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Robust Bayesian Inference in Finite Population Sampling under Balanced Loss Function

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Abstract

In this paper we develop Bayes and empirical Bayes estimators of the finite population mean with the assumption of posterior linearity rather than normality of the superpopulation under the balanced loss function. We compare the performance of the optimal Bayes estimator with ones of the classical sample mean and the usual Bayes estimator under the squared error loss with respect to the posterior expected losses, risks and Bayes risks when the underlying distribution is normal as well as when they are binomial and Poisson.

Keywords: Balanced loss function, Bayes risk, empirical Bayes, finite population mean, posterior expected loss, posterior linearity, risk function.

1. Introduction

Consider a finite population $\mathcal{U}$ with units labeled $1, 2, \ldots, N$. Let y_i denote the value of a single characteristic attached to the unit i. The vector $\mathbf{y} = (y_1, \ldots, y_N)^T$ is the unknown state of nature, and is assumed to belong to $\Theta = R^N$. A subset s of $\{1, 2, \ldots, N\}$ is called a sample. Let $n(s)$ denote the number of elements belonging to s. The set of all possible samples is denoted by S. A design is a function p on S such that $p(s) \in [0, 1]$ for all $s \in S$ and $\sum_{s \in S} p(s) = 1$. Given $\mathbf{y} \in \Theta$ and $s = \{i_1, \ldots, i_{n(s)}\}$ with $1 \le i_1 < \cdots < i_{n(s)} \le N$, let $y(s) = \{y_{i_1}, \ldots, y_{i_{n(s)}}\}$. One of the main objectives in sample surveys is to draw inference about $\mathbf{y}$ or some function (real or vector valued) $\gamma(\mathbf{y})$ of $\mathbf{y}$ on the basis of s and $y(s)$. For simplicity, only the case where $p(s) > 0$ if and only if $n(s) = n$ will be considered. This amounts to considering only fixed samples of size n.

A Bayesian approach for finite population sampling was initiated by Hill (1968) and Ericson (1969): subsequently, a large body of literature has grown in this area. Ericson (1988), Bolfarine and Zacks (1992), Ghosh and Meeden (1997), and Mukhopadhyay (2000) provide an up-to-date account of Bayesian literature in finite population sampling. In most of the Bayesian literature in survey sampling, the loss is assumed to be squared error.

However, squared error loss is primarily designed to reflect precision of estimation. As an alternative, we consider in this paper balanced loss functions (BLF) first introduced by Zellner (1988, 1992). This loss is a weighted average of two losses, one the squared distance between the parameters and their estimates, and the other the squared distance between estimates and data. The latter reflects the goodness of fit of the estimates. Recently Ghosh *et al.* (2008) considered Bayes estimations under random effects normal ANOVA model setup under BLF.

Ghosh and Meeden (1986) considered an empirical Bayes estimation of the finite population mean assuming a normal superpopulation model. The normality assumption in the superpopulation model

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was relaxed by Ghosh and Lahiri (1987). They assumed that the posterior expectation of a finite population mean is a linear function of the sample observations. Such a property is referred to as posterior linearity. This assumption is met when the superpopulation is normal (see Ericson, 1969); however, there are other situations when the same assumption holds (see, *e.g.*, Diaconis and Ylvisaker, 1979; Goldstein, 1975; Hartigan, 1969).

The set up considered in this paper is a finite population sampling with the assumption of posterior linearity rather than normality of the superpopulation under the balanced loss function. In Section 2, starting with the assumption of posterior linearity, the optimal Bayes estimator of the finite population mean under BLF is derived. We also compare the performance of the optimal Bayes estimator with ones of the sample mean and the usual Bayes estimator under the squared error loss using the posterior expected losses. Moreover, we seek the dominant conditions for typical estimators by the optimal Bayes estimator with respect to the risk function. In Section 3, we examine the performance of the proposed estimator with ones of the sample mean and the usual Bayes estimator under the squared error loss in terms of Bayes risk, theoretically and empirically. We also compare the Bayes risk of the empirical Bayes estimator with the typical estimators. In Section 4, we summarize the results.

2. Bayes Estimation of the Finite Population Mean under the BLF

2.1. Notation, model and assumption

Let r denote the set of nonsample units. Recall that s denotes a sample. Let $\bar{y}_s = \sum_{i\in s} y_i/n$ and $\bar{y}_r = \sum_{i\in r} y_i/(N-n)$ denote the means in the sample units and nonsample units, respectively. Also denote $\mathbf{y}_s = (y_i, i \in s)$.

Our objective is to estimate of the finite population mean $\gamma = N^{-1}\sum_{i=1}^{N} y_i$. It is possible to express a quantity of interest γ as $\gamma = [n\,\bar{y}_s + (N-n)\,\bar{y}_r]/N$. It is a combination of the known mean for the seen part plus the unknown mean for the unseen part. Estimation in finite population sampling can be thought of as predicting the unseen from the seen. So, we can define a predictor $\tilde{\gamma}$ for γ as:

$$\tilde{\gamma} = \frac{1}{N}\left[n\,\bar{y}_s + (N-n)\,\tilde{\bar{y}}_r\right] = (1-f)\bar{y}_s + f\,\tilde{\bar{y}}_r,$$

where $\tilde{\bar{y}}_r$ is a predictor of $\bar{y}_r$ and $f = (N-n)/N$ is the finite population correction factor. Estimating γ is then equivalent to predicting the value $\sum_{i\in r} y_i$ (hence $\bar{y}_r$).

Now, we assume the superpopulation model as:

(i) $y_i \,|\, \theta \overset{iid}{\sim} \text{pdf } f(\cdot\,|\,\theta)$ with $E[y_i\,|\,\theta] = \theta$ and $V[y_i\,|\,\theta] = \mu_2(\theta),\ i = 1,\ldots,N$.

(ii) θ has prior $\pi(\theta)$ with $E(\theta) = \mu$ and $V(\theta) = \tau^2$.

(iii) $0 < \tau^2 < \infty$ and $0 < \sigma^2 = E[\mu_2(\theta)] < \infty$.

The basic assumption of this model is that the posterior expectation of mean parameter θ (superpopulation mean) is a linear function of sample observations, without normality. That is,

$$E(\theta\,|\,\boldsymbol{y}_s) = \sum_{i\in s} a_i y_i + b, \tag{2.1}$$

where a_i's and b are some constants. From the work of Goldstein (1975), (2.1) leads to

$$E(\theta\,|\,\boldsymbol{y}_s) = a\bar{y}_s + b, \tag{2.2}$$

where a and b are some constants and $\bar{y}_s = n^{-1}\sum_{i\in s} y_i$. Hereafter, we will use (2.2) rather than (2.1). From Hartigan (1969), it follows that

$$a = \frac{\tau^2}{\tau^2 + \sigma^2/n} = \frac{n}{M+n} = 1 - B \quad \text{and} \quad b = B\mu,$$

where $M = \sigma^2/\tau^2$ and $B = M/(M+n)$.

2.2. Optimal Bayes estimator under the BLF

First, we consider two typical estimators of the finite population mean. One is the classical estimator of γ, which is $\tilde{\gamma}_C = \bar{y}_s$. The other is the Bayes estimator of γ under the squared error loss in the assumed model given by $\tilde{\gamma}_B = E[\gamma \,|\, \boldsymbol{y}_s] = (1-f)\bar{y}_s + f[(1-B)\bar{y}_s + B\mu]$.

Now we formulate the BLF with respect to γ in the finite population setup, denoted by $L_B(\tilde{\gamma}, \gamma)$, as:

$$L_B(\tilde{\gamma}, \gamma) = \frac{\omega\left(\boldsymbol{y}_s - \tilde{\theta}\mathbf{1}_n\right)'\left(\boldsymbol{y}_s - \tilde{\theta}\mathbf{1}_n\right)}{n} + (1-\omega)(\tilde{\gamma} - \gamma)^2, \tag{2.3}$$

where $\tilde{\gamma}$ and $\tilde{\theta}$ are some estimators of γ and θ, respectively. Here ω is the relative weight given to the goodness of fit portion such that $0 \le \omega \le 1$, and $1-\omega$ is the relative weight given to the precision of estimation portion of the loss function. Since $\tilde{\gamma} = N^{-1}[n\bar{y}_s + (N-n)\tilde{\theta}]$, the BLF given in (2.3) can be alternately written as

$$L_B(\tilde{\gamma}, \gamma) = \omega \sum_{i\in s} \frac{\left(y_i - \tilde{\theta}\right)^2}{n} + (1-\omega) f^2 \left(\tilde{\theta} - \bar{y}_r\right)^2. \tag{2.4}$$

Hence it follows that

$$L_B(\tilde{\gamma}, \gamma) = \omega\left[\hat{\sigma}^2 + \left(\tilde{\theta} - \bar{y}_s\right)^2\right] + (1-\omega) f^2 \left(\tilde{\theta} - \bar{y}_r\right)^2, \tag{2.5}$$

where $\hat{\sigma}^2 = \sum_{i\in s}(y_i - \bar{y}_s)^2/n$.

Next, to find the optimal Bayes estimator of γ, we consider the posterior expected loss under the BLF as follows:

$$\begin{aligned}\rho(\tilde{\gamma}, \gamma) &= E_{\theta|\boldsymbol{y}_s}\left[L_B(\tilde{\gamma}, \gamma)\right] \\ &= \omega\hat{\sigma}^2 + (1-\omega) f^2 \upsilon + \omega\left(\tilde{\theta} - \bar{y}_s\right)^2 + (1-\omega) f^2 \left(\tilde{\theta} - \bar{\theta}\right)^2,\end{aligned} \tag{2.6}$$

where $\upsilon = V[\bar{y}_r \,|\, \boldsymbol{y}_s]$ is the posterior variance of $\bar{y}_r$ and $\bar{\theta}$ is the posterior mean of θ. On completing the square on $\tilde{\theta}$ from (2.6), it follows that

$$\rho(\tilde{\gamma}, \gamma) = \omega\hat{\sigma}^2 + (1-\omega) f^2 \upsilon + \left[\frac{\omega(1-\omega) f^2}{\omega + (1-\omega) f^2}\right]\left(\bar{y}_s - \bar{\theta}\right)^2 + \left(\omega + (1-\omega) f^2\right)\left(\tilde{\theta} - \tilde{\theta}^*\right)^2, \tag{2.7}$$

with

$$\tilde{\theta}^* = \frac{\omega\bar{y}_s + (1-\omega) f^2 \bar{\theta}}{\omega + (1-\omega) f^2}.$$

Here, align estimator $\tilde{\theta}^*$ that minimizes the posterior expected loss is the optimal estimator of θ under the BLF. Let $\omega^* = \omega/(\omega + (1-\omega) f^2)$. Then it is also equal to

$$\tilde{\theta}^* = \omega^* \bar{y}_s + (1-\omega^*)\bar{\theta}. \tag{2.8}$$

Hence the optimal Bayes estimator of γ under the BLF, denoted by $\tilde{\gamma}_{BLF}$, is obtained by

$$\tilde{\gamma}_{BLF} = ((1-f) + f\omega^*)\,\bar{y}_s + f(1-\omega^*)\,\bar{\theta}\,. \tag{2.9}$$

2.3. Comparisons between posterior expected losses relative to the BLF

To compare the optimal Bayes estimator $\tilde{\gamma}_{BLF}$ with typical estimators $\tilde{\gamma}_C$ and $\tilde{\gamma}_B$ in terms of the posterior expected loss under the BLF, we compute the posterior expected loss of each estimator relative to the BLF and relative losses of each of typical estimators with respect to the optimal Bayes estimator.

For each typical estimator, the posterior expected loss with respect to the BLF is given by

$$\rho(\tilde{\gamma}_C, \gamma) = \omega\hat{\sigma}^2 + (1-\omega) f^2 \upsilon + (1-\omega) f^2 \left(\bar{y}_s - \bar{\theta}\right)^2$$

and

$$\rho(\tilde{\gamma}_B, \gamma) = \omega\hat{\sigma}^2 + (1-\omega) f^2 \upsilon + \omega \left(\bar{y}_s - \bar{\theta}\right)^2,$$

using the loss given in (2.6). Also, from (2.7), we get the posterior expected loss of the optimal Bayes estimator with respect to the BLF as follows:

$$\rho(\tilde{\gamma}_{BLF}, \gamma) = \omega\hat{\sigma}^2 + (1-\omega) f^2 \upsilon + \omega^*(1-\omega) f^2 \left(\bar{y}_s - \bar{\theta}\right)^2.$$

Then, differences between posterior expected losses are given by

$$\begin{aligned}\Delta_\rho(\tilde{\gamma}_C, \tilde{\gamma}_{BLF}) &= \rho(\tilde{\gamma}_C, \gamma) - \rho(\tilde{\gamma}_{BLF}, \gamma) \\ &= (1-\omega)(1-\omega^*) f^2 \left(\bar{y}_s - \bar{\theta}\right)^2\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}\Delta_\rho(\tilde{\gamma}_B, \tilde{\gamma}_{BLF}) &= \rho(\tilde{\gamma}_C, \gamma) - \rho(\tilde{\gamma}_{BLF}, \gamma) \\ &= \omega\omega^* \left(\bar{y}_s - \bar{\theta}\right)^2.\end{aligned} \tag{2.11}$$

Also, relative losses of the classical estimator over the optimal Bayes estimator, denoted by $RL(\tilde{\gamma}_C, \tilde{\gamma}_{BLF})$, and the Bayes estimator under the squared error loss over the optimal Bayes estimator, denoted by $RL(\tilde{\gamma}_B, \tilde{\gamma}_{BLF})$, are given by

$$\begin{aligned}RL(\tilde{\gamma}_C, \tilde{\gamma}_{BLF}) &= \frac{\Delta_\rho(\tilde{\gamma}_C, \tilde{\gamma}_{BLF})}{\rho(\tilde{\gamma}_{BLF}, \gamma)} \\ &= \frac{(1-\omega)(1-\omega^*) f^2 z^2}{1 + \omega^*(1-\omega) f^2 z^2}\end{aligned} \tag{2.12}$$

Table 1: Relative losses for various of ω, f and z^2

	z^2	$f = 0.8$					$f = 0.9$				
		$\omega = .00$	.25	.50	.75	1.00	$\omega = .00$	.25	.50	.75	1.00
$RL(\tilde{\gamma}_C, \tilde{\gamma}_{BLF})$	0.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.5	0.320	0.146	0.057	0.013	0.000	0.405	0.198	0.082	0.020	0.000
	1.0	0.640	0.271	0.104	0.025	0.000	0.810	0.366	0.148	0.037	0.000
	2.0	1.280	0.475	0.180	0.045	0.000	1.620	0.636	0.250	0.065	0.000
	4.0	2.560	0.762	0.281	0.074	0.000	3.240	1.008	0.383	0.105	0.000
$RL(\tilde{\gamma}_B, \tilde{\gamma}_{BLF})$	0.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.5	0.000	0.040	0.139	0.290	0.500	0.000	0.033	0.124	0.273	0.500
	1.0	0.000	0.074	0.255	0.546	1.000	0.000	0.062	0.226	0.509	1.000
	2.0	0.000	0.129	0.439	0.978	2.000	0.000	0.108	0.382	0.896	2.000
	4.0	0.000	0.207	0.685	1.619	4.000	0.000	0.171	0.583	1.442	4.000

and

$$RL(\tilde{\gamma}_B, \tilde{\gamma}_{BLF}) = \frac{\Delta_\rho\left(\tilde{\gamma}_B, \tilde{\gamma}_{BLF}\right)}{\rho\left(\tilde{\gamma}_{BLF}, \gamma\right)} = \frac{\omega\omega^* z^2}{1 + \omega^*(1-\omega) f^2 z^2}, \tag{2.13}$$

where $\upsilon_a^2 = \omega\hat{\sigma}^2 + (1-\omega) f^2 \upsilon$ and $z^2 = (\bar{y}_s - \bar{\theta})^2 / \upsilon_a^2$.

Table 1 gives relative losses of $\tilde{\gamma}_C$ over $\tilde{\gamma}_{BLF}$ and $\tilde{\gamma}_B$ over $\tilde{\gamma}_{BLF}$. Variable choices of ω, f, and z^2 are considered. For example, $RL(\tilde{\gamma}_C, \tilde{\gamma}_{BLF}) = 0.250$ and $RL(\tilde{\gamma}_B, \tilde{\gamma}_{BLF}) = 0.382$ in the case of $f = 0.9$, $z^2 = 2.0$ and $\omega = 0.5$. That means posterior expected losses in such a case are inflated by 25% and 38.2% using $\tilde{\gamma}_C$ and $\tilde{\gamma}_B$ rather than $\tilde{\gamma}_{BLF}$, respectively.

We can simply see that differences between posterior expected losses, $\Delta_\rho(\tilde{\gamma}_C, \tilde{\gamma}_{BLF})$ and $\Delta_\rho(\tilde{\gamma}_B, \tilde{\gamma}_{BLF})$, depend on ω, f and $(\bar{y}_s - \bar{\theta})$ from (2.10) and (2.11). For given values of ω and f, as the difference between the sample mean $\bar{y}_s$ and the posterior mean $\bar{\theta}$ increases, $\Delta_\rho(\tilde{\gamma}_C, \tilde{\gamma}_{BLF})$ and $\Delta_\rho(\tilde{\gamma}_B, \tilde{\gamma}_{BLF})$ are larger. The larger z^2, the larger $RL(\tilde{\gamma}_B, \tilde{\gamma}_{BLF})$ and $RL(\tilde{\gamma}_B, \tilde{\gamma}_{BLF})$. When $z^2 = 0$, especially, that means $\tilde{\theta} = \bar{y}_s = \bar{\theta}$, both of relative losses are equal to zero. Also, for the case $\omega = 1$, reflecting only goodness of fit, the optimal Bayes estimator $\tilde{\gamma}_{BLF}$ is equivalent to $\tilde{\gamma}_C$. Whereas, for the case $\omega = 0$, considering only precision of estimation, $\tilde{\gamma}_{BLF}$ is equivalent to $\tilde{\gamma}_B$. Therefore, $RL(\tilde{\gamma}_C, \tilde{\gamma}_{BLF})$ and $RL(\tilde{\gamma}_B, \tilde{\gamma}_{BLF})$ in these cases are equal to zero. Furthermore, $RL(\tilde{\gamma}_C, \tilde{\gamma}_{BLF})$ is decreasing monotonically in ω and $RL(\tilde{\gamma}_B, \tilde{\gamma}_{BLF})$ is increasing monotonically in ω. On the contrary, $\tilde{\gamma}_{BLF}$ is getting better gradually than $\tilde{\gamma}_C$ as the value of f increases and $\tilde{\gamma}_{BLF}$ is getting better gradually than $\tilde{\gamma}_B$ as the value of f decreases.

Form (2.10)–(2.13), it is clear that the difference between posterior expected losses (hence, relative losses) are not negative since $0 \le \omega \le 1$ and $0 \le \omega^* \le 1$. That means the optimal Bayes estimator $\tilde{\gamma}_{BLF}$ is superior to the Bayes estimator under the squared error loss, $\tilde{\gamma}_B$ as well as the classical estimator $\tilde{\gamma}_C$ in terms of the posterior expected loss. From Table 1, it is easy to check that $\tilde{\gamma}_{BLF}$ is better than $\tilde{\gamma}_C$ and $\tilde{\gamma}_B$ except that ω is equal to zero or one.

2.4. Conditions for dominance related to risks

The risk function for any estimator $\tilde{\gamma}$ of γ associated with the BLF is defined as

$$R(\tilde{\gamma}, \gamma) = E_{\mathbf{y}_s|\theta}\left[L_B(\tilde{\gamma}, \gamma)\right].$$

In this section, we derive risk functions of $\tilde{\gamma}_C$, $\tilde{\gamma}_B$ and $\tilde{\gamma}_{BLF}$ relative to the balanced loss function. Then we seek dominant conditions for typical estimators, $\tilde{\gamma}_C$ and $\tilde{\gamma}_B$ by the optimal Bayes estimator $\tilde{\gamma}_{BLF}$ with respect to the risk function.

First, the risk function of the classical estimator $\tilde{\gamma}_C$ with respect to the BLF is given by

$$R(\tilde{\gamma}_C, \gamma) = \omega E_{\boldsymbol{y}_s|\theta}\left[\hat{\sigma}^2\right] + (1-\omega) f^2 E_{\boldsymbol{y}_s|\theta}\left[(\bar{y}_s - \bar{y}_r)^2\right]$$
$$= \frac{\mu_2(\theta)}{n}\left[\omega(n-1) + (1-\omega) f\right],$$

since $E_{\boldsymbol{y}_s|\theta}[\hat{\sigma}^2] = (n-1)/n\,\mu_2(\theta)$ and $E_{\boldsymbol{y}_s|\theta}[(\bar{y}_s - \bar{y}_r)^2] = \mu_2(\theta)/n f^{-1}$.

Next, we get the risk function of $\tilde{\gamma}_B$ with respect to the BLF as follows:

$$R(\tilde{\gamma}_B, \gamma) = \omega E_{\boldsymbol{y}_s|\theta}\left[\hat{\sigma}^2\right] + \omega E_{\boldsymbol{y}_s|\theta}\left[\left(\bar{\theta} - \bar{y}_s\right)^2\right] + (1-\omega) f^2 E_{\boldsymbol{y}_s|\theta}\left[\left(\bar{\theta} - \bar{y}_r\right)^2\right].$$

Since $E_{\boldsymbol{y}_s|\theta}[(\bar{\theta} - \bar{y}_s)^2] = B^2[\mu_2(\theta)/n + (\theta-\mu)^2]$ and $E_{\boldsymbol{y}_s|\theta}[(\bar{\theta} - \bar{y}_r)^2] = B^2(\theta-\mu)^2 + \mu_2(\theta)/n\,[(1-B)^2 + n/(N-n)]$, it follows that

$$R(\tilde{\gamma}_B, \gamma) = \frac{\mu_2(\theta)}{n}\left[\omega(n-1) + \omega B^2 + (1-\omega) f^2\left\{(1-B)^2 + \frac{n}{N-n}\right\}\right] + \left(\omega + (1-\omega) f^2\right) B^2(\theta-\mu)^2.$$

Also, the risk function of $\tilde{\gamma}_{BLF}$ with respect to the BLF is given by

$$R(\tilde{\gamma}_{BLF}, \gamma) = \omega E_{\boldsymbol{y}_s|\theta}\left[\hat{\sigma}^2\right] + \omega E_{\boldsymbol{y}_s|\theta}\left[(1-\omega^*)\left(\bar{\theta} - \bar{y}_s\right)\right]^2 + (1-\omega) f^2 E_{\boldsymbol{y}_s|\theta}\left[\omega^* \bar{y}_s + (1-\omega^*)\,\bar{\theta} - \bar{y}_r\right]^2.$$

Since $E_{\boldsymbol{y}_s|\theta}[(1-\omega^*)(\bar{\theta} - \bar{y}_s)]^2 = C^2[\mu_2(\theta)/n + (\theta-\mu)^2]$ and $E_{\boldsymbol{y}_s|\theta}[\omega^* \bar{y}_s + (1-\omega^*)\bar{\theta} - \bar{y}_r]^2 = C^2(\theta - \mu)^2 + \mu_2(\theta)/n\,[(1-C)^2 + n/(N-n)]$, we finally get

$$R(\tilde{\gamma}_{BLF}, \gamma) = \frac{\mu_2(\theta)}{n}\left[\omega(n-1) + \omega C^2 + (1-\omega) f^2\left\{(1-C)^2 + \frac{n}{N-n}\right\}\right] + \left(\omega + (1-\omega) f^2\right) C^2(\theta-\mu)^2,$$

where $C = (1-\omega^*)B$.

Now, we examine conditions for dominance of $\tilde{\gamma}_C$ and $\tilde{\gamma}_B$ by $\tilde{\gamma}_{BLF}$ with respect to the risks, respectively. Let $\delta^2 = (\theta-\mu)^2/(\mu_2(\theta)/n)$, then the differences between risk functions is obtained by

$$\Delta_R(\tilde{\gamma}_C, \tilde{\gamma}_{BLF}) = R(\tilde{\gamma}_C, \gamma) - R(\tilde{\gamma}_{BLF}, \gamma)$$
$$= \frac{\mu_2(\theta)}{n}(1-\omega)(1-\omega^*) f^2 B^2\left[\frac{2-B}{B} - \delta^2\right]$$

and

$$\Delta_R(\tilde{\gamma}_B, \tilde{\gamma}_{BLF}) = R(\tilde{\gamma}_B, \gamma) - R(\tilde{\gamma}_{BLF}, \gamma)$$
$$= \frac{\mu_2(\theta)}{n}\omega(2-\omega^*)B^2\left[(1+\delta^2) - \frac{2(1-\omega^*)}{B(2-\omega^*)}\right].$$

Hence, the condition for dominance of $\tilde{\gamma}_C$ by $\tilde{\gamma}_{BLF}$ is $\delta^2 < (2-B)/B$. Also, the dominant condition for $\tilde{\gamma}_B$ by $\tilde{\gamma}_{BLF}$ is $\delta^2 > 2(1-\omega^*)/(B(2-\omega^*)) - 1$. It can be shown that the smaller δ^2, the more easily $\tilde{\gamma}_{BLF}$ dominates $\tilde{\gamma}_C$. The small δ^2 can be obtained by either the closeness of θ to the prior mean or the larger variability in $\bar{y}_s$. However, the larger δ^2, the more easily $\tilde{\gamma}_{BLF}$ dominates $\tilde{\gamma}_B$.

3. Monte Carlo Simulation for Bayes Risks

3.1. Bayes risks

To examine the superior performance of the proposed optimal Bayes estimator in terms of Bayes risk, we compute Bayes risks of each estimator under the BLF. We also illustrate the superiority of the optimal Bayes estimator $\tilde{\gamma}_{BLF}$ through the Monte Carlo simulation.

For each estimator, the Bayes risk with respect to the BLF is given by

$$r(\tilde{\gamma}_C, \gamma) = \frac{\sigma^2}{n}\left[\omega(n-1) + (1-\omega)f\right],$$
$$r(\tilde{\gamma}_B, \gamma) = \frac{\sigma^2}{n}\left[\omega(n-1) + \omega B^2 + (1-\omega)f^2\left\{(1-B)^2 + \frac{n}{N-n}\right\} + \left(\omega + (1-\omega)f^2\right)B(1-B)\right],$$

and

$$r(\tilde{\gamma}_{BLF}, \gamma) = \frac{\sigma^2}{n}\left[\omega(n-1) + \omega C^2 + (1-\omega)f^2\left\{(1-C)^2 + \frac{n}{N-n}\right\} + \left(\omega + (1-\omega)f^2\right)C(1-C-\omega^*)\right].$$

Thus, differences between Bayes risks are obtained by

$$\Delta_r\left(\tilde{\gamma}_C, \tilde{\gamma}_{BLF}\right) = \frac{\sigma^2}{n}B(1-\omega)\left(1-\omega^*\right)f^2 \tag{3.1}$$

and

$$\Delta_r\left(\tilde{\gamma}_B, \tilde{\gamma}_{BLF}\right) = \frac{\sigma^2}{n}B\omega\omega^*. \tag{3.2}$$

It is easy to show that $\Delta_r(\tilde{\gamma}_C, \tilde{\gamma}_{BLF})$ is strictly positive except for $\omega = 1$ and $\Delta_r(\tilde{\gamma}_B, \tilde{\gamma}_{BLF})$ is also strictly positive except for $\omega = 0$ from (3.1) and (3.2). It means that the optimal Bayes estimator, $\tilde{\gamma}_{BLF}$, is superior to typical estimators $\tilde{\gamma}_C$ and $\tilde{\gamma}_B$ in terms of the Bayes risk. Moreover, $\Delta_r(\tilde{\gamma}_C, \tilde{\gamma}_{BLF})$ is monotonically decreasing in ω. However, $\Delta_r(\tilde{\gamma}_B, \tilde{\gamma}_{BLF})$ is monotonically increasing in ω.

3.2. Empirical Bayes estimation

Generally, one or all of the parameter σ^2, τ^2 (hence M) and μ are unknown and require to be estimated from the data. We consider the most realistic situation when M and μ are both unknown (and need to be estimated from the data). Ghosh and Meeden (1986) used the estimators from the analysis of variance to estimate the parameter M. In an empirical Bayes framework, it is assumed that we are at the m^{th} stage of sampling procedure and the data are available from $(m-1)$-previous samplings from the same (similar type of) population. The population size and a sample size at the j^{th} stage are denoted by N_j and n_j, respectively. Let $y_i^{(j)}$ denote the characteristic of interest associated with i^{th} unit at the j^{th} stage$(i = 1, \ldots, N_j;\ j = 1, \ldots, m)$. Without loss of generality, the sample values are denoted by $y_1^{(j)}, \ldots, y_{n_j}^{(j)}$ and the nonsample values are denoted by $y_{n_j+1}^{(j)}, \ldots, y_{N_j}^{(j)}$ at the j^{th} stage.

First, we estimate M. Let $n_T = \sum_{j=1}^m n_j$, $\bar{y}^{(j)} = \sum_{i=1}^{n_j} \bar{y}_i^{(j)}/n_j$ and $\bar{y}_{\cdot\cdot} = \sum_{j=1}^m n_j\bar{y}^{(j)}/n_T$. Define $\text{BMS} = \sum_{j=1}^m n_j(\bar{y}^{(j)} - \bar{y}_{\cdot\cdot})^2/(m-1)$ and $\text{WMS} = \sum_{j=1}^m \sum_{i=1}^{n_j}(y_i^{(j)} - \bar{y}^{(j)})^2/(n_T - m)$. Accordingly, a consistent estimator of $M^{-1} = \tau^2/\sigma^2$ is $\max\{0, (\text{BMS}/\text{WMS} - 1)(m-1)g^{-1}\}$, where $g = n_T - (\sum_{j=1}^m n_j^2)/n_T$. Ghosh and Meeden (1986) modified the above estimator slightly by multiplying BMS/WMS by

Table 2: Empirical estimates and $PCTIMP$s in the Cities population, $f = 0.9$

ω	$\tilde{\gamma}_C$	$\tilde{\gamma}_{B.Em}$	$\tilde{\gamma}_{BLF.Em}$	$PCTIMP_{C.Em}$	$PCTIMP_{B.Em}$
0.00	0.304	0.274	0.274	68.724	0.000
0.25	0.304	0.274	0.283	8.919	1.646
0.50	0.304	0.274	0.291	2.129	3.254
0.75	0.304	0.274	0.298	0.348	4.705
1.00	0.304	0.274	0.304	0.000	6.019

$(m-1)/(m-3)$ so that the resulting estimator is identical with the James-Stein estimator when $n_1 = \cdots = n_m = n$. Therefore, they proposed the estimator

$$\hat{M}^{-1} = \max\left\{0, \left(\frac{(m-1)\text{BMS}}{(m-3)\text{WMS}} - 1\right)(m-1)g^{-1}\right\}, \quad m \geqq 4.$$

Then, the corresponding estimator of $B_j = M/(M+n_j) = (1+n_jM^{-1})^{-1}$ is $\hat{B}_j = (1+n_j\hat{M}^{-1})^{-1}$ $(j = 1, \ldots, m)$. Next, they proposed the empirical Bayes estimator of μ based on the principle of maximum likelihood

$$\hat{\mu} = \begin{cases} \dfrac{\sum_{j=1}^m \left(1-\hat{B}_j\right)\bar{y}^{(j)}}{\sum_{j=1}^m \left(1-\hat{B}_j\right)}, & \text{if } \hat{M}^{-1} \neq 0, \\ m^{-1}\displaystyle\sum_{j=1}^m \bar{y}^{(j)}, & \text{if } \hat{M}^{-1} = 0. \end{cases}$$

An empirical Bayes predictor of population mean under the BLF, denoted by $\tilde{\gamma}_{BLF.Em}$ is obtained by substituting the estimator $\hat{\mu}$ and $\hat{B}(\equiv \hat{B}_m)$ for μ and B respectively. Therefore,

$$\tilde{\gamma}_{BLF.Em} = (1-f)\bar{y}_s + f\left[\omega^*\bar{y}_s + (1-\omega^*)\left\{\left(1-\hat{B}\right)\bar{y}_s + \hat{B}\hat{\mu}\right\}\right]. \tag{3.3}$$

In addition, note that an empirical Bayes predictor of γ under the squared error loss function, denoted by $\tilde{\gamma}_{B.Em}$ is given by

$$\tilde{\gamma}_{B.Em} = (1-f)\bar{y}_s + f\left[\left(1-\hat{B}\right)\bar{y}_s + \hat{B}\hat{\mu}\right]. \tag{3.4}$$

Now, we perform the analysis of real data set from one of the six real populations that are used in Royall and Cumberland (1981). In the population named Cities, the variable y is the 1970 population, in millions, of 125 US cities with 1960 population between 100,000 and 1,000,000. The problem is to estimate the population mean in 1970, $\gamma = \sum_{i=1}^{125} y_i/125$ which is 0.2903 that can be obtained from the complete population. To estimate $\tilde{\gamma}_B$ and $\tilde{\gamma}_{BLF}$, we need to know the $M(=\sigma^2/\tau^2)$ and μ. So, we will estimate the values from the data. We consider 1950 and 1960 populations as well as 1970 population in 125 cities and generated one artificial population having similar mean and variance with y because of $m \geqq 4$. We selected 10% simple random sample from the population ($f = 0.9$). We have reported our analysis for one sample for illustration purpose. Table 2 provides $\tilde{\gamma}_C$, $\tilde{\gamma}_{B.Em}$ and $\tilde{\gamma}_{BLF.Em}$ for various ω in case $f = 0.9$ and $n_j = 13$ $(j = 1, \ldots, 4)$. The table also provides the percentage risk improvements $PCTIMP_{C.Em}$ and $PCTIMP_{B.Em}$. Note that we obtained $\hat{M}^{-1} = 0.0232$, $\hat{B} = 0.7686$ and $\hat{\mu} = 0.2609$ for the sample. The results report that $\tilde{\gamma}_{BLF.Em}$ is superior to both $\tilde{\gamma}_C$ and $\tilde{\gamma}_{B.Em}$ in terms of the Bayes risks. We also compare the Bayes risks of these empirical estimators through a Monte Carlo simulation study in the following subsection.

Table 3: The percentage improvements of $\tilde{\gamma}_{BLF}$ over $\tilde{\gamma}_C$ and $\tilde{\gamma}_B$ in normal case

τ^2	ω	$f = 0.8$		$f = 0.9$		$f = 0.95$	
		$PCTIMP_C$	$PCTIMP_B$	$PCTIMP_C$	$PCTIMP_B$	$PCTIMP_C$	$PCTIMP_B$
0.5	.00	2.954	0.000	5.416	0.000	10.353	0.000
	.25	0.070	0.016	0.499	0.062	1.373	0.167
	.50	0.015	0.029	0.070	0.120	0.326	0.417
	.75	0.003	0.044	0.013	0.165	0.055	0.639
	1.00	0.000	0.056	0.000	0.213	0.000	0.830
1.0	.00	1.400	0.000	2.875	0.000	5.758	0.000
	.25	0.030	0.011	0.277	0.036	0.729	0.097
	.50	0.006	0.018	0.032	0.066	0.192	0.205
	.75	0.002	0.022	0.007	0.088	0.028	0.331
	1.00	0.000	0.028	0.000	0.113	0.000	0.447
2.0	.00	0.649	0.000	1.345	0.000	3.365	0.000
	.25	0.016	0.005	0.100	0.011	0.378	0.051
	.50	0.004	0.008	0.025	0.025	0.102	0.099
	.75	0.001	0.012	0.003	0.044	0.016	0.176
	1.00	0.000	0.014	0.000	0.057	0.000	0.232

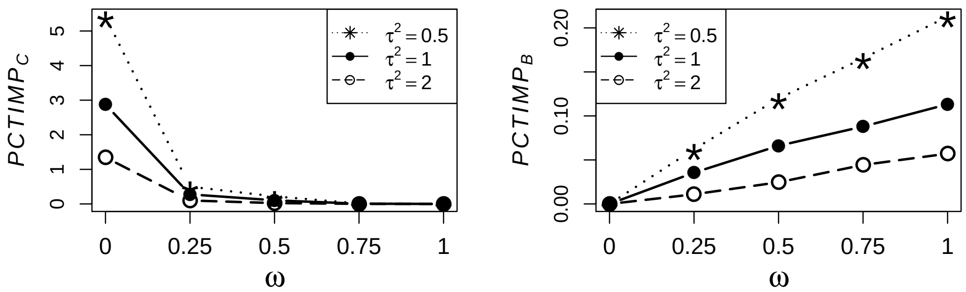

Figure 1: *PCTIMPs of $\tilde{\gamma}_{BLF}$ over $\tilde{\gamma}_C$ and $\tilde{\gamma}_B$ for various ω and τ^2 ($f = 0.9$) in normal case*

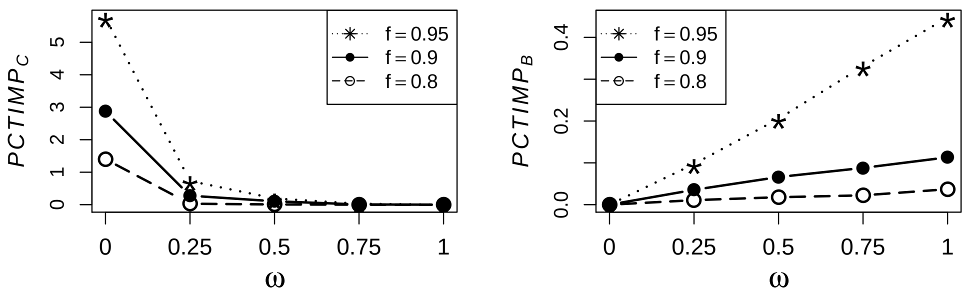

Figure 2: *PCTIMPs of $\tilde{\gamma}_{BLF}$ over $\tilde{\gamma}_C$ and $\tilde{\gamma}_B$ for various ω and f ($\tau^2 = 1.0$) in normal case*

3.3. Monte Carlo simulations

Monte Carlo simulations are performed to compare the Bayes risks of $\tilde{\gamma}_{BLF}$ with those of $\tilde{\gamma}_C$ and $\tilde{\gamma}_B$ under the BLF. The simulated Bayes risks are the losses given in (2.3) averaged over 10,000 repetitions of the experiment for each parameters as well as ω and f. We considered binomial, Poisson and normal cases with the assumption of posterior linearity. The percentage risk improvement of $\tilde{\gamma}_{BLF}$ over $\tilde{\gamma}_C$ and $\tilde{\gamma}_B$, denoted by $PCTIMP_C$ and $PCTIMP_B$ repectively, are provided based on the simulated

Table 4: The percentage improvements of $\tilde{\gamma}_{BLF}$ over $\tilde{\gamma}_C$ and $\tilde{\gamma}_B$ in binomial case

(p, q)	ω	$f = 0.8$		$f = 0.9$		$f = 0.95$	
		$PCTIMP_C$	$PCTIMP_B$	$PCTIMP_C$	$PCTIMP_B$	$PCTIMP_C$	$PCTIMP_B$
	.00	1.264	0.000	2.871	0.000	6.034	0.000
	.25	0.036	0.008	0.183	0.026	0.732	0.101
(0.5, 0.5)	.50	0.007	0.016	0.039	0.057	0.171	0.235
	.75	0.001	0.022	0.007	0.083	0.034	0.326
	1.00	0.000	0.028	0.000	0.114	0.000	0.440
	.00	2.766	0.000	5.925	0.000	11.145	0.000
	.25	0.066	0.018	0.362	0.044	1.436	0.170
(1.0, 1.0)	.50	0.014	0.034	0.078	0.116	0.329	0.420
	.75	0.002	0.046	0.014	0.157	0.057	0.623
	1.00	0.000	0.055	0.000	0.214	0.000	0.853
	.00	5.366	0.000	11.106	0.000	19.992	0.000
	.25	0.119	0.041	0.681	0.098	2.434	0.346
(2.0, 2.0)	.50	0.027	0.063	0.152	0.201	0.640	0.724
	.75	0.004	0.086	0.025	0.309	0.103	1.109
	1.00	0.000	0.105	0.000	0.403	0.000	1.507
	.00	3.841	0.000	6.020	0.000	13.516	0.000
	.25	0.091	0.018	0.475	0.044	1.672	0.226
(0.5, 2.0)	.50	0.017	0.039	0.098	0.127	0.434	0.494
	.75	0.003	0.055	0.017	0.203	0.065	0.756
	1.00	0.000	0.067	0.000	0.261	0.000	0.992
	.00	3.190	0.000	6.946	0.000	15.029	0.000
	.25	0.061	0.033	0.427	0.062	1.655	0.250
(2.0, 0.5)	.50	0.019	0.037	0.094	0.144	0.446	0.486
	.75	0.002	0.055	0.015	0.204	0.067	0.776
	1.00	0.000	0.069	0.000	0.266	0.000	1.016
	.00	7.219	0.000	15.746	0.000	26.587	0.000
	.25	0.182	0.054	0.992	0.123	3.407	0.473
(2.0, 4.0)	.50	0.043	0.084	0.221	0.285	0.794	1.005
	.75	0.006	0.126	0.034	0.442	0.137	1.519
	1.00	0.000	0.157	0.000	0.580	0.000	1.997

Bayes risks of each estimator. Here, the percentage risk improvement of $\tilde{\gamma}_{BLF}$ over $\tilde{\gamma}$ is defined as

$$PCTIMP = \frac{r(\tilde{\gamma}, \gamma) - r(\tilde{\gamma}_{BLF}, \gamma)}{r(\tilde{\gamma}, \gamma)} \times 100.$$

First, we consider the normal case. Without loss of generality, let μ be zero and let σ^2 be one. The Bayes risks are calculated for various ω, f and τ^2. Table 3 reports the percentage risk improvement $PCTIMP_C$ and $PCTIMP_B$ of $\tilde{\gamma}_{BLF}$ over one of typical estimators $\tilde{\gamma}_C$ and $\tilde{\gamma}_B$ for various choices of f as well as ω and τ^2. Figure 1 and Figure 2 illustrate the results for easy verification. As stated previous numerical results, if $\omega = 1$, that is, reflecting only goodness of fit, then $\tilde{\gamma}_{BLF}$ is equivalent to $\tilde{\gamma}_C$ and $PCTIMP_C$ is equal to zero. However if $\omega = 0$, that is, reflecting only the precision of estimator, then $\tilde{\gamma}_{BLF}$ is equivalent to $\tilde{\gamma}_B$ and $PCTIMP_B$ is equal to zero. $PCTIMP_C$ and $PCTIMP_B$ are greater than zero except for ω equal to zero or one. It shows that the optimal estimator $\tilde{\gamma}_{BLF}$ is better than both $\tilde{\gamma}_C$ and $\tilde{\gamma}_B$ in terms of the Bayes risks. In addition, $PCTIMP_C$ is monotonically decreasing in ω and $PCTIMP_B$ is monotonically increasing in ω.

Table 3 show that both $PCTIMP_C$ and $PCTIMP_B$ decrease naturally with τ^2. It's because higher values of τ^2 imply greater uncertainty about the prior. Moreover, we observe that the $PCTIMP$s are getting larger as values of f increase from Table 3, since large values of the finite population corrector f means a small n and smaller n with better $\tilde{\gamma}_{BLF}$ than typical estimates.

Table 5: The percentage improvements of $\tilde{\gamma}_{BLF}$ over $\tilde{\gamma}_C$ and $\tilde{\gamma}_B$ in Poisson case

(α, p)	ω	$f = 0.8$		$f = 0.9$		$f = 0.95$	
		$PCTIMP_C$	$PCTIMP_B$	$PCTIMP_C$	$PCTIMP_B$	$PCTIMP_C$	$PCTIMP_B$
(0.5, 0.5)	.00	0.901	0.000	1.304	0.000	2.552	0.000
	.25	0.018	0.004	0.069	0.023	0.503	0.007
	.50	0.003	0.009	0.031	0.018	0.115	0.092
	.75	0.001	0.012	0.003	0.047	0.015	0.173
	1.00	0.000	0.014	0.000	0.057	0.000	0.233
(0.5, 1.0)	.00	0.621	0.000	1.563	0.000	3.761	0.000
	.25	0.015	0.006	0.074	0.021	0.449	0.028
	.50	0.004	0.007	0.020	0.030	0.065	0.142
	.75	0.001	0.012	0.004	0.041	0.016	0.169
	1.00	0.000	0.014	0.000	0.056	0.000	0.232
(0.5, 3.0)	.00	0.791	0.000	1.636	0.000	3.688	0.000
	.25	0.016	0.006	0.078	0.019	0.360	0.063
	.50	0.002	0.010	0.018	0.033	0.092	0.115
	.75	0.001	0.012	0.003	0.043	0.016	0.174
	1.00	0.000	0.014	0.000	0.056	0.000	0.231
(1.0, 0.5)	.00	1.777	0.000	2.808	0.000	5.497	0.000
	.25	0.032	0.009	0.179	0.023	0.894	0.056
	.50	0.005	0.019	0.049	0.042	0.133	0.258
	.75	0.002	0.026	0.008	0.081	0.026	0.348
	1.00	0.000	0.028	0.000	0.107	0.000	0.438
(1.0, 1.0)	.00	1.450	0.000	2.662	0.000	5.424	0.000
	.25	0.025	0.015	0.162	0.037	0.689	0.131
	.50	0.005	0.019	0.053	0.041	0.251	0.153
	.75	0.002	0.022	0.009	0.080	0.026	0.347
	1.00	0.000	0.028	0.000	0.112	0.000	0.435
(1.0, 3.0)	.00	1.175	0.000	2.578	0.000	5.801	0.000
	.25	0.040	0.006	0.166	0.031	0.724	0.108
	.50	0.008	0.015	0.033	0.063	0.193	0.197
	.75	0.002	0.022	0.007	0.086	0.029	0.332
	1.00	0.000	0.028	0.000	0.110	0.000	0.429
(3.0, 0.5)	.00	4.345	0.000	9.900	0.000	17.245	0.000
	.25	0.082	0.036	0.509	0.076	1.713	0.359
	.50	0.024	0.043	0.123	0.148	0.418	0.656
	.75	0.005	0.059	0.022	0.229	0.078	0.920
	1.00	0.000	0.080	0.000	0.315	0.000	1.179
(3.0, 1.0)	.00	4.361	0.000	7.645	0.000	16.804	0.000
	.25	0.094	0.029	0.467	0.097	2.079	0.261
	.50	0.017	0.051	0.106	0.179	0.504	0.544
	.75	0.003	0.067	0.020	0.234	0.087	0.873
	1.00	0.000	0.080	0.000	0.312	0.000	1.163
(3.0, 3.0)	.00	3.549	0.000	8.693	0.000	15.950	0.000
	.25	0.096	0.028	0.517	0.083	1.931	0.259
	.50	0.019	0.051	0.121	0.145	0.465	0.612
	.75	0.003	0.068	0.017	0.253	0.077	0.904
	1.00	0.000	0.082	0.000	0.312	0.000	1.178

Next, we consider the binomial case. In this case, we first generate θ from a $Beta(p, q)$, and then we generate y_i's from $Bin(1, \theta)$. Various choices of p and q are considered. Table 4 provides Monte Carlo findings. Lastly, the Poisson case is considered. First the θ is generated from a Gamma(α, p), and then the y_i's are generated from Poisson(θ). Various types of gamma distributions are considered in Table 5. Similar results appear in both cases with normal case.

Furthermore, we compare the Bayes risks of the empirical Bayes estimator $\tilde{\gamma}_{BLF.Em}$ given in (3.3)

Table 6: Bayes risks of the empirical Bayes estimators and $PCTIMP$s in normal case, $f = 0.95$

τ^2	ω	$r(\tilde{\gamma}_C, \gamma)$	$r(\tilde{\gamma}_{B.Em}, \gamma)$	$r(\tilde{\gamma}_{BLF.Em}, \gamma)$	$PCTIMP_{C.Em}$	$PCTIMP_{B.Em}$
	.00	0.093	0.088	0.088	5.497	0.000
	.25	0.296	0.294	0.293	0.966	0.181
0.5	.50	0.498	0.498	0.497	0.316	0.283
	.75	0.699	0.702	0.698	0.054	0.501
	1.00	0.899	0.906	0.899	0.000	0.686
	.00	0.097	0.094	0.094	3.462	0.000
	.25	0.294	0.293	0.292	0.662	0.075
1.0	.50	0.499	0.500	0.499	0.143	0.198
	.75	0.706	0.708	0.706	0.027	0.319
	1.00	0.900	0.904	0.900	0.000	0.397
	.00	0.096	0.094	0.094	2.098	0.000
	.25	0.295	0.294	0.294	0.261	0.074
2.0	.50	0.497	0.497	0.496	0.081	0.111
	.75	0.700	0.701	0.700	0.016	0.167
	1.00	0.904	0.906	0.904	0.000	0.212

Table 7: Bayes risks of the empirical Bayes estimators and $PCTIMP$s in binomial case, $f = 0.95$

(p, q)	ω	$r(\tilde{\gamma}_C, \gamma)$	$r(\tilde{\gamma}_{B.Em}, \gamma)$	$r(\tilde{\gamma}_{BLF.Em}, \gamma)$	$PCTIMP_{C.Em}$	$PCTIMP_{B.Em}$
	.00	0.011	0.011	0.011	1.277	0.000
	.25	0.037	0.037	0.037	0.400	0.509
(0.5, 0.5)	.50	0.061	0.062	0.061	0.096	0.654
	.75	0.087	0.088	0.087	0.007	0.930
	1.00	0.111	0.112	0.111	0.000	0.877
	.00	0.016	0.015	0.015	7.448	0.000
	.25	0.049	0.049	0.048	1.322	0.257
(1.0, 1.0)	.50	0.082	0.082	0.082	0.385	0.453
	.75	0.116	0.117	0.116	0.058	0.713
	1.00	0.151	0.153	0.151	0.000	0.979
	.00	0.019	0.016	0.016	14.126	0.000
	.25	0.059	0.058	0.058	2.605	0.177
(2.0, 2.0)	.50	0.100	0.100	0.099	0.619	0.666
	.75	0.139	0.141	0.139	0.108	1.072
	1.00	0.180	0.183	0.180	0.000	1.446
	.00	0.011	0.010	0.010	9.686	0.000
	.25	0.034	0.033	0.033	1.554	0.396
(0.5, 2.0)	.50	0.057	0.057	0.057	0.388	0.684
	.75	0.080	0.081	0.080	0.061	0.977
	1.00	0.103	0.104	0.103	0.000	1.278
	.00	0.011	0.010	0.010	8.332	0.000
	.25	0.034	0.034	0.034	1.432	0.417
(2.0, 0.5)	.50	0.057	0.057	0.057	0.416	0.692
	.75	0.079	0.080	0.079	0.060	1.027
	1.00	0.103	0.105	0.103	0.000	1.187
	.00	0.018	0.015	0.015	17.512	0.000
	.25	0.057	0.055	0.055	3.348	0.161
(2.0, 4.0)	.50	0.095	0.095	0.094	0.835	0.742
	.75	0.132	0.134	0.132	0.147	1.347
	1.00	0.173	0.176	0.173	0.000	1.815

under the BLF with those of other considered estimators, $\tilde{\gamma}_C$ and $\tilde{\gamma}_{B.Em}$ given in (3.4). Table 6 reports the simulated Bayes Risks of $\tilde{\gamma}_C$, $\tilde{\gamma}_{B.Em}$ and $\tilde{\gamma}_{BLF.Em}$, and the percentage risk improvement of $\tilde{\gamma}_{BLF.Em}$ over $\tilde{\gamma}_C$, denoted by $PCTIMP_{C.Em}$ and the percentage risk improvement of $\tilde{\gamma}_{BLF.Em}$ over $\tilde{\gamma}_{B.Em}$, denoted by $PCTIMP_{B.Em}$ in normal case ($f = 0.95$) which $m = 5$ and $n_1 = 5$, $n_2 = 12$, $n_3 = 8$, $n_4 = 15$

Table 8: Bayes risks of the empirical Bayes estimators and $PCTIMP$s in Poisson case, $f = 0.95$

(α, p)	ω	$r(\tilde{\gamma}_C, \gamma)$	$r(\tilde{\gamma}_{B.Em}, \gamma)$	$r(\tilde{\gamma}_{BLF.Em}, \gamma)$	$PCTIMP_{C.Em}$	$PCTIMP_{B.Em}$
	.00	0.092	0.090	0.090	1.602	0.000
	.25	0.290	0.289	0.289	0.220	0.109
(0.5, 0.5)	.50	0.500	0.501	0.500	0.056	0.153
	.75	0.685	0.686	0.685	0.012	0.191
	1.00	0.904	0.906	0.904	0.000	0.247
	.00	0.193	0.190	0.190	1.708	0.000
	.25	0.598	0.596	0.596	0.338	0.051
(0.5, 1.0)	.50	0.976	0.977	0.976	0.068	0.137
	.75	1.388	1.391	1.388	0.016	0.174
	1.00	1.763	1.767	1.763	0.000	0.233
	.00	0.576	0.566	0.566	1.735	0.000
	.25	1.797	1.792	1.791	0.332	0.045
(0.5, 3.0)	.50	2.962	2.963	2.959	0.077	0.118
	.75	4.212	4.219	4.212	0.012	0.172
	1.00	5.487	5.499	5.487	0.000	0.218
	.00	0.048	0.046	0.046	2.853	0.000
	.25	0.145	0.145	0.145	0.462	0.172
(1.0, 0.5)	.50	0.245	0.245	0.244	0.142	0.253
	.75	0.362	0.363	0.362	0.027	0.329
	1.00	0.438	0.440	0.438	0.000	0.438
	.00	0.094	0.091	0.091	3.075	0.000
	.25	0.293	0.291	0.291	0.554	0.119
(1.0, 1.0)	.50	0.496	0.497	0.496	0.143	0.207
	.75	0.689	0.691	0.689	0.025	0.329
	1.00	0.879	0.883	0.879	0.000	0.436
	.00	0.303	0.292	0.292	3.517	0.000
	.25	0.874	0.869	0.868	0.598	0.083
(1.0, 3.0)	.50	1.482	1.483	1.480	0.134	0.226
	.75	2.109	2.115	2.108	0.022	0.316
	1.00	2.708	2.719	2.708	0.000	0.400
	.00	0.016	0.015	0.015	9.496	0.000
	.25	0.049	0.049	0.049	1.834	0.072
(3.0, 0.5)	.50	0.085	0.085	0.084	0.404	0.429
	.75	0.115	0.116	0.115	0.070	0.704
	1.00	0.148	0.150	0.148	0.000	0.976
	.00	0.033	0.030	0.030	9.105	0.000
	.25	0.098	0.097	0.097	1.406	0.197
(3.0, 1.0)	.50	0.169	0.169	0.168	0.460	0.322
	.75	0.233	0.235	0.233	0.071	0.654
	1.00	0.302	0.305	0.302	0.000	0.919
	.00	0.093	0.084	0.084	9.518	0.000
	.25	0.293	0.288	0.288	1.768	0.058
(3.0, 3.0)	.50	0.498	0.497	0.495	0.426	0.368
	.75	0.709	0.713	0.709	0.073	0.638
	1.00	0.876	0.884	0.876	0.000	0.934

and $n_5 = 10$. Results for binomial and Poisson cases are shown in Table 7 and Table 8. It is easy to check in all cases that $\tilde{\gamma}_{BLF.Em}$ performs better than $\tilde{\gamma}_C$ and $\tilde{\gamma}_{B.Em}$.

4. Concluding Remarks

We have considered the balanced loss function in the Bayesian inferential problem for finite population sampling. The balanced loss function is a broadened loss function that incorporates both the goodness of fit criterion and the precision of estimation criterion. We relaxed the normality assumption with the assumption of posterior linearity.

The Bayes and empirical Bayes estimators of the finite population mean under the BLF have a certain robustness property in the sense that they can be motivated with the basic assumption of posterior linearity rather than the normality of the superpopulation. The proposed estimators of the finite population mean under the BLF enjoy a clear superiority over the posterior mean and the sample mean in terms of the posterior expected loss and Bayes risk.

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