k_{n+1}/\varepsilon)^2 + w_i^2}}(1 + e^{k_{n+1}\varepsilon^{-1}t})\right)\|P\|$.

Dut to $k_{n+1} < 0$, $\sigma(\varepsilon) \to \left(\sum_{i=1}^{n} \frac{|k_i|}{\varepsilon^{3-i}} \frac{a_i}{\sqrt{(k_{n+1}/\varepsilon)^2 + w_i^2}}\right)\|P\| < 0$ as $t \to \infty$. From (16), it is obvious that there exists a positive constant b^* such that $\|E_\varepsilon X\| \leq b^*$ as $t \to \infty$ for any $X(0)$ because $\dot{V}(X) \leq 0$ when $\|E_\varepsilon X\| \geq \sigma(\varepsilon)$. Thus, $\|E_\varepsilon X\|$ is globally ultimately bounded from (11) and (16) from [5]. Then, it is obvious that $\|x\|$ is globally ultimately bounded.

In the following example, we illustrate the construction of $M(\varepsilon)$.

Example A: Consider the following system

$$\dot{x}_1 = x_2 + \frac{x_2^2 \sin x_1}{1 + 10|x_2|}$$
$$\dot{x}_2 = u \tag{17}$$

Suppose that there exists a nonzero measurement noise $s_1(t)$ and $s_2(t) = 0$. The existing control methods in [1-4,6-9] are not applicable to the system (17). Select $K_1 = [-1,-3,-3]$. From (4), we have

$$u = x_3 + e^{-3\varepsilon^{-1}t}\left(-\frac{1}{\varepsilon^3}\int_0^t s_1(\tau)e^{3\tau}d\tau - \frac{3}{\varepsilon^2}\int_0^t s_2(\tau)e^{3\tau}d\tau\right) \tag{18}$$

where

$$x_3 = e^{-3\varepsilon^{-1}t}\left(-\frac{1}{\varepsilon^3}\int_0^t x_1(\tau)e^{3\tau}d\tau - \frac{3}{\varepsilon^2}\int_0^t x_2(\tau)e^{3\tau}d\tau\right) \tag{19}$$

and $\dot{x}_3 = K_1(\varepsilon)X$.

Then, we have $P = \begin{bmatrix} 2.3125 & 1.9375 & 0.5000 \\ 1.9375 & 3.2500 & 0.8125 \\ 0.5000 & 0.8125 & 0.4375 \end{bmatrix}$.

From Assumption 3, using $\dfrac{x_2^2 \sin x_1}{1+10|x_2|} \leq 0.1|x_2|$, we obtain

$$X^T E_\varepsilon P E_\varepsilon \delta_1(t,x,u) = X^T E_\varepsilon P \begin{bmatrix} \dfrac{x_2^2 \sin x_1}{1+10|x_2|} & 0 & 0 \end{bmatrix}$$

$$\leq 0.1(2.3125|x_1||x_2| + 1.9375\varepsilon|x_2|^2 + 0.5000\varepsilon^2|x_2||x_3|) \quad (20)$$

$$= |X|^T E_\varepsilon M(\varepsilon) E_\varepsilon |X|$$

where

$$M(\varepsilon) = 0.05\varepsilon^{-1} \begin{bmatrix} 0 & 2.3125 & 0 \\ 2.3125 & 3.8750 & 0.5000 \\ 0 & 0.5000 & 0 \end{bmatrix} \quad (21)$$

Then, from the condition (11) in Theorem 1, it is obvious That $h\varepsilon^{-1}I - 2M(\varepsilon) > 0$ with $h = 0.5$ for any $\varepsilon > 0$.

Lemma 1: If Assumption 1 holds. Then, Assumption 3 holds, not vice versa.

Proof: Note that

$$\sum_{i=1}^n \varepsilon^{i-1} |\delta_i(t,x,u)| \leq c\varepsilon^{-2} \sum_{i=1}^n |x_i| \quad (22)$$

Then, it is clear from Lemma 1 in [3].

III. APPLICATION EXAMPLE

We consider the inertial wheel pendulum [8]

$$\begin{aligned} \dot{x}_1 &= x_2 + m_0 \sin m_1 x_1 + m_2 x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \end{aligned} \quad (23)$$

where $|m_0| \leq 0.01$, $|m_1| \leq 1$, and $|m_2| \leq 0.01$. Suppose that there exists a nonzero measurement noise $s_1(t)$ and $s_2(t) = s_3(t) = 0$.

For new features, only the bound of the parameters m_2 is known. Due to the term $m_0 \sin m_1 x_1 + m_2 x_2$ and the noise $s_1(t)$, the existing controllers in [1-4,6-9] can not be applicable to the system (23).

Select $K_1 = [-1, -4, -6, -4]$. From (4), we have

$$u = x_4 + e^{-4\varepsilon^{-1}t}\left(-\dfrac{1}{\varepsilon^4}\int_0^t s_1(\tau)e^{4\tau}d\tau - \dfrac{6}{\varepsilon^3}\int_0^t s_2(\tau)e^{4\tau}d\tau \right.$$
$$\left. - \dfrac{4}{\varepsilon^2}\int_0^t s_3(\tau)e^{4\tau}d\tau\right) \quad (24)$$

where

$$x_4 = e^{-4\varepsilon^{-1}t}\left(-\dfrac{1}{\varepsilon^4}\int_0^t x_1(\tau)e^{4\tau}d\tau - \dfrac{6}{\varepsilon^3}\int_0^t x_2(\tau)e^{4\tau}d\tau \right.$$
$$\left. - \dfrac{4}{\varepsilon^2}\int_0^t x_3(\tau)e^{4\tau}d\tau\right) \quad (25)$$

and $\dot{x}_4 = K_1(\varepsilon)X$.

Then, we have $P = \begin{bmatrix} 3.1250 & 4.0000 & 2.3750 & 0.5000 \\ 4.0000 & 8.3750 & 5.5000 & 1.1250 \\ 2.3750 & 5.5000 & 5.1250 & 1.0000 \\ 0.5000 & 1.1250 & 1.0000 & 0.3750 \end{bmatrix}$.
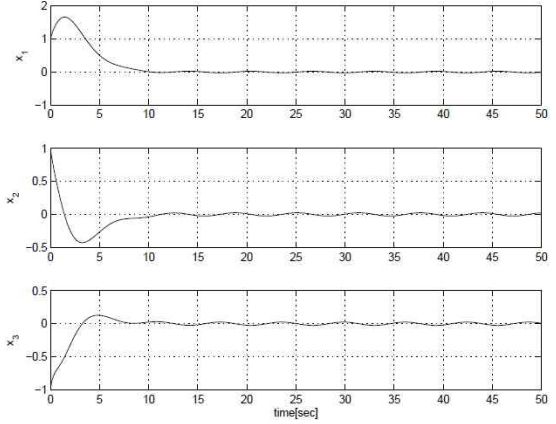

그림 1. 제어 시스템 (23)의 상태 궤적.
Fig. 1. State trajectories of the controlled system (23).

By applying Assumption 3, using $m_0 \sin m_1 x_1 \leq |m_0||m_1 x_1|$, we obtain

$$X^T E_\varepsilon P E_\varepsilon \delta_1(t,x,u)$$
$$= X^T E_\varepsilon P [m_0 \sin m_1 x_1 + m_2 x_2 \quad 0 \quad 0 \quad 0]^T$$
$$\leq 0.0050(3.1250|x_1|^2 + 4.0000\varepsilon|x_1||x_2|)$$
$$\quad + 0.0050(2.3750\varepsilon^2|x_1||x_3| + 0.5000\varepsilon^3|x_1||x_4|) \quad (26)$$
$$\quad + 0.0050(3.1250|x_1||x_2| + 4.0000\varepsilon|x_2|^2)$$
$$\quad + 0.0050(2.3750\varepsilon^2|x_2||x_3| + 0.5000\varepsilon^3|x_2||x_4|)$$
$$= |X|^T E_\varepsilon M(\varepsilon) E_\varepsilon |X|$$

where

$$M(\varepsilon) = 0.0050 \begin{bmatrix} 6.250 & 4.0000 & 2.3750 & 0.5000 \\ 4.0000 & 0 & 0 & 0 \\ 2.3750 & 0 & 0 & 0 \\ 0.5000 & 0 & 0 & 0 \end{bmatrix}?$$
$$+ \dfrac{0.0050}{\varepsilon} \begin{bmatrix} 0 & 3.1250 & 0 & 0 \\ 3.1250 & 8.0000 & 2.3750 & 0.5000 \\ 0 & 2.3750 & 0 & 0 \\ 0 & 0.5000 & 0 & 0 \end{bmatrix} \quad (27)$$

Then, from the condition (11) in Theorem 1, it is obvious That $h\varepsilon^{-1}I - 2M(\varepsilon) > 0$ with $h = 0.9$ for any $0 < \varepsilon < 4.5707$. The simulation results with $\varepsilon = 1$ are shown in Fig. 1.

IV. CONCLUSION

We have presented a new control design method using the matrix inequality approach. The considered nonlinearity is not restricted to feedforward forms, but more general nonlinearities. Consequently, the measurement feedback control problem for more general nonlinearities can be solved. We show the generalized features of our method over the existing controllers via an application example.

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