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Nonparametric Bayesian estimation on the exponentiated inverse Weibull distribution with record values†

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Abstract

The inverse Weibull distribution (IWD) is the complementary Weibull distribution and plays an important role in many application areas. In Bayesian analysis, Soland's method can be considered to avoid computational complexities. One limitation of this approach is that parameters of interest are restricted to a finite number of values. This paper introduce nonparametric Bayesian estimator in the context of record statistics values from the exponentiated inverse Weibull distribution (EIWD). In stead of Soland's conjugate piror, stick-breaking prior is considered and the corresponding Bayesian estimators under the squared error loss function (quadratic loss) and LINEX loss function are obtained and compared with other estimators. The results may be of interest especially when only record values are stored.

Keywords: Exponentiated inverse Weibull distribution, nonparametric Bayesian estimation, record statistics, stick-breaking prior.

1. Introduction

The inverse Weibull distribution (IWD) is the complementary Weibull distribution and plays an important role in many applications including the dynamic components of diesel engines, the times to breakdown of an insulating fluid subject to the action of constant tensioin and flood data (Nelson, 1982; Maswadah, 2003). Also, it has been used quite extensively when the data indicate a monotone hazard function beacuse of the flexibility of the pdf and its corresponding hazard function. Studies for the inverse Weibull distribution were conducted by many authors. Calabria and Pulcini (1994) studied Bayes 2-sample prediction for the inverse Weibull distribution. Mahmoud *et al.* (2003) considered the order statistics arising from the inverse Weibull distribution and derived the exact expression for the single moments of order statistics. They also obtained the variances and covariances based on the moments of order statistics.

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The probability density function (pdf) and cumulative distribution function (cdf) of the random variable X having the exponentiated inverse Weibull distribution (EIWD) are given by

$$f(x;\alpha,\beta,\gamma) = \frac{\alpha\gamma}{\beta^\gamma}\exp\left(-\alpha(\beta x)^{-\gamma}\right)x^{-\gamma-1} \tag{1.1}$$

and

$$F(x;\alpha,\beta,\gamma) = \exp\left(-\alpha(\beta x)^{-\gamma}\right), \quad x>0,\ \alpha,\ \beta,\ \gamma>0. \tag{1.2}$$

The kth moment of this distribution that was introduced by Ali *et al.* (2007) is

$$E\left(X^k\right) = \frac{\alpha^{k/\gamma}}{\beta^k}\Gamma\left(1-\frac{k}{\gamma}\right), \quad \gamma>k. \tag{1.3}$$

Therefore, the mean and the variance of the exponentiated inverse Weibull distribution can be written as follows.

$$E\left(X\right) = \frac{\alpha^{1/\gamma}}{\beta}\Gamma\left(1-\frac{1}{\gamma}\right) \tag{1.4}$$

and

$$Var\left(X\right) = \frac{\alpha^{2/\gamma}}{\beta^2}\left[\Gamma\left(1-\frac{2}{\gamma}\right) - \left\{\Gamma\left(1-\frac{1}{\gamma}\right)\right\}^2\right] \quad \text{for} \quad \gamma>2. \tag{1.5}$$

It is clear that both the mean (1.4) and the variance (1.5) increase as α increases, when $\gamma > 2$. From (1.2), the reliability function of the exponentiated inverse Weibull distribution is given by

$$R(t) = 1-F(t) = 1-\exp\left(-\alpha(\beta t)^{-\gamma}\right), \quad t>0. \tag{1.6}$$

Note that the inverse Weibull distribution is a special case of (1.1) when $\alpha = 1$.

Chandler (1952) introduced the study of record values and documented many of the basic properties of records. Record values arise in many real-life situations involving weather, sports, economics and life tests. Record model is very related to the order statistics model, both of which appear in many statistical applications and are widely used in statistical modeling and inference beacuse it can be viewed as order statistics from a sample whose size is determined by the values and the order of occurence of observations. In particular, Balakrishnan *et al.* (1992) established some recurrence relations for the single and double moments of lower record values from Gumble distribution. Soliman *et al.* (2006) obtained Bayes estimators based on record statistics for two unknown parameters of the Weibull distribution. Recently, Sultan (2008) derived the Bayes estimators and obtained the estimators of the reliability and hazard functions for the unkonwn parameters of the inverse Weibull distribution based on lower record values.

Kim *et al.* (2012) proposed a Bayesian estimator in the context of record statistics values from the exponentiated inverse Weibull distribution using Soland's method (1969). This can be done with the evaluation of hyperparameters. To avoid this, we consider more flexible prior distributions. In general, nonparametric model can accommodate much more flexible forms and can easily deal with skewness, multimodality, etc. Here, for unknown parameters β and γ, we consider a nonparametric mode that are based on general class of priors that is called stick-breaking priors. Under three types loss functions, we derive the Bayes estimators in the context of record statistics values from the exponentiated inverse Weibull distribution. We also analyze application examples to illustrate the application of different derived estimators. Finally, in the estimated risks, the Bayes estimators are compared with the MLEs through Monte Carlo simulations.

2. Maximum likelihood estimation

In this section, we consider the MLEs of the unknown parameters and reliability function $R(t)$ in an exponentiated inverse Weibull distribution based on lower record values. Let $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed (iid) random variables with cdf $F(x)$ and pdf $f(x)$. Setting $Y_n = \min(X_1, X_2, \ldots, X_n)$, $n \geq 1$, we say that X_j is a lower record and denoted by $X_{L(j)}$ if $Y_j < Y_{j-1}$, $j > 1$. The indices at which the lower record values occur are given by the record times $\{L(n),\, n \geq 1\}$, where $L(n) = \min\{j | j > L(n-1),\, X_j < X_{L(n-1)}\}$, $n > 1$, with $L(1) = 1$. The corresponding likelihood function of the first n lower record values, $x_{L(1)}, \ldots, x_{L(n)}$ is

$$L = f(x_{L(n)}) \prod_{i=1}^{n-1} \frac{f(x_{L(i)})}{F(x_{L(i)})}. \tag{2.1}$$

Suppose we observe n lower record values $x_{L(1)}, \ldots, x_{L(n)}$ from the exponentiated inverse Weibull distribution with pdf (1.1). It follows, from (1.1), (1.2), and (2.1), that

$$L(\alpha,\, \beta,\, \gamma) = \left(\frac{\alpha\gamma}{\beta^\gamma}\right)^n \exp\left(-\frac{\alpha}{(\beta x_{L(n)})^\gamma}\right) \prod_{i=1}^{n} x_{L(i)}^{-\gamma-1}. \tag{2.2}$$

As a property of lower record values, its kth moment can be obtained by

$$E\left(X_{L(n)}^k\right) = \frac{\alpha^{k/\gamma}}{\beta^k} \frac{\Gamma(n - k/\gamma)}{\Gamma(n)}, \quad \gamma > k. \tag{2.3}$$

Now, we derive the MLEs of the parameters of the exponentiated inverse Weibull distribution when record values are given as data. From (2.2), the natural logarithm of the likeliood function is given by

$$\log L(\alpha,\, \beta,\, \gamma) = n \log \alpha - n\gamma \log \beta + n \log \gamma - \frac{\alpha}{(\beta x_{L(n)})^\gamma} - (\gamma + 1) \sum_{i=1}^{n} \log x_{L(i)}. \tag{2.4}$$

From the log-likelihood funcion (2.4), we obtain the likelihood equations for α, β, and γ as

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - \left(\frac{1}{\beta x_{L(n)}}\right)^\gamma = 0, \tag{2.5}$$

$$\frac{\partial \log L}{\partial \beta} = -\frac{\gamma n}{\beta} - \frac{\alpha\gamma}{\beta} \left(\frac{1}{\beta x_{L(n)}}\right)^\gamma = 0, \tag{2.6}$$

and

$$\frac{\partial \log L}{\partial \gamma} = \frac{n}{\gamma} - n \log \beta + \frac{\alpha \log \left(\beta x_{L(n)}\right)}{\left(\beta x_{L(n)}\right)^{\gamma}} - \sum_{i=1}^{n} \log x_{L(i)} = 0. \tag{2.7}$$

By solving the above equations, we can find the following MLEs of the unknown parameters α, β, and γ.

$$\hat{\alpha} = n \left(\hat{\beta} x_{L(n)}\right)^{\hat{\gamma}}, \tag{2.8}$$

$$\hat{\beta} = \left(\frac{\hat{\alpha}}{n}\right)^{1/\hat{\gamma}} x_{L(n)}^{-1} \tag{2.9}$$

and

$$\hat{\gamma} = \frac{n}{n \log \hat{\beta} + \sum_{i=1}^{n} \log x_{L(i)} - \hat{\alpha}(\hat{\beta} x_{L(n)})^{-\hat{\gamma}} \log(\hat{\beta} x_{L(n)})}. \tag{2.10}$$

The MLE $\hat{\gamma}$ in (2.10), in conjunction with the MLE $\hat{\beta}$ in (2.9), reduces to

$$\hat{\gamma} = \frac{n}{\sum_{i=1}^{n} \log x_{L(i)} - n \log x_{L(n)}}. \tag{2.11}$$

By the invariance property of the MLE, we can obtain the MLE of reliability function $R(t)$ to be

$$\hat{R}(t) = 1 - \exp\left(-\frac{\hat{\alpha}}{(\hat{\beta} t)^{\hat{\gamma}}}\right). \tag{2.12}$$

3. Bayesian estimation

In this section, we estimate α, β, γ, and $R(t)$, by considering both symmetric loss function and asymmetric loss function, and discuss method for obtaining hyperparameters. The LINEX loss function (LLF) is asymmetric loss functions while the squared error loss function (SELF) is a symmetric loss function assigning equal losses to overestimation and underestimation. The LLF was introduced by Varian (1975) and got a lot of popularity due to Zellner (1986). It may be expressed as $L(\triangle) \propto \exp(c\triangle) - c\triangle - 1$, $c \neq 0$, where $\triangle = \hat{\theta} - \theta$. The sign and magnitude of the shape parameter c represents the direction and degree of symmetry, respectively. When c is positive, the overestimation is more serious than underestimation and the situation is reverse when c is negative.

3.1. Parametric Bayesian analysis

A natural conjugate prior for the parameter α is a gamma prior as follows.

$$\pi(\alpha) = \frac{b^a}{\Gamma(a)} \alpha^{a-1} e^{-\alpha b}, \quad \alpha > 0,\, a,\, b > 0. \tag{3.1}$$

Now, we need to specify a general joint prior for α, β and γ which may leads to computational complexities. To avoid this problem, we consider Soland's method. Soland (1969) considered a family of joint prior distribution that places continuous distribution on the scale parameter and discrete distributions on the shape parameter. Kim *et al.* (2012) expanded the method employed by Soland (1969). Suppose that β and γ are restricted to a finite number of values $\beta_1, \beta_2, \ldots, \beta_J$ and $\gamma_1, \gamma_2, \ldots, \gamma_K$ with prior probabilities $\eta_1, \eta_2, \ldots, \eta_J$ and $\zeta_1, \zeta_2, \ldots, \zeta_K$, respectively. That is,

$$\pi(\beta_j) = P[\beta = \beta_j] = \eta_j, \quad j = 1, 2, \ldots, J \tag{3.2}$$

and

$$\pi(\gamma_k) = P[\gamma = \gamma_k] = \zeta_k, \quad k = 1, 2, \ldots, K. \tag{3.3}$$

Now, assume that the conditional α upon $\beta = \beta_j$ and $\gamma = \gamma_k$, $j = 1, 2, \ldots, J$ and $k = 1, 2, \ldots, K$ has a gamma$\,(a_{jk}, b_{jk})$ prior with pdf

$$\pi(\alpha|\beta = \beta_j, \gamma = \gamma_k) = \frac{b_{jk}^{a_{jk}}}{\Gamma(a_{jk})}\alpha^{a_{jk}-1}e^{-\alpha b_{jk}}, \quad \alpha > 0,\ a_{jk},\, b_{jk} > 0. \tag{3.4}$$

Then, the conditional posterior of $\alpha|\beta = \beta_j, \gamma = \gamma_k$ and the mrginal joint posterior of (β, γ) can be obtained by

$$\pi(\alpha|\beta = \beta_j, \gamma = \gamma_k, \mathbf{x}) = \frac{\left(b_{jk} + (\beta_j x_{L(n)})^{-\gamma_k}\right)^{n+a_{jk}}}{\Gamma(n + a_{jk})}\alpha^{n+a_{jk}-1}e^{-\alpha(b_{jk}+(\beta_j x_{L(n)})^{-\gamma_k})} \tag{3.5}$$

and

$$\pi_M(\beta_j, \gamma_k|\mathbf{x}) = G(\beta, \gamma)\frac{\eta_j\zeta_k b_{jk}^{a_{jk}}\Gamma(n + a_{jk})\gamma_k^n \prod_{i=1}^n x_{L(i)}^{-\gamma_k-1}}{\Gamma(a_{jk})\beta_j^{\gamma_k n}\left(b_{jk} + (\beta_j x_{L(n)})^{-\gamma_k}\right)^{n+a_{jk}}}, \tag{3.6}$$

where $G(\beta, \gamma)$ is the normalizing constant given by

$$G^{-1}(\beta, \gamma) = \sum_{j=1}^{J}\sum_{k=1}^{K}\frac{\eta_j\zeta_k b_{jk}^{a_{jk}}\Gamma(n + a_{jk})\gamma_k^n \prod_{i=1}^n x_{L(i)}^{-\gamma_k-1}}{\Gamma(a_{jk})\beta_j^{\gamma_k n}\left(b_{jk} + (\beta_j x_{L(n)})^{-\gamma_k}\right)^{n+a_{jk}}}. \tag{3.7}$$

Note that $\alpha|\beta = \beta_j, \gamma = \gamma_k$ has a Gamma$\,\left(n + a_{jk},\, b_{jk} + (\beta_j x_{L(n)})^{-\gamma_k}\right)$.

Using (3.5) and (3.6), we can obtain the marginal posterior of α as

$$\pi_{M_2}(\alpha|\mathbf{x}) = \sum_{j=1}^{J}\sum_{k=1}^{K}\pi(\alpha|\beta = \beta_j, \gamma_k, \mathbf{x})\pi_M(\beta_j, \gamma_k|\mathbf{x}). \tag{3.8}$$

From (3.6) and (3.8), the Bayes estimators of α, β, γ, and $R(t)$ based on the SELF are derived, respectively, as

$$
\begin{aligned}
\hat{\alpha}_s &= \int_0^\infty \alpha \pi_{M_2}(\alpha|\mathbf{x}) d\alpha = \sum_{j=1}^{J}\sum_{k=1}^{K} \pi_M(\beta_j,\, \gamma_k|\mathbf{x}) \int_0^\infty \alpha \pi(\alpha|\beta=\beta_j,\, \gamma=\gamma_k,\, \mathbf{x}) d\alpha \\
&= \sum_{j=1}^{J}\sum_{k=1}^{K} \pi_M(\beta_j,\, \gamma_k|\mathbf{x}) \frac{n+a_{jk}}{b_{jk}+(\beta_j x_{L(n)})^{-\gamma_k}}, \qquad (3.9)\\
\hat{\beta}_s &= \sum_{j=1}^{J}\sum_{k=1}^{K} \beta_j \pi_M(\beta_j,\, \gamma_k|\mathbf{x}), \qquad (3.10)\\
\hat{\gamma}_s &= \sum_{j=1}^{J}\sum_{k=1}^{K} \gamma_k \pi_M(\beta_j,\, \gamma_k|\mathbf{x}), \qquad (3.11)
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{R}_s(t) &= \int_0^\infty \pi_{M_2}(\alpha|\mathbf{x}) R(t) d\alpha = \int_0^\infty \pi_{M_2}(\alpha|\mathbf{x}) \left(1 - e^{-\alpha(\beta_j t)^{-\gamma_k}}\right) d\alpha \\
&= \sum_{j=1}^{J}\sum_{k=1}^{K} \pi_M(\beta_j,\, \gamma_k|\mathbf{x}) \int_0^\infty \pi(\alpha|\beta=\beta_j,\, \gamma=\gamma_k,\, \mathbf{x}) \left(1 - e^{-\alpha(\beta_j t)^{-\gamma_k}}\right) d\alpha \\
&= 1 - \sum_{j=1}^{J}\sum_{k=1}^{K} \pi_M(\beta_j,\, \gamma_k|\mathbf{x}) \left(1 + \frac{(\beta_j t)^{-\gamma_k}}{b_{jk}+(\beta_j x_{L(n)})^{-\gamma_k}}\right)^{-(n+a_{jk})}. \qquad (3.12)
\end{aligned}
$$

Similarly, the Bayes estimators of α, β, γ, and $R(t)$ based on the LLF are obtained by

$$
\begin{aligned}
\hat{\alpha}_L &= -\frac{1}{c} \log \int_0^\infty \pi_{M_2}(\alpha|\mathbf{x}) e^{-c\alpha} d\alpha \\
&= -\frac{1}{c} \log \left[\sum_{j=1}^{J}\sum_{k=1}^{K} \pi_M(\beta_j,\, \gamma_k|\mathbf{x}) \left(1 + \frac{c}{b_{jk}+(\beta_j x_{L(n)})^{-\gamma_k}}\right)^{-(n+a_{jk})} \right], \qquad (3.13)\\
\hat{\beta}_L &= -\frac{1}{c} \log \left[\sum_{j=1}^{J}\sum_{k=1}^{K} \pi_M(\beta_j,\, \gamma_k|\mathbf{x}) e^{-c\beta_j} \right], \qquad (3.14)\\
\hat{\gamma}_L &= -\frac{1}{c} \log \left[\sum_{j=1}^{J}\sum_{k=1}^{K} \pi_M(\beta_j,\, \gamma_k|\mathbf{x}) e^{-c\gamma_k} \right], \qquad (3.15)
\end{aligned}
$$

and

$$\hat{R}_L(t) = -\frac{1}{c}\log\int_0^\infty \pi_{M_2}(\alpha|\mathbf{x})e^{-cR(t)}d\alpha$$
$$= -\frac{1}{c}\log\left[e^{-c}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{m=0}^{\infty}\pi_M(\beta_j,\,\gamma_k|\mathbf{x})\frac{c^m}{m!}\left(1+\frac{m(\beta_j t)^{-\gamma_k}}{b_{jk}+(\beta_j x_{L(n)})^{-\gamma_k}}\right)^{-(n+a_{jk})}\right]$$
$$= 1-\frac{1}{c}\log\left[\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{m=0}^{\infty}\pi_M(\beta_j,\,\gamma_k|\mathbf{x})\frac{c^m}{m!}\left(1+\frac{m(\beta_j t)^{-\gamma_k}}{b_{jk}+(\beta_j x_{L(n)})^{-\gamma_k}}\right)^{-(n+a_{jk})}\right]. \quad (3.16)$$

In order to apply the methods discussed in this section, we should first extract the values of $(\beta_j,\,\eta_j)$, $(\gamma_k,\,\zeta_k)$ and the hyperparameters $(a_{jk},\,b_{jk})$ in the conjugate prior (3.4). For each choice of $(a_{jk},\,b_{jk})$, it is difficult to find the prior of α conditioned on each value of β_j and γ_k. An alternative method for obtaining the values $(a_{jk},\,b_{jk})$ can be based on the expected value of the reliability function $R(t)$ conditional on $\beta=\beta_j$ and $\gamma=\gamma_k$, which is given using (3.4) by

$$E\left[R(t)|\beta=\beta_j,\,\gamma=\gamma_k\right] = \frac{b_{jk}^{a_{jk}}}{\Gamma(a_{jk})}\int_0^\infty \left(1-\exp\left(-\alpha(\beta_j t)^{-\gamma_k}\right)\right)\alpha^{a_{jk}-1}e^{-\alpha b_{jk}}d\alpha$$
$$= 1-\left(1+\frac{(\beta_j t)^{-\gamma_k}}{b_{jk}}\right)^{-a_{jk}}, \quad t>0. \quad (3.17)$$

If we are able to specify two values $(t_1,\,R(t_1))$ and $(t_2,\,R(t_2))$ from prior beliefs about the distribution, the values of a_{jk} and b_{jk} can be obtained numerically from (3.17). Otherwise, a nonparametric procedure can be used to estimate the corresponding two different values of $R(t)$. We use mid-point estimator for $R(t)$ as a nonparametric method. Note that the Bayes estimators are in implicit form, so it can be solved by using numerical method such as Newton-Raphson.

3.2. Nonparametric Bayesian analysis

To avoid difficulties casued by finding values $(a_{jk},\,b_{jk})$, we consider more flexible prior distributions. In general, nonparametric model can accommodate much more flexible forms and can easily deal with skewness, multimodality, etc. Here, for unknown parameters β and γ, we consider a nonparametric mode that are based on general class of priors that is called stick-breaking priors. The stick-breaking priors are almost surely discrete random probability measure $\mathcal{P}$ represented generally as

$$\mathcal{P}(\cdot) = \sum_{k=1}^{N} p_k\delta_{Z_k}(\cdot), \quad (3.18)$$

where $\delta_{Z_k}(\cdot)$ denotes a discrete measure concentrated at Z_k, and p_k denote random weights chosen to be independent of Z_k and such that $0\le p_k\le 1$ and $\sum_{k=1}^{N}p_k=1$ almost surely. It is also assumed that Z_k are iid random variable with a distribution H. Note that stick-breaking priors can be constructed using a finite or an infinite number of terms, $1\le N\le\infty$.

The random weights p_k can be constructed as follows:

$$p_1 = V_1 \quad \text{and} \quad p_k = V_k \prod_{j=1}^{k-1}(1 - V_j), \quad k \geq 2, \tag{3.19}$$

where V_k are independent Beta(a_k,b_k) random variables for $a_k, b_k > 0$.

In our setting, we suppose that

$$x_i \overset{ind}{\sim} f(x|\alpha, \beta_i, \gamma_i), \quad i = 1, \ldots, n \tag{3.20}$$

$$\alpha \sim \pi(\alpha) \tag{3.21}$$

$$\beta_i \overset{ind}{\sim} \mathcal{P} \tag{3.22}$$

$$\gamma_i \overset{ind}{\sim} \mathcal{P} \tag{3.23}$$

where the prior can be characterized by a generalized Pólya urn mechanism. Then the Gibbs sampler involves drawing samples from the posterior of hierarchical model formed by marginalizing over the prior, which is known as prediction rule. Here we consider a simple finite dimensional Dirichlet prior, a special case of stick-breaking random measure $a_k = 1 - a$ and $b_k = b + ka$, where $0 \leq a < 1$ and $b > -a$. For the base distribution H, a gamma distribtuion is considered.

Pólya urn Gibbs sampler is a direct extension of the widely used Pólya urn sampler developed by Escobar (1994), MacEachern (1994), and Escobar and West (1995) for fitting the Ferguson (1973) Dirichlet process. One limitation of Pólya urn Gibbs sampler is to include complicate numerical integration in the nonconjugate case.To deal with this particular problem, we consider the blocked Gibbs algorithm which makes use of blocked updates for parameters. The work in Ishwaran and Zarepour (2000) can be extended straightforwardly to derive the required conditional distributions. In the blocked Gibbs sampling, let K_i be a classification variables to identify the β_k associated with each x_i. That is, β_k equals to Z_{K_i}. Let $\{K_1^*, \ldots, K_m^*\}$ denote the set of current m unique values of $\{K_1, \ldots, K_n\}$. The random weight $(p_1, \ldots, p_N)$ can then be updated by the conjugacy of the generalized Dirichlet distribution to multinomial as follows: let

- simulate $Z_k \overset{ind}{\sim} H$ for each $k \in \{K_1, \ldots, K_n\} \setminus \{K_1^*, \ldots, K_m^*\}$
- simulate $Z_{K_j^*}$ from the density proportional to $H(dZ_{K_j^*}) \prod_{\{i:K_i = K_j^*\}} f(x_i|\alpha, Z_{K_j^*}, \gamma_i)$
- simulate K from $\sum_{k=1}^{N} p_{k,i}\delta_k(\cdot)$ for $i = 1, \ldots, n$, where $p_{k,i} \propto p_i f(x_i|\alpha, Z_k, \gamma_i)$ for $k = 1, \ldots, N$
- simulate p_k for $k = 1, \ldots, N$ such that

$$p_1^* = V_1^* \quad \text{and} \quad p_k^* = V_k^* \prod_{j=1}^{k-1}(1 - V_j^*), \quad k = 2, \ldots, N-1$$

  and

$$V_k^* \sim \text{Beta}\left(a_k + M_k, b_k + \sum_{l=k+1}^{N} M_l\right), \quad k = 1, \ldots, N-1.$$

Note that M_k is the number of K_i values that equal to k and $Z_{K_i} = \beta_i$ (Ishwaran and James, 2001).

Similarly, γ can be updated in each Gibbs iteration. Updating other parameters can proceed by a Metropolis-Hasting algorithm.

4. Application

We present two examples to illustrate the methods of inference discussed in the previous sections.

4.1. Real data

Consider the real data given by Dumonceaux and Antle (1973) which represent the maximum flood level (in millions of cubic feet per second) of the Susquehenna River at Harrisburg, Pennsylvenia over a 20 four–year period (1890–1969). This data given in Table 1 has been utilized by some authors such as Maswadah (2003) and Sultan (2008). Maswadah (2003) showed that this real data follow an inverse Weibull distribution giving a rough indication of the goodness of fit for the model.

Table 4.1 The maximum flood level over a 20 four–year period (1890–1969)

0.654	0.613	0.315	0.449	0.297	0.402	0.379	0.423	0.379	0.324
0.269	0.740	0.418	0.412	0.494	0.416	0.338	0.392	0.484	0.265

During this period, 6 lower records of the maximum flood level are observed, they are

$$0.654, \quad 0.613, \quad 0.315, \quad 0.297, \quad 0.269, \quad 0.265.$$

In this example, we use gamma prior for the parameter α and discrete priors for the parameters β and γ. The values of β_j, γ_k and the hyperparameters of the gamma prior (3.4) are derived by the following steps. First, we estimate two values of the reliability function using the mid-point estimator for $R(t_i = x_{L(i)}) = (n - i + 0.5)/n, \ i = 1, 2, \ldots, n$. Here, we assume that the relaibility for $t_1 = 0.613$ and $t_2 = 0.269$ are, respectively, $R(t_1) = 0.25$, and $R(t_2) = 0.75$. Next, we obtain the MLE $\hat{\gamma} = 2.93565$ from (2.11) based on the above 6 lower record values when $\beta = 1$. Finally, we assume that $\gamma_k = 2.6(0.1)3.2$ and $\beta_j = 0.8(0.1)1.2$. So, the values of the hyperparameters a_{jk} and b_{jk} at each value of β_j and γ_k are obtained by solving the following equations using Newton-Raphson method.

$$1 - \left(1 + \frac{(\beta_j \, 0.613)^{-\gamma_k}}{b_{jk}}\right)^{-a_{jk}} = 0.25 \tag{4.1}$$

and

$$1 - \left(1 + \frac{(\beta_j \, 0.269)^{-\gamma_k}}{b_{jk}}\right)^{-a_{jk}} = 0.75. \tag{4.2}$$

The values of the hyperparameters and posterior probabilities obtained for each β_j and γ_k can be refered in Kim *et al.* (2012). A simple stick-breaking random measure with parameters

$a = 0$ and $b = \alpha$ for nonparameteric Bayesian estimation. By using these values, the ML estimates, and the Bayes estimates of α, β, γ, and $R(t)$ are calculated. The results are given in Table 2. We see that the MLEs are nearly equal to the Bayes estimates.

Table 4.2 Estimates of α, β, γ, and $R(t = 0.5)$ for real data

	MLE	BS	NPBS	BL			NPBL		
				$c = 0.5$	$c = 1.5$	$c = 2.5$	$c = 0.5$	$c = 1.5$	$c = 2.5$
α	0.12162	0.13884	0.12783	0.13684	0.13310	0.12965	0.12624	0.12521	0.12295
β	1.00000	1.00000	1.00000	0.99500	0.98507	0.97533	0.98753	0.98173	0.97452
γ	2.93565	2.90122	2.91453	2.89134	2.87179	2.85291	2.90134	2.87339	2.86529
$R(t)$	0.60565	0.58662	0.58983	0.58106	0.56982	0.55847	0.58326	0.57142	0.56245

BS: parametric Bayesian estimator under SELF, BL: parametric Bayesian estimator under LLF, NPBS: nonparametric Bayesian estimator under SELF, NPBL: nonparametric Bayesian estimator under LLF

To check the goodness of fit for the exponentiated inverse Weibull distribution with $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$, we conduct a simple test. A simple plot of 6 lower records of maximum flood level against the expected values of the first exponentiated inverse Weibull lower record values indicates a very strong correlation (0.895). Besides, we have nearly the same results for the Bayes estimates of α, β, and γ. Therefore, the assumption that these record values are from the exponentiated inverse Weibull distribution seems quite reasonable. The data given in Table 3 are the expected values of the first exponentiated inverse Weibull lower record values with the MLEs $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$.

Table 4.3 Expected values and real data for the simple plot

i	1	2	3	4	5	6
$E\left(X_{L(i)}\right)$	0.66710	0.43986	0.36494	0.32350	0.29595	0.27579
Real Data	0.654	0.613	0.315	0.297	0.269	0.265

4.2. Simulation study

To assess the performance of the MLEs and the Bayes estimators, we simulate the estimated risks of all derived estimators through Monte Carlo simulation method when the parameters β and γ are known. The following procedure is required to obtain the estimated risks. After setting $E(\alpha)$ and $Var(\alpha)$ from the prior density (3.1), we obtain the hyperparameters a and b of the gamma prior (3.1) by solving them. Note that $E(\alpha)$ is the actual value for α. We generate the lower record values from the exponentiated inverse Weibull distribution with $\alpha = E(\alpha)$. By using these value, we can finally obtain the Bayes estimators. The estimated risks for each estimator are calculated as the average of their squared deviations for 10,000 repetitions. It is expressed as

$$\frac{1}{n}\sum_{i=1}^{n}\left(\theta_t - \hat{\theta}\right)^2.$$

Here θ_t and $\hat{\theta}$ is the actual value and the estimate of θ, respectively.

In general, it is difficult to judge which is better the Bayes or ML estimators through a set of sample. A simulation study is conducted to see the efficiency of the Bayes and ML estimation methods in terms of estimated risks. The estimated risks for each estimator are calculated as the average of their squared deviations for 10,000 repetitions accroding to method disscussed in the Section 3. Samples of lower record values with size $n = 10$, are genetated from the exponentiated inverse Weibull distribution with $\alpha = 0.05$, $\beta = 0.6$, and $\gamma = 1.2$. For $\beta = 0.6$ and $\gamma = 1.2$, we consider the prior over the interval (0.1, 1.0) and (0.7, 1.6) by the discrete priors with β and γ taking the 10 values, each with probability 0.1. To obtain the Bayes estimates, we first calculate two values of the reliability function $R(t_2 = x_{L(2)})$ and $R(t_9 = x_{L(9)})$ and then can obtain the hyperparameters a_{jk} and b_{jk} using the expected value of the $R(t)$ in (3.16). The posterior probabilities are easily calculated from a_{jk} and b_{jk} at each value of β_j and γ_k. Through these steps, we obtain the Bayes estimates. By 10,000 repeating this procedure, the estimated risks for α, β, γ, and $R(t)$ are obtained. For $\alpha = 0.05$, $\beta = 2$, and $\gamma = 1.5$, the same simulation method is carry out. The results are presented in Table 4. From the table, we can see that the Bayes estimators are generally better than their corresponding MLEs. For α and $R(t)$, the Bayes estimators relative to asymmetric loss function are more efficient than the Bayes estimators under symmetric loss function such as SELF. Also, the estimated risks of them decrease as c increases. For β and γ, the symmetric Bayes estimators are more efficient than the asymmetric Bayes estimators. Not only that but, their estimated risks rather increase as c increases.

Table 4.4 The estimated risks of α, β, γ, and $R(t)$ when record values of size is 10

Actual values : (α, β, γ, $R(t=0.5)$)=(0.05, 0.6, 1.2, 0.19107)									
	MLE	BS	NPBS	BL			NPBL		
				$c=0.5$	$c=1.5$	$c=2.5$	$c=0.5$	$c=1.5$	$c=2.5$
α	0.51779	0.21478	0.22478	0.15282	0.09004	0.05991	0.16513	0.10534	0.07465
β	0.02719	0.00250	0.00574	0.00826	0.027400	0.05048	0.01035	0.02964	0.06135
γ	0.20427	0.07541	0.09634	0.08267	0.09711	0.11086	0.09142	0.10733	0.12386
$R(t)$	0.41108	0.27623	0.31453	0.26283	0.23540	0.20828	0.29453	0.26853	0.23536
Actual values : (α, β, γ, $R(t=0.5)$)=(0.05, 2, 1.5, 0.04877)									
	MLE	BS	NPBS	BL			NPBL		
				$c=0.5$	$c=1.5$	$c=2.5$	$c=0.5$	$c=1.5$	$c=2.5$
	0.69794	0.14711	0.19383	0.12347	0.09815	0.08557	0.18783	0.14352	0.10353
	0.02173	0.00250	0.00567	0.01226	0.02196	0.03307	0.02124	0.042353	0.06737
	0.31917	0.07057	0.09812	0.07915	0.09196	0.11086	0.08912	0.10346	0.13286
	0.25340	0.08046	0.10434	0.07190	0.06673	0.06204	0.09123	0.08783	0.07623

5. Concluding remarks

In this paper, we develop nonparametric Bayes estimators in the context of record statistics values from the exponentiated inverse Weibull distribution. In general, the existence of Bayes estimators is not always guaranteed under non-informative prior distrbution. To avoid difficulties casued by finding values (a_{jk}, b_{jk}) in the Soland's method, we consider more flexible prior distributions. In general, nonparametric model can accommodate much more flexible forms and can easily deal with skewness, multimodality, etc. Here, for unknown parameters β and γ, we consider a nonparametric mode that are based on general class of priors that is called stick-breaking priors. We derive the Bayes estimators for unknown parameters and reliability function $R(t)$. Their corresponding MLEs are also obtained. The MLEs are compared with Bayes estimators based on the symmetric and two types asymmetric loss functions in terms of estimated risks. Our result show that the Bayes estimators superior to the MLEs. Especially, the asymmetric Bayes estimators are generally better than the symmetric Bayes estimators provided using a suitable value of c and d.

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