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Estimation for the extreme value distribution under progressive Type-I interval censoring

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Abstract

In this paper, we propose some estimators for the extreme value distribution based on the interval method and mid-point approximation method from the progressive Type-I interval censored sample. Because log-likelihood function is a non-linear function, we use a Taylor series expansion to derive approximate likelihood equations. We compare the proposed estimators in terms of the mean squared error by using the Monte Carlo simulation.

Keywords: Approximate maximum likelihood estimator, extreme value distribution, progressive Type-I interval censoring.

1. Introduction

The probability density function (pdf) and cumulative distribution function (cdf) of the random variable X having an extreme value distribution with the location parameter μ and the scale parameter σ are given by

$$f(x) = \frac{1}{\sigma}\exp\left(\frac{x-\mu}{\sigma}\right)\exp\left[-\exp\left(\frac{x-\mu}{\sigma}\right)\right], -\infty < x < \infty,\ \mu > 0,\ \sigma > 0, \tag{1.1}$$

and

$$F(x) = 1 - \exp\left[-\exp\left(\frac{x-\mu}{\sigma}\right)\right],\ -\infty < x < \infty. \tag{1.2}$$

In most cases of censoring, estimators of parameters may not be obtained as closed form by the maximum likelihood method. As the log-likelihood functions do not admit closed form, it will be useful to consider an approximation to the likelihood functions which provide us with estimators of closed form. The approximate maximum likelihood estimating method was originally developed by Balakrishnan (1989) for the purpose of exactly estimating the scale parameter in the Rayleigh distribution. Kang *et al.* (2001) obtained the approximation maximum likelihood estimators (AMLEs) for the parameters in the three-parameter Weibull distribution. Kang (2003) presented approximate MLEs for exponential distribution under

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multiple Type-II censoring. Kang *et al.* (2014) presented goodness-of-fit test for the logistic distribution based on multiply Type-II censored samples.

Aggarwala (2001) introduced the progressive Type-I interval censored sample. This progressive Type-I interval censoring is carried out as follows. Suppose n units put a life test at the same time at time $T_0 = 0$. Units are confirmed by interval $T_1, T_2, \ldots, T_m$, where T_m is the end point in the experiment. R_i censored items randomly failed at the censoring time T_i, $i = 1, \ldots, m$.

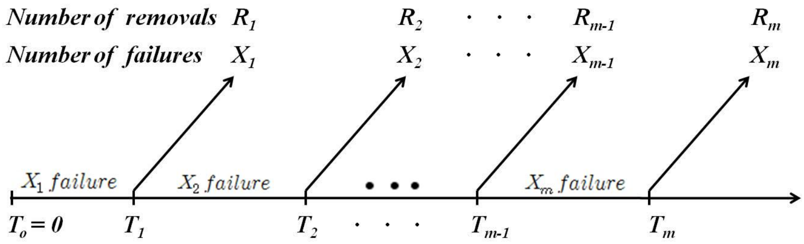

Figure 1.1 Scheme of a progressive Type-I interval censoring

More concretely, X_1 is the number of failure units in $(0, T_1]$, and R_1 is the number of removed units resulting in $n - X_1$ remaining units. Further X_2 is the number of failure units in the second interval $(T_1, T_2]$, and R_2 is the number of removed units, resulting in $n - X_1 - X_2$ remaining units. Thus, X_m is the final number of failure units in $(T_{m-1}, T_m]$, and all the remaining units $n - \sum_{i=1}^{m} X_i - \sum_{i=1}^{m} R_i = R_m$ are removed at time T_m. A schematic representation of the progressive Type-I interval censoring is presented in Figure 1.1 (see Ng and Wang, 2009).

Chen and Lio (2010) studied statistical estimator for the parameters of generalized exponential distribution. They proposed the mid-point estimators and derived the likelihood function. Shin *et al.* (2010) presented parameter estimation for exponential distribution under progressive Type-I interval censoring. Recently, Cho *et al.* (2013) studied the estimation for the generalized exponential distribution under progressive Type-I interval censoring.

In this paper, we derive several AMLEs of the location parameter and scale parameter in a extreme value distribution under progressive Type-I interval censoring. We also compare the proposed estimators in the sense of the bias and mean squared error (MSE) for different combination of values of the parameters, sample size, and censoring scheme. In section 2, we obtain the MLE and AMLEs of the parameters in the extreme value distribution based on progressive Type-I interval censored sample. In section 3, we simulate the MSEs of all proposed estimators through Monte Carlo simulation method and compare the performances of the proposed estimators for several censoring schemes.

2. Estimation for parameters

2.1. Mid-point approximation method

Suppose a progressive Tyep-I interval censored sample is collected, being with a random sample of n units with a continuous life time distibution (1.2). Then, based on the observed data, Aggarwala (2001) gave the joint likelihood function as follows

$$\begin{aligned} L(\theta) =& C[F(T_1,\beta;\theta)]^{X_1}[1-F(T_1,\beta;\theta)]^{R_1} \\ & \times [F(T_2,\beta;\theta)-F(T_1,\beta;\theta)]^{X_2}[1-F(T_2,\beta;\theta)]^{R_2} \\ & \times \cdots \times [F(T_m,\beta;\theta)-F(T_{m-1},\beta;\theta)]^{X_m}[1-F(T_m,\beta;\theta)]^{R_m} \\ =& C\prod_{i=1}^{m}[F(T_i,\beta;\theta)-F(T_{i-1},\beta;\theta)]^{X_i}[1-F(T_i,\beta;\theta)]^{R_i}, \end{aligned} \tag{2.1}$$

where $C = n(n-1-R_1)(n-2-R_1-R_2)\cdots(n-m+1-R_1-\cdots-R_{m-1})$ and $T_0 = 0$.

Ng and Wang (2009) introduced the mid-point estimators that are obtained by assuming the X_i failures ocured at the center of the interval $M_i = \frac{1}{2}(T_{i-1}+T_i)$ and R_i censored items failed at the censoring time T_i.

By putting $Z_i = \frac{M_i-\mu}{\sigma}$, $S_i = \frac{T_i-\mu}{\sigma}$, the likelihood function can be rewritten as

$$L = C\prod_{i=1}^{m}\left[\frac{1}{\sigma}f(z_i)\right]^{X_i}[1-F(s_i)]^{R_i}, \tag{2.2}$$

where $f(z) = e^z\exp(-e^z)$ and $F(z) = 1-\exp(-e^z)$ are the pdf and the cdf of the standard extreme value distribution. Therefore, we obtain the likelihood equations as follows

$$\frac{\partial \ln L}{\partial \mu} \simeq -\frac{1}{\sigma}\left[\sum_{i=1}^{m}X_i\frac{f'(z_i)}{f(z_i)} - \sum_{i=1}^{m}R_i\frac{f(s_i)}{1-F(s_i)}\right] = 0 \tag{2.3}$$

and

$$\frac{\partial \ln L}{\partial \sigma} \simeq -\frac{1}{\sigma}\left[\sum_{i=1}^{m}X_i\left(1+\frac{f'(z_i)z_i}{f(z_i)}\right) - \sum_{i=1}^{m}R_i\frac{f(s_i)s_i}{1-F(s_i)}\right] = 0. \tag{2.4}$$

We may expand the following functions in Taylor series around the points $\xi_i = F^{-1}(p_i)$,

$$\frac{f'(z_i)}{f(z_i)},\ \frac{f(s_i)}{1-F(s_i)}. \tag{2.5}$$

First, we can approximate these functions by

$$\frac{f'(z_i)}{f(z_i)} \simeq -e^{\xi_i}z_i + 1 - e^{\xi_i}(1-\xi_i), \tag{2.6}$$

$$\frac{f(s_i)}{1-F(s_i)} \simeq e^{\xi_i}s_i + e^{\xi_i}(1-\xi_i). \tag{2.7}$$

By substituting the equations (2.6) and (2.7) into the equations (2.3) and (2.4), we obtain the approximate likelihood equations for μ and σ as follows

$$\begin{aligned} \frac{\partial \ln L}{\partial \mu} \simeq & -\frac{1}{\sigma}\left[\sum_{i=1}^{m}X_i\{-e^{\xi_i}z_i+1-e^{\xi_i}(1-\xi_i)\} - \sum_{i=1}^{m}R_i\{e^{\xi_i}s_i+e^{\xi_i}(1-\xi_i)\}\right] \\ = & \ 0 \end{aligned} \tag{2.8}$$

and

$$\frac{\partial \ln L}{\partial \sigma} \simeq -\frac{1}{\sigma}\left[\sum_{i=1}^{m} X_i + \sum_{i=1}^{m} X_i \left\{-e^{\xi_i} z_i + 1 - e^{\xi_i}(1-\xi_i)\right\} z_i \right.$$
$$\left. - \sum_{i=1}^{m} R_i \left\{e^{\xi_i} s_i + e^{\xi_i}(1-\xi_i)\right\} s_i \right]$$
$$= -\frac{1}{\sigma}\left[\sigma^2 \sum_{i=1}^{m} X_i + \left[\sum_{i=1}^{m} X_i\{1-e^{\xi_i}(1-\xi_i)\}(M_i-\mu) - \sum_{i=1}^{m} R_i e^{\xi_i}(1-\xi_i)(T_i-\mu)\right]\sigma \right.$$
$$\left. - \sum_{i=1}^{m} X_i e^{\xi_i}(M_i-\mu)^2 - \sum_{i=1}^{m} R_i e^{\xi_i}(T_i-\mu)^2 \right] = 0. \tag{2.9}$$

Upon solving the equations (2.8) and (2.9) for σ, we can derive an approximate estimator of σ as follows

$$\hat{\sigma}_{M_1} = \frac{-N + \sqrt{N^2 - 4J\sum_{i=1}^{m} X_i}}{2\sum_{i=1}^{m} X_i} \tag{2.10}$$

where

$$N = \sum_{i=1}^{m} X_i\{1-e^{\xi_i}(1-\xi_i)\}(M_i-\hat{\mu}_m) - \sum_{i=1}^{m} R_i e^{\xi_i}(1-\xi_i)(T_i-\hat{\mu}_m),$$
$$J = -\sum_{i=1}^{m} X_i e^{\xi_i}(M_i-\hat{\mu}_m)^2 - \sum_{i=1}^{m} R_i e^{\xi_i}(T_i-\hat{\mu}_m)^2.$$

Since J is always negative, $\hat{\sigma}_{M_1}$ is greater than 0.

Second, we can also approximate these functions by

$$\frac{f'(z_i)z_i}{f(z_i)} \simeq a_{1i} z_i + b_{1i}, \tag{2.11}$$
$$\frac{f(s_i)s_i}{1-F(s_i)} \simeq c_{1i} s_i + d_{1i}, \tag{2.12}$$

where

$$a_{1i} = 1 - (1+\xi_i)e^{\xi_i}, \quad b_{1i} = \xi_i^2 e^{\xi_i},$$
$$c_{1i} = (\xi_i + 1)e^{\xi_i}, \quad d_{1i} = -\xi_i^2 e^{\xi_i}.$$

By substituting the equations (2.11) and (2.12) into the equation (2.4), we obtain the approximate likelihood equations for σ as follows

$$\begin{aligned}
\frac{\partial \ln L}{\partial \sigma} \simeq& -\frac{1}{\sigma}\left[\sum_{i=1}^{m} X_i + \sum_{i=1}^{m} X_i(a_{1i}z_i + b_{1i}) - \sum_{i=1}^{m} R_i(c_{1i}s_i + d_{1i})\right] \\
=& -\frac{1}{\sigma}\left[\left[\sum_{i=1}^{m} X_i + \sum_{i=1}^{m} X_i b_{1i} - \sum_{i=1}^{m} R_i d_{1i}\right]\sigma + \sum_{i=1}^{m} X_i a_{1i}(M_i - \hat{\mu}) \right. \\
&\left. - \sum_{i=1}^{m} R_i c_{1i}(T_i - \hat{\mu})\right] = 0. \qquad (2.13)
\end{aligned}$$

Upon solving the equations (2.8) and (2.13) for σ, we can derive an approximate estimator of σ as follows

$$\hat{\sigma}_{M_2} = -\frac{F_2 - F_3\hat{\mu_m}}{F_1}, \qquad (2.14)$$

where

$$\begin{aligned}
F_1 =& \sum_{i=1}^{m} X_i + \sum_{i=1}^{m} X_i a_{1i} - \sum_{i=1}^{m} R_i c_{1i}, \\
F_2 =& \sum_{i=1}^{m} X_i a_{1i} M_i - \sum_{i=1}^{m} R_i c_{1i} T_i, \\
F_3 =& \sum_{i=1}^{m} X_i a_{1i} - \sum_{i=1}^{m} R_i c_{1i}.
\end{aligned}$$

From the equations (2.8) and (2.13) for μ, we can obtain the estimator of μ as follows

$$\hat{\mu}_M = \frac{E_2F_1 - E_3F_2}{E_1F_1 - E_3F_3}, \qquad (2.15)$$

where

$$\begin{aligned}
E_1 =& \sum_{i=1}^{m} X_i e^{\xi_i} + \sum_{i=1}^{m} R_i e^{\xi_i}, \\
E_2 =& \sum_{i=1}^{m} X_i e^{\xi_i} M_i + \sum_{i=1}^{m} R_i e^{\xi_i} T_i, \\
E_3 =& \sum_{i=1}^{m} X_i \left\{1 - e^{\xi_i}(1 - \xi_i)\right\} - \sum_{i=1}^{m} R_i e^{\xi_i}(1 - \xi_i).
\end{aligned}$$

2.2. Approximate maximum likelihood estimation

The likelihood function can be rewritten as

$$L = C\prod_{i=1}^{m} \left[F(s_i) - F(s_{i-1})\right]^{X_i} \left[1 - F(s_i)\right]^{Ri}. \qquad (2.16)$$

Hence,

$$\frac{\partial \ln L}{\partial \mu} \simeq -\frac{1}{\sigma}\left[\sum_{i=1}^{m} X_i \frac{f(s_i)-f(s_{i-1})}{F(s_i)-F(s_{i-1})} - \sum_{i=1}^{m} R_i \frac{f(s_i)}{1-F(s_i)}\right] = 0 \tag{2.17}$$

and

$$\frac{\partial \ln L}{\partial \sigma} \simeq -\frac{1}{\sigma}\left[\sum_{i=1}^{m} X_i \frac{f(s_i)s_i-f(s_{i-1})s_{i-1}}{F(s_i)-F(s_{i-1})} - \sum_{i=1}^{m} R_i \frac{f(s_i)s_i}{1-F(s_i)}\right] = 0. \tag{2.18}$$

First, we can approximate these functions by

$$\frac{f(s_i)}{F(s_i)-F(s_{i-1})} \simeq a_{2i} + b_{2i}s_i + c_{2i}s_{i-1}, \tag{2.19}$$

$$\frac{f(s_{i-1})}{F(s_i)-F(s_{i-1})} \simeq a_{3i} + b_{3i}s_i + c_{3i}s_{i-1}, \tag{2.20}$$

and

$$\frac{f(s_i)-f(s_{i-1})}{F(s_i)-F(s_{i-1})} \simeq a_{4i} + b_{4i}s_i + c_{4i}s_{i-1}, \tag{2.21}$$

$$\frac{f(s_i)}{1-F(s_i)} \simeq e^{\xi_i}(1-\xi_i) + e^{\xi_i}s_i, \tag{2.22}$$

where

$$\begin{aligned}
a_{2i} &= \frac{1-(1-e^{\xi_i})\xi_i}{p_i-p_{i-1}} f(\xi_i) + \frac{f(\xi_i)\xi_i - f(\xi_{i-1})\xi_{i-1}}{(p_i-p_{i-1})^2} f(\xi_i),\\
b_{2i} &= \frac{1-e^{\xi_i}}{p_i-p_{i-1}} f(\xi_i) - \left(\frac{f(\xi_i)}{p_i-p_{i-1}}\right)^2,\\
c_{2i} &= \frac{f(\xi_i)f(\xi_{i-1})}{(p_i-p_{i-1})^2},\\
a_{3i} &= \frac{1-(1-e^{\xi_i})\xi_{i-1}}{p_i-p_{i-1}} f(\xi_{i-1}) + \frac{f(\xi_i)\xi_i - f(\xi_{i-1})\xi_{i-1}}{(p_i-p_{i-1})^2} f(\xi_{i-1}),\\
b_{3i} &= -\frac{f(\xi_i)f(\xi_{i-1})}{(p_i-p_{i-1})^2},\\
c_{3i} &= \frac{1-e^{\xi_{i-1}}}{p_i-p_{i-1}} f(\xi_{i-1}) + \left(\frac{f(\xi_{i-1})}{p_i-p_{i-1}}\right)^2,\\
a_{4i} &= a_{2i}-a_{3i},\ b_{4i} = b_{2i}-b_{3i},\ c_{4i} = c_{2i}-c_{3i}.
\end{aligned}$$

By substituting the equations (2.21) and (2.22) into the equation (2.17), we obtain the approximate likelihood equations for μ as follows

$$\begin{aligned}
\frac{\partial \ln L}{\partial \mu} &\simeq -\frac{1}{\sigma}\left[\sum_{i=1}^{m} X_i(a_{4i}+b_{4i}s_i+c_{4i}s_{i-1}) - \sum_{i=1}^{m} R_i\left\{e^{\xi_i}(1-\xi_i)+e^{\xi_i}s_i\right\}\right]\\
&= -\mu A_1 + A_2 + A_3\sigma \qquad (2.23)\\
&= 0,
\end{aligned}$$

where

$$A_1 = \sum_{i=1}^{m} X_i b_{4i} + \sum_{i=1}^{m} X_i c_{4i} - \sum_{i=1}^{m} R_i e^{\xi_i},$$
$$A_2 = \sum_{i=1}^{m} X_i b_{4i} T_i + \sum_{i=1}^{m} X_i c_{4i} T_{i-1} - \sum_{i=1}^{m} R_i e^{\xi_i} T_i,$$
$$A_3 = \sum_{i=1}^{m} X_i a_{4i} - \sum_{i=1}^{m} R_i e^{\xi_i}(1-\xi_i).$$

And by substituting the equations (2.19), (2.20) and (2.22) into the equation (2.18), we obtain the approximate likelihood equations for σ as follows

$$\frac{\partial \ln L}{\partial \sigma} \simeq -\frac{1}{\sigma}\left[\sum_{i=1}^{m} X_i \left\{(a_{2i} + b_{2i}s_i + c_{2i}s_{i-1})s_i - (a_{3i} + b_{3i}s_i + c_{3i}s_{i-1})s_{i-1}\right\} - \sum_{i=1}^{m} R_i \left\{e^{\xi_i}(1-\xi_i) + e^{\xi_i}s_i\right\} s_i\right] = 0. \quad (2.24)$$

Upon solving the equations (2.23) and (2.24) for σ, we can derive an approximate estimator of σ as follows

$$\hat{\sigma}_1 = \frac{B_2 + B_3\hat{\mu} - A_1\hat{\mu}^2}{B_1 - A_3\hat{\mu}}, \quad (2.25)$$

where

$$B_1 = \sum_{i=1}^{m} X_i a_{2i} T_i - \sum_{i=1}^{m} X_i a_{3i} T_{i-1} - \sum_{i=1}^{m} R_i e^{\xi_i}(1-\xi_i) T_i,$$
$$B_2 = -\sum_{i=1}^{m} X_i b_{2i} T_i^2 - \sum_{i=1}^{m} X_i c_{2i} T_i T_{i-1} + \sum_{i=1}^{m} X_i b_{3i} T_i T_{i-1} + \sum_{i=1}^{m} X_i c_{3i} T_{i-1}^2 + \sum_{i=1}^{m} R_i e^{\xi_i} T_i^2,$$
$$B_3 = 2\sum_{i=1}^{m} X_i b_{2i} T_i + \sum_{i=1}^{m} X_i c_{2i}(T_{i-1} + T_i) - \sum_{i=1}^{m} X_i b_{3i}(T_{i-1} + T_i) - 2\sum_{i=1}^{m} X_i c_{3i} T_{i-1} - 2\sum_{i=1}^{m} R_i e^{\xi_i} T_i.$$

Second, we can also approximate these functions by

$$\frac{f(s_i)s_i}{F(s_i) - F(s_{i-1})} \simeq a_{5i} + b_{5i}s_i + c_{5i}s_{i-1}, \quad (2.26)$$
$$\frac{f(s_{i-1})s_{i-1}}{F(s_i) - F(s_{i-1})} \simeq a_{6i} + b_{6i}s_i + c_{6i}s_{i-1}, \quad (2.27)$$

$$\frac{f(s_i)s_i - f(s_{i-1})s_{i-1}}{F(s_i) - F(s_{i-1})} \simeq a_{7i} + b_{7i}s_i + c_{7i}s_{i-1}, \tag{2.28}$$

$$\frac{f(s_i)s_i}{1 - F(s_i)} \simeq -e^{\xi_i}\xi_i^2 + e^{\xi_i}(1+\xi_i)s_i, \tag{2.29}$$

where

$$\begin{aligned}
a_{5i} &= -\frac{(1-e^{\xi_i})\xi_i}{p_i - p_{i-1}} f(\xi_i)\xi_i + \frac{f(\xi_i)\xi_i - f(\xi_{i-1})\xi_{i-1}}{(p_i - p_{i-1})^2} f(\xi_i)\xi_i, \\
b_{5i} &= \frac{(1-e^{\xi_i})\xi_i + 1}{p_i - p_{i-1}} f(\xi_i) - \xi_i \left(\frac{f(\xi_i)}{p_i - p_{i-1}}\right)^2, \\
c_{5i} &= \frac{f(\xi_i) f(\xi_{i-1})\xi_i}{(p_i - p_{i-1})^2}, \\
a_{6i} &= -\frac{(1-e^{\xi_{i-1}})\xi_{i-1}}{p_i - p_{i-1}} f(\xi_{i-1})\xi_{i-1} + \frac{f(\xi_i)\xi_i - f(\xi_{i-1})\xi_{i-1}}{(p_i - p_{i-1})^2} f(\xi_{i-1})\xi_{i-1}, \\
b_{6i} &= -\frac{f(\xi_i) f(\xi_{i-1})\xi_{i-1}}{(p_i - p_{i-1})^2}, \\
c_{6i} &= \frac{(1-e^{\xi_{i-1}})\xi_{i-1} + 1}{p_i - p_{i-1}} f(\xi_{i-1}) + \xi_{i-1} \left(\frac{f(\xi_{i-1})}{p_i - p_{i-1}}\right)^2, \\
a_{7i} &= a_{5i} - a_{6i}, \ b_{7i} = b_{5i} - b_{6i}, \ c_{7i} = c_{5i} - c_{6i}.
\end{aligned}$$

By substituting the equations (2.28) and (2.29) into the equation (2.18), we obtain the approximate likelihood equations for σ as follows

$$\begin{aligned}
\frac{\partial \ln L}{\partial \sigma} &\simeq -\frac{1}{\sigma}\left[\sum_{i=1}^{m} X_i(a_{7i} + b_{7i}s_i + c_{7i}s_{i-1}) - \sum_{i=1}^{m} R_i \left\{-e^{\xi_i}\xi_i^2 + e^{\xi_i}(1+\xi_i)s_i\right\}\right] \\
&= 0.
\end{aligned} \tag{2.30}$$

Upon solving the equations (2.23) and (2.30) for σ, we can derive an approximate estimator of σ as follows

$$\hat{\sigma}_2 = \frac{C_3\hat{\mu} - C_2}{C_1}, \tag{2.31}$$

where

$$\begin{aligned}
C_1 &= \sum_{i=1}^{m} X_i a_{7i} + \sum_{i=1}^{m} R_i e^{\xi_i}\xi_i^2, \\
C_2 &= \sum_{i=1}^{m} X_i b_{7i} T_i + \sum_{i=1}^{m} X_i c_{7i} T_{i-1} - \sum_{i=1}^{m} R_i e^{\xi_i}(1+\xi_i) T_i, \\
C_3 &= \sum_{i=1}^{m} X_i b_{7i} + \sum_{i=1}^{m} X_i c_{7i} - \sum_{i=1}^{m} R_i e^{\xi_i}(1+\xi_i).
\end{aligned}$$

We can also obtain the following estimators by using equations (2.23), (2.25) and (2.31);

$$\hat{\mu}_{I_1} = \frac{A_2B_1 + A_3B_2}{A_1B_1 + A_2A_3 - A_3B_3}, \tag{2.32}$$

$$\hat{\mu}_{I_2} = \frac{A_2C_1 - A_3C_2}{A_1C_1 - A_3C_3}, \tag{2.33}$$

$$\hat{\sigma}_{I_1} = \frac{C_3\hat{\mu}_{i1} - C_2}{C_1}, \tag{2.34}$$

$$\hat{\sigma}_{I_2} = \frac{B_2 + B_3\hat{\mu}_{i2} - A_1\hat{\mu}_{i2}^2}{B_1 - A_3\hat{\mu}_{i2}}. \tag{2.35}$$

3. Simulation study

We compare the proposed estimators in the sense of the mean squared errors throught Monte Carlo simulation for various censoring schemes. The simulation procedure is repeated 5,000 times for the sample size n=30, 50 and various choice of censoring.

By Aggarwala (2001), the progressive Type-I interval censored samples were generated by using the following algorithm in which $X_1 \sim \text{Bin}(n, F(t_1))$ for $i = 2, 3, \ldots, m$,

$$\begin{aligned} X_i | X_{i-1}, \ldots, X_1, R_{i-1}, \ldots, R_1 & \\ \sim \text{Bin}\left(n - \sum_{j=1}^{i-1}(X_j + R_j), \frac{F(T_i) - F(T_{i-1})}{1 - \sum_{j=1}^{i-1}[F(T_j) - F(T_{j-1})]}\right) & \\ = \text{Bin}\left(n - \sum_{j=1}^{i-1}(X_j + R_j), \frac{F(T_i) - F(T_{i-1})}{1 - F(T_{i-1})}\right). & \end{aligned} \tag{3.1}$$

The binomial random variables of m units were generated by the following algorithm step.

step 1. Initialize $i = 0, xsum = 0, rsum = 0$.

step 2. $i = i + 1$.

step 3. If $i = m$, exit the algorithm.

step 4. Generate X_i with Bin $\sim \left(n - xsum - rsum, \frac{F(T_i) - F(T_{i-1})}{[1 - F(T_{i-1})]}\right)$.

step 5. Caculate $R_i = floor[p_i(n - xsum - rsum - X_i)]$.

step 6. Set $xsum = xsum + X_i, \quad rsum = rsum + R_i$.

step 7. Go to step 2.

Here $p_1, p_2, \cdots, p_{m-1} (p_m = 1)$ and $R_m = n - \sum_{i=1}^{m} X_i - \sum_{i=1}^{m} R_i$ are fixed values.

We simulated progressive Type-I interval censored data with equal width time interval (0.5 time units) with 9 censoring points.

$$p_1 = (0.5, 0.5, 0.5, 0.5, 0.0, 0.0, 0.0, 0.0, 1.0),$$
$$p_2 = (0.5, 0.5, 0.25, 0.25, 0.0, 0.0, 0.0, 0.0, 1.0).$$

Table 3.1 The MSEs for the proposed estimators of the shape parameter μ

p	n	σ	μ	$MSE(\hat{\mu}_{M_1})$	$MSE(\hat{\mu}_{I_1})$	$MSE(\hat{\mu}_{I_2})$
p_1	30	1.0	0.0	0.2981	0.1018	0.0093
		0.4	0.7	0.0132	1.1278	0.7282
	50	1.0	0.0	0.3211	0.1118	0.0112
		0.4	0.7	0.0078	1.1759	0.8241
p_2	30	1.0	0.0	0.3427	0.1246	0.0134
		0.4	0.7	0.0071	1.1846	0.8396
	50	1.0	0.0	0.3239	0.1138	0.0113
		0.4	0.7	0.0078	1.1760	0.8241

Table 3.2 The MSEs for the proposed estimators of the shape parameter σ

p	n	σ	μ	$MSE(\hat{\sigma}_{M_1})$	$MSE(\hat{\sigma}_{M_2})$	$MSE(\hat{\sigma}_{I_1})$	$MSE(\hat{\sigma}_{I_2})$
p_1	30	1.0	0.0	1.3378	0.8683	0.0007	0.6384
		0.4	0.7	0.2826	0.1235	0.0063	0.0431
	50	1.0	0.0	1.3226	0.8721	0.0033	0.6585
		0.4	0.7	0.2800	0.1232	0.0081	0.0443
p_2	30	1.0	0.0	1.3420	0.8680	0.0009	0.6361
		0.4	0.7	0.2826	0.1235	0.0063	0.2697
	50	1.0	0.0	1.330	0.8713	0.0045	0.6540
		0.4	0.7	0.2801	0.1232	0.0081	0.2767

4. Concluding remarks

In most cases of censored samples, the MLE does not provide explict estimators. By using mid-point approximation method and approximation maximum likelihood estimation method, we obtain estimators having the closed form of the parameters for the extreme value distribution under progressive Type-I interval censoring, MSEs of $\hat{\mu}_{I_1}$ and $\hat{\mu}_{I_2}$ increase with increasing μ and decreasing σ. $\hat{\mu}_{M_1}$ decrease with increasing μ and decreasing σ. MSEs of $\hat{\sigma}_{M_1}$, $\hat{\sigma}_{M_2}$ and $\hat{\sigma}_{I_2}$ decrease with increasing μ and decreasing σ. $\hat{\mu}_{I_2}$ is more efficient than $\hat{\mu}_{M_1}$ and $\hat{\mu}_{I_1}$ for $\mu = 0$ and $\sigma = 1$ but $\hat{\mu}_{M_1}$is more efficient than the interval estimators $\hat{\mu}_{I_1}$ and $\hat{\mu}_{I_2}$ for $\mu = 0.7$ and $\sigma = 0.4$. For the estimator of σ, $\hat{\sigma}_{I_1}$ is always more efficient than the estimators $\hat{\sigma}_{M_1}$, $\hat{\sigma}_{M_2}$, and $\hat{\sigma}_{I_2}$. The interval estimation is better than the mid-point estimation.

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