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A Note on Exponential Inequalities of ψ-Weakly Dependent Sequences

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Abstract

Two exponential inequalities are established for a wide class of general weakly dependent sequences of random variables, called *ψ-weakly dependent process* which unify weak dependence conditions such as mixing, association, Gaussian sequences and Bernoulli shifts. The ψ-weakly dependent process includes, for examples, stationary ARMA processes, bilinear processes, and threshold autoregressive processes, and includes essentially all classes of weakly dependent stationary processes of interest in statistics under natural conditions on the process parameters. The two exponential inequalities are established on more general conditions than some existing ones, and are proven in simpler ways.

Keywords: Weak dependence, exponential inequality, Bernstein-type inequality, partial sum of random variables.

1. Introduction

In this paper, we establish exponential inequalities for a wide class of general weakly dependent sequences of random variables. An exponential inequality for the partial sum $S_n = \sum_{t=1}^{n} X_t$ of random variables $\{X_t\}$ is useful in many probabilistic derivations and convergence theorems. New exponential inequalities are developed for a general weakly dependent sequence of stationary random variables, which is called ψ-weak dependence, proposed by Doukhan and Louhichi (1999).

The ψ-weakly dependent process generalizes mixings and other weakly dependent random variables. It was shown, by Ango Nze *et al.* (2002) that the ψ-weak dependence unifies weak dependence conditions such as mixing, association, Gaussian sequences and Bernoulli shifts. According to Ango Nze and Doukhan (2004), stationary ARMA processes, bilinear processes, and threshold autoregressive processes are all ψ-weakly dependent processes and the ψ-weakly dependent sequences essentially include all classes of weakly dependent stationary processes of interest in statistics under natural conditions on the process parameters.

Recently many studies have been done on ψ-weakly dependent processes. Since Doukhan and Louhichi (1999) which was the original work of the ψ-weak dependence, Coulon-Prieur and Doukhan (2000), Doukhan and Louhichi (2001), Dedecker and Prieur (2004) Kallabis and Neumann (2006), Doukhan and Neumann (2007, 2008), Hwang and Shin (2011, 2012a, 2012b, 2013a) among others presented probabilistic properties and statistical inferences.

In particular, as for the inequalities of ψ-weakly dependent processes, Doukhan and Louhichi (1999) established moment inequalities such as the Marcinkiewicz-Zygmund, Rosenthal and exponential inequalities, Dedecker and Prieur (2004) the Bennett-type inequality, Kallabis and Neumann

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(2006) the Bernstein-type inequality. Doukhan and Neumann (2007) improved results for Bernstein-type and Rosenthal-type inequalities, and Hwang and Shin (2013a) for Roussas-Ioannides-type inequalities.

In this paper, two new exponential inequalities are developed for a class of ψ-weak dependence under some mild conditions. The conditions required for the exponential inequalities are weak dependence coefficients of order $o(r^{-2})$ and finiteness of asymptotic norm of the process. These conditions are simpler than those works such as Kallabis and Neumann (2006) and Doukhan and Neumann (2007). Proofs of the exponential inequalities are concise thanks to the results of Hwang and Shin (2013a).

The remaining of the paper is organized as follows. Section 2 describes the notion of the ψ-weakly dependent processes and Section 3 presents the main results of two exponential inequalities. Section 4 gives proofs.

2. ψ-Weak Dependence

The definition of ψ-weak dependence makes explicit the asymptotic independence between "past" and "future". In terms of the time series, for convenient functions g and h, it is assumed that $\mathrm{Cov}(g_{\text{past}}, h_{\text{future}})$ is small when the distance between the "past" and the "future" is sufficiently large. Asymptotics are expressed in terms of the distance between indices of the initial time series in the "past" and the "future" terms; the convergence is not assumed to hold uniformly on the dimension of the marginal involved.

We introduce some classes of functions to define the notion of the weak dependence. Let $\mathbb{L}^\infty = \bigcup_{n=1}^{\infty} \mathbb{L}^\infty(\mathbb{R}^n)$, the set of real-valued and bounded functions on the space $\mathbb{R}^n$ for $n = 1, 2, \ldots$. Consider a function $g : \mathbb{R}^n \to \mathbb{R}$ where $\mathbb{R}^n$ is equipped with its l_1-norm (*i.e.* $\|(x_1, \ldots, x_n)\|_1 = |x_1| + \cdots + |x_n|$) and define the Lipschitz modulus of g,

$$\mathrm{Lip}(g) = \sup_{x \neq y} \frac{|g(x) - g(y)|}{\|x - y\|_1}.$$

Let

$$\mathcal{L} = \bigcup_{n=1}^{\infty} \mathcal{L}_n, \quad \text{where } \mathcal{L}_n = \{g \in \mathbb{L}^\infty(\mathbb{R}^n);\ \mathrm{Lip}(g) < \infty,\ \|g\|_\infty \leq 1\}.$$

The class $\mathcal{L}$ is sometimes used together with the following functions $\psi = \psi_0, \psi_1, \psi_2, \eta, \kappa$ and λ, where $\psi_0(g, h, n, m) = 4\|g\|_\infty \|h\|_\infty$, $\psi_1(g, h, n, m) = \min(n, m)\mathrm{Lip}(g)\mathrm{Lip}(h)$, $\psi_2(g, h, n, m) = 4(n + m)\min\{\mathrm{Lip}(g), \mathrm{Lip}(h)\}$, $\eta(g, h, n, m) = n\mathrm{Lip}(g) + m\mathrm{Lip}(h)$, $\kappa(g, h, n, m) = nm\mathrm{Lip}(g)\mathrm{Lip}(h)$, $\lambda(g, h, n, m) = n\mathrm{Lip}(g) + m\mathrm{Lip}(h) + nm\mathrm{Lip}(g)\mathrm{Lip}(h)$, for functions g and h defined on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. See Doukhan and Neumann (2007) and Dedecker *et al.* (2007).

Definition 1. *(Doukhan and Louhichi, 1999) The sequence $\{X_t\}_{t \in \mathbb{Z}}$ is called $(\theta, \mathcal{L}, \psi)$-weakly dependent, (simply, ψ-weakly dependent), if there exists a sequence $\theta = (\theta_r)_{r \in \mathbb{Z}}$ decreasing to zero at infinity and a function ψ with arguments $(g, h, n, m) \in \mathcal{L}_n \times \mathcal{L}_m \times \mathbb{N}^2$ such that for n-tuple $(i_1, \ldots, i_n)$ and m-tuple $(j_1, \ldots, j_m)$ with $i_1 \leq \cdots \leq i_n < i_n + r \leq j_1 \leq \cdots \leq j_m$, one has*

$$\left| Cov(g(X_{i_1}, \ldots, X_{i_n}), h(X_{j_1}, \ldots, X_{j_m})) \right| \leq \psi(g, h, n, m)\theta_r.$$

According to Doukhan and Louhichi (1999), strong mixing is ψ_0-weakly dependent, associated sequences are ψ_1-weakly dependent, and Bernoulli shifts and Markov processes are ψ_2-weakly dependent.

3. Main Results

In this section we present two exponential inequalities in Theorems 1 and 2, which require some mild assumptions of the weak dependence coefficients condition and asymptotically finite norm condition. In proving Theorems 1 and 2, moment inequality results of Hwang and Shin (2013a) are applied with an exponential function, which satisfy conditions of moment inequality in Hwang and Shin (2013a).

Theorem 1. *Let $\{X_t\}$ be a stationary sequence of ψ-weakly dependent random variables with mean zero and with ψ-weak dependence coefficient sequence $\{\theta_r\}$. If $\theta_r = o(r^{-2})$, and*

$$\lim_{n\to\infty} \|X_t\|_\gamma = \lim_{n\to\infty} (E|X_\gamma|)^{\frac{1}{\gamma}} < \infty, \quad \text{where } \gamma \sim \sqrt{n} \tag{3.1}$$

then for any $t > 0$ and for sufficiently large n, we have

$$P\left(\frac{1}{n}\left|\sum_{i=1}^{n} X_i\right| \geq t\right) \leq C_0 \log n \exp\left(-\frac{n^{\frac{1}{4}}}{2\log n} t\right).$$

Theorem 2. *Let $\{X_t\}$ be a stationary sequence of ψ-weakly dependent random variables with mean zero and with ψ-weak dependence coefficient sequence $\{\theta_r\}$. If $\theta_r = o(r^{-2})$, and*

$$\lim_{n\to\infty} \|X_t\|_\gamma = \lim_{n\to\infty} (E|X_\gamma|)^{\frac{1}{\gamma}} < \infty, \quad \text{where } \gamma \sim \sqrt{n}$$

then for any $t > 0$ and for sufficiently large n, we have

$$P(|S_n| \geq t) \leq C_0 \log n \exp\left(-\frac{t^2}{A_n + B_n t^{\epsilon}}\right),$$

where A_n can be chosen as any number greater than or equal to σ_n^2 and $B_n = n^{3/4}\log n / A_n$ for some constant $C_0 > 0$ and for any $0 < \epsilon \leq 1$.

Remark 1. Similar types of inequalities of weakly dependent processes can be found in Kallabis and Neumann (2006) and Doukhan and Neumann (2007). In Theorem 2.1 of Kallabis and Neumann (2006), condition $P(|X_t| \leq M) = 1$ is assumed, while in Theorem 1 of Doukhan and Neumann (2007), condition $E|X_t|^k \leq (k!)^\nu M^k$ for all k, for some finite M and ν, is assumed, and in Theorem 3 of Doukhan and Neumann (2007), condition $E|X_t|^{p-2} \leq M^{p-2}$ for some p is assumed, instead of finiteness of asymptotic norm of condition (3.1) in our Theorems 1 and 2. Note that $P(|X_t| \leq M) = 1$ and $E|X_t|^{p-2} \leq M^{p-2}$ imply condition (3.1), and condition $E|X_t|^k \leq (k!)^\nu M^k$ is more general than condition (3.1). Thus, our condition of $\lim_{\gamma\to\infty} \|X_t\|_\gamma < \infty$ can be regarded as an intermediate one between these two conditions of Theorem 2.1 of Kallabis and Neumann (2006) and Theorem 1 of Doukhan and Neumann (2007). As seen in the proofs of our theorem and those of Kallabis and Neumann (2006) and Doukhan and Neumann (2007), proof on condition (3.1) is simpler than those of Theorem 2.1 of Kallabis and Neumann (2006) and Theorem 1 of Doukhan and Neumann (2007).

Remark 2. As seen in the proof of Theorem 1, we can generalize exponential inequalities as: Choose sequences β, γ and τ tending to ∞ as $n \to \infty$ such that $\beta\gamma\tau \sim n$ and $\tau/(\beta\gamma^2) \to 0$. If the ψ-weak dependence process has norm condition $\|X_t\|_\gamma < \infty$ for sufficiently large n, and the weak dependence coefficients θ satisfies $\gamma\theta_{\beta\tau}^{1/2} \to 0$, then for any $t > 0$ and for sufficiently large n, we have

$$P\left(\frac{1}{n}\left|\sum_{i=1}^{n} X_i\right| \geq t\right) \leq c_0 \beta \exp\left(-\left(\frac{\tau}{\beta}\right)^{\frac{1}{2}} \frac{t}{2}\right)$$

and

$$P(|S_n| \geq t) \leq c_0\beta \exp\left(-\frac{t^2}{A_n + B_n t^{\epsilon}}\right),$$

where A_n is chosen as any number greater than or equal to σ_n^2 and $B_n = n(\beta/\tau)^{1/2}/A_n$ for some constant $c_0 > 0$ and for any $0 < \epsilon \leq 1$.

Remark 3. The exponential inequality result in Theorem 2 is used for almost a complete convergence of the kernel density estimators and consequently almost complete convergence of kernel mode estimators and its convergence rate for ψ-weakly dependent sequences in Hwang and Shin (2013b).

4. Proofs

To prove Theorems 1 and 2 we review a result of moment inequality established in Hwang and Shin (2013a). Let $\{X_t\}$ be a sequence of ψ-weakly dependent random variables with the weak dependence coefficient sequence (θ_r). Let A and B be disjoint finite sets of indices such that distance between A and B is greater than or equal to r, and let $\xi = h(X_j : j \in A)$ and $\eta = g(X_k : k \in B)$ where $h(x_j : j \in A)$ and $g(x_k : k \in B)$ are some real-valued functions. We make the following assumptions:

(i) ψ is bounded for the class of g, h such that $\|h\|_\infty, \|g\|_\infty, \mathrm{Lip}(g)$, and $\mathrm{Lip}(h)$ are all bounded,

(ii) $E|\xi|^p$, $E|\eta|^q < \infty$ for some p, $q > 1$ with $1/p + 1/q < 1$,

(iii) $E|X_j|^p < \infty$ for all $j \in A$; $E|X_k|^q < \infty$ for all $k \in B$,

(iv) $D_i h(x_j, j \in A) < \infty$ for all $x_j, j \in A$; $D_i g(x_k, k \in B) < \infty$ for all $x_k, k \in B$,

(v) $M_h^D := \max_{i\in A} \max_{|x_j|\leq M} |D_i h(x_j, j \in A)| = O(M_h)$ as $M \to \infty$,
$N_g^D := \max_{i\in B} \max_{|x_k|\leq N} |D_i g(x_k, k \in B)| = O(N_g)$ as $N \to \infty$,
where

$$D_i f(x_1, \ldots, x_n) = \limsup_{y_i \to x_i} \frac{|f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)|}{|y_i - x_i|},$$
$$M_h = \max \left| h\left(x_j^M : j \in A\right)\right|, \quad N_g = \max \left| g\left(x_k^N : k \in B\right)\right|,$$
$$x_j^M = x_j \text{ if } |x_j| \leq M; \; x_j^M = M \text{ if } x_j > M; \; x_j^M = -M \text{ if } x_j < -M,$$

and x_k^N is defined similarly. See Hwang and Shin (2013a) for some remarks on the conditions above. Now we state the Roussas-Ioannides-type inequality.

Lemma 1. *(Hwang and Shin, 2013a) Let $\{X_t\}$ be a sequence of ψ-weakly dependent random variables with the weak dependence coefficient sequence (θ_r).*

(a) Let $\xi = h(X_j : j \in A)$ and $\eta = g(X_k : k \in B)$ where h and g are some real-valued functions, and A and B are finite disjoint sets of indices such that distance between A and B is greater than or equal to r. Under conditions (i)–(v) above, we have

$$|E(\xi\eta) - (E\xi)(E\eta)| \leq C \cdot \theta_r^{1-\frac{1}{p}-\frac{1}{q}} \|\xi\|_p \|\eta\|_q$$

for some constant C not depending on r.

(b) Let A_i, $(i = 1, 2, \ldots, \gamma)$, be mutually disjoint finite sets of indices such that distance between A_{i+1} and A_i is greater than or equal to r for $i = 1, \ldots, \gamma - 1$. Let $\xi_i = h_i(X_{j_i} : j_i \in A_i)$ for some real-valued functions h_i such that $\|X_{j_i}\|_{p_i} < \infty$ for $j_i \in A_i$, and

$$E|\xi_i|^{p_i} < \infty \text{ with } p_i > 1, \quad (i = 1, \ldots, \gamma), \quad \text{and} \quad \frac{1}{p_1} + \cdots + \frac{1}{p_\gamma} =: \frac{1}{q_\gamma} < 1.$$

Under conditions (i), (iv), (v) above for ψ and $\xi = \xi_i$, $(i = 1, \ldots, \gamma)$, we have

$$\left| E\left[\prod_{i=1}^{\gamma} \xi_i\right] - \prod_{i=1}^{\gamma} E[\xi_i] \right| \le B(\gamma - 1)\theta_r^{1-\frac{1}{q_\gamma}} \prod_{i=1}^{\gamma} \|\xi_i\|_{p_i}$$

for some constant B not depending on r.

Proof of Theorem 1: To prove Theorem 1 we apply Lemma 1(b) above. Under the conditions $\theta_r = o(r^{-2})$ and $\|X_t\|_\gamma < \infty$ where $\gamma \sim \sqrt{n}$ for sufficiently large n, we apply Lemma 1(b) for function $h(x_1, \ldots, x_\tau) = \exp(1/n^{3/4} \sum_{i=1}^{\tau} x_i)$ for $\tau \sim \sqrt{n}/\log n$. Let $\gamma = \gamma_n = \lfloor \sqrt{n} \rfloor$, $\beta = \beta_n = \lfloor \log n \rfloor$ and $\tau = \tau_n = \lfloor \sqrt{n}/\log n \rfloor$, where $\lfloor x \rfloor$ is the integer part of x. Note that $\beta\gamma\tau \sim n$. For $i = 1, \ldots, \gamma$, and $j = 1, \ldots, \beta$, let

$$U_{j,i} = X_{[\beta(i-1)+(j-1)]\tau+1} + \cdots + X_{[\beta(i-1)+j]\tau},$$

and $W_n = X_{\beta\gamma\tau+1} + \cdots + X_n$. For $j = 1, \ldots, \beta$, let $\bar{U}_j = 1/n \sum_{i=1}^{\gamma} U_{j,i}$ and $\bar{W} = 1/n\, W_n$. Since $1/n \sum_{i=1}^{n} X_i = \bar{U}_1 + \cdots + \bar{U}_\beta + \bar{W}$, we consider

$$P\left(\frac{1}{n}\left|\sum_{i=1}^{n} X_i\right| \ge t\right) \le \sum_{j=1}^{\beta} P\left(\left|\bar{U}_j\right| \ge \frac{t}{\beta+1}\right) + P\left(\left|\bar{W}\right| \ge \frac{t}{\beta+1}\right) \tag{4.1}$$

for any $t > 0$. Let $\alpha = \sqrt{\tau \log n} \sim n^{1/4}$. For each j, we have

$$P\left(\bar{U}_j \ge \frac{t}{\beta+1}\right) = P\left(e^{\alpha\bar{U}_j} \ge e^{\frac{\alpha t}{\beta+1}}\right) \le e^{-\frac{\alpha t}{\beta+1}} E\left[e^{\alpha\bar{U}_j}\right]$$

$$\le e^{-\frac{\alpha t}{\beta+1}} \left[\left| E\left(e^{\frac{\alpha}{n}\sum_{i=1}^{\gamma} U_{j,i}}\right) - \prod_{i=1}^{\gamma} E\left(e^{\frac{\alpha}{n} U_{j,i}}\right)\right| + \prod_{i=1}^{\gamma} E\left(e^{\frac{\alpha}{n} U_{j,i}}\right)\right].$$

In order to find an upper bound of $\left|E(e^{\alpha/n \sum_{i=1}^{\gamma} U_{j,i}}) - \prod_{i=1}^{\gamma} E(e^{\alpha/n\, U_{j,i}})\right|$, we apply Lemma 1(b) above. Note that letting $p = 2\gamma$ and $\bar{X}_\tau = \sum_{i=1}^{\tau} X_i/\tau$, we have

$$E\left|h(X_1, \ldots, X_\tau)\right|^p = E\left|e^{\frac{\alpha}{n} U_{1,1}}\right|^p = E\exp\left(\frac{2\gamma\sqrt{\tau \log n}}{n}\tau\bar{X}_\tau\right) < \infty$$

for sufficiently large n since $2\gamma\sqrt{\tau \log n}\sqrt{\tau}/n \sim 1/\sqrt{\log n} \to 0$, $\sqrt{\tau}\bar{X}_\tau = O_p(1)$ and thus $(2\gamma\sqrt{\tau \log n}/n)\tau\bar{X}_\tau = o_p(1)$ as $n \to \infty$. By Lemma 1(b), we have

$$\left| E\left(e^{\frac{\alpha}{n}\sum_{i=1}^{\gamma} U_{j,i}}\right) - \prod_{i=1}^{\gamma} E\left(e^{\frac{\alpha}{n} U_{j,i}}\right)\right| \le c(\gamma - 1)\theta_{(\beta-1)\tau}^{1-\frac{\gamma}{p}} \left(E\left|e^{\frac{\alpha}{n} U_{1,1}}\right|^p\right)^{\frac{\gamma}{p}} = c(\gamma - 1)\theta_{(\beta-1)\tau}^{\frac{1}{2}}$$

for some generic constant $c > 0$.

Using equality $e^x \leq 1 + x + x^2$ ($|x| \leq 1/2$) and noting that $E(U_{j,i}) = 0$, $E(U_{j,i}^2) = \sum_{j=1}^{\tau} EX_j^2 + 2\sum_{i<j}^{\tau} \mathrm{Cov}(X_i, X_j)$, we observe

$$\prod_{i=1}^{\gamma} E\left(e^{\frac{\alpha}{n}U_{j,i}}\right) \leq \prod_{i=1}^{\gamma} E\left(1 + \frac{\alpha}{n}U_{j,i} + \frac{\alpha^2}{n^2}U_{j,i}^2\right) \leq \prod_{i=1}^{\gamma}\left(1 + c\frac{\alpha^2\tau^2}{n^2}\right) \tag{4.2}$$

and

$$\frac{\alpha^2\tau^2}{n^2} \sim \frac{\tau\log n}{n^2}\frac{n}{(\log n)^2} \sim \frac{1}{\sqrt{n}(\log n)^2} \sim \frac{1}{b_n\gamma},$$

where $b_n = (\log n)^2 \to \infty$ as $n \to \infty$. Thus the last term in (4.2) is asymptotically same as $(1 + c/(b_n\gamma))^\gamma \sim e^{c/b_n} \to 1$ as $n \to \infty$. Hence,

$$P\left(\bar{U}_j \geq \frac{t}{\beta+1}\right) \leq e^{-\frac{\alpha t}{\beta+1}}\left[c(\gamma-1)\theta_{(\beta-1)\tau}^{\frac{1}{2}} + 1\right] \leq ce^{-\frac{\alpha t}{\beta+1}},$$

where the last inequality holds since $(\gamma - 1)\theta_{(\beta-1)\tau}^{1/2} \sim \sqrt{n}\theta_{\beta\tau}^{1/2} \to 0$ for $\theta_r = o(r^{-2})$. The above inequalities hold as $U_{j,i}$ are replaced by $-U_{j,i}$, and thus we obtain $P(|\bar{U}_j| \geq t/(\beta+1)) = P(\bar{U}_j \geq t/(\beta+1)) + P(-\bar{U}_j \geq t/(\beta+1)) \leq ce^{-\alpha t/(\beta+1)}$. It is clear that $P(|\bar{W}| \geq t/(\beta+1)) \to 0$ as $n \to \infty$. Therefore, in (4.1), we obtain

$$P\left(\frac{1}{n}\left|\sum_{i=1}^{n} X_i\right| \geq t\right) \leq C_0\beta\exp\left(-\frac{\alpha}{2\beta}t\right)$$

for sufficiently large n. We complete the proof. □

Proof of Theorem 2: By Theorem 1, for sufficiently large n we have

$$P(|S_n| \geq nt_0) \leq C_0\log n\exp(-\varphi_n t_0)$$

for any $t_0 > 0$, where $\varphi_n = n^{1/4}/(2\log n)$. Equivalently, for any $t > 0$, we have

$$P(|S_n| \geq t) \leq C_0\log n\exp\left(-\frac{\varphi_n}{n}t\right).$$

Note that sequence φ_n increases in n; consequently, for given fixed $t > 0$ there exists N such that $t < \varphi_n$ for all $n \geq N$. Also note that $\sigma_n^2 = O(n)$ and

$$\frac{\varphi_n}{\sigma_n^2} \sim \frac{\varphi_n}{n} = \frac{1}{2n^{\frac{3}{4}}\log n}.$$

We observe, for $0 < \epsilon \leq 1$,

$$\frac{t^2}{A_n + B_n t^\epsilon} \leq \frac{\varphi_n t}{A_n + B_n t^\epsilon} \leq \frac{\varphi_n t}{\sigma_n^2 + B_n t^\epsilon} = \frac{\varphi_n t}{\sigma_n^2 + t^\epsilon n^{\frac{3}{4}}\log n/A_n}$$
$$\sim \frac{\varphi_n t}{\sigma_n^2 + \sigma_n^2 t^\epsilon/(2A_n\varphi_n)} = \frac{\varphi_n t}{\sigma_n^2[1 + t^\epsilon/(2A_n\varphi_n)]} \sim \frac{\varphi_n t}{\sigma_n^2} \sim \frac{\varphi_n}{n}t$$

and thus

$$\exp\left(-\frac{\varphi_n}{n}t\right) \leq \exp\left(-\frac{t^2}{A_n + B_n t^\epsilon}\right).$$

Therefore, the desired inequality holds for sufficiently large n. □

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