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AREA OF TRIANGLES ASSOCIATED WITH A CURVE

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Abstract. It is well known that the area U of the triangle formed by three tangents to a parabola X is half of the area T of the triangle formed by joining their points of contact. In this article, we study some properties of U and T for strictly convex plane curves. As a result, we establish a characterization for parabolas.

1. Introduction

Let $X = X(s)$ be a unit speed smooth curve in the plane $\mathbb{R}^2$ with nonvanishing curvature, and let $A = X(s)$, $A_i = X(s + h_i)$, $i = 1, 2$, be three distinct neighboring points on X. Denote by ℓ, ℓ_1, ℓ_2 the tangent lines passing through the points A, A_1, A_2 and by B, B_1, B_2 the intersection points $\ell_1 \cap \ell_2$, $\ell \cap \ell_1$, $\ell \cap \ell_2$, respectively. It is well known that the area $U(s, h_1, h_2) = |\triangle BB_1B_2|$ of the triangle formed by three tangents to a parabola is half of the area $T(s, h_1, h_2) = |\triangle AA_1A_2|$ of the triangle formed by joining their points of contact ([1]).

The present article studies whether this property exhaustively characterizes parabolas.

Usually, a regular plane curve $X : I \to \mathbb{R}^2$ defined on an open interval is called *convex* if, for all $t \in I$, the trace $X(I)$ lies entirely on one side of the closed half-plane determined by the tangent line at $X(t)$ ([2]).

Hereafter, we will say that a simple convex curve X in the plane $\mathbb{R}^2$ is *strictly convex* if the curve is smooth (that is, of class $C^{(3)}$) and is of positive curvature κ with respect to the unit normal N pointing to the convex side. Hence, in this case we have $\kappa(s) = \langle X''(s), N(X(s))\rangle > 0$, where $X(s)$ is an arclength parametrization of X.

For a smooth function $f : I \to \mathbb{R}$ defined on an open interval, we will also say that f is *strictly convex* if the graph of f has positive curvature κ with respect

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to the upward unit normal N. This condition is equivalent to the positivity of $f''(x)$ on I.

Suppose that X is a strictly convex curve in the plane $\mathbb{R}^2$ with the unit normal N pointing to the convex side. For a fixed point $P = A \in X$, and for a sufficiently small $h > 0$, consider the line m passing through $P + hN(P)$ which is parallel to the tangent ℓ of X at P. Let us denote by A_1 and A_2 the points where the line m intersects the curve X. We denote by $L_P(h)$ the length $|A_1A_2|$ of the chord A_1A_2.

Let us denote by ℓ_1, ℓ_2 the tangent lines passing through the points A_1, A_2 and by B, B_1, B_2 the intersection points $\ell_1 \cap \ell_2$, $\ell \cap \ell_1$, $\ell \cap \ell_2$, respectively. We denote by $T_P(h), U_P(h)$ the area $| \triangle AA_1A_2|$, $| \triangle BB_1B_2|$, of triangles, respectively. Then, obviously we have $T_P(h) = \frac{h}{2}L_P(h)$.

In this paper, first of all, in Section 2 we prove the following:

Theorem 1. *Let X denote a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$. Then we have*

$$\lim_{h\to 0} \frac{T_P(h)}{h\sqrt{h}} = \frac{\sqrt{2}}{\sqrt{\kappa(P)}} \tag{1.1}$$

and

$$\lim_{h\to 0} \frac{U_P(h)}{h\sqrt{h}} = \frac{\sqrt{2}}{2\sqrt{\kappa(P)}}. \tag{1.2}$$

Next in Section 3, using Theorem 1 we characterize parabolas as follows.

Theorem 2. *Let $X = X(s)$ denote a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$. Suppose that for all s and sufficiently small $h_i, i = 1, 2$, the curve X satisfies*

$$U(s, h_1, h_2) = \lambda(s)T(s, h_1, h_2). \tag{1.3}$$

Then, we have $\lambda(s) = \frac{1}{2}$ and X is an open part of a parabola.

In [8], Krawczyk showed that for a strictly convex $C^{(4)}$ curve $X = X(s)$ in the plane $\mathbb{R}^2$, the following holds:

$$\lim_{h_1,h_2\to 0} \frac{T(s, h_1, h_2)}{U(s, h_1, h_2)} = 2. \tag{1.4}$$

His application of (1.4) states that if a strictly convex $C^{(4)}$ curve $X = X(s)$ in the plane $\mathbb{R}^2$ satisfies (1.3), then $\lambda(s) = \frac{1}{2}$ and X is an open part of the graph of a quadratic polynomial.

But, for example, consider a function $f(x)$ given by

$$y = \frac{2\sqrt{a}cx + 1 - \sqrt{4\sqrt{a}cx + 1}}{2c^2}, \tag{3.14}$$

where $a, c > 0$. Then, the function f is defined on $I = (-\frac{1}{4\sqrt{a}c}, \infty)$. Its graph X is strictly convex and satisfies (1.3) with $\lambda = \frac{1}{2}$. Note that X is not the graph of a quadratic polynomial, but an open part of the parabola given in (3.16) in Section 3.

In [6], the first author and Y. H. Kim established five characterizations of parabolas, which are the converses of well-known properties of parabolas originally due to Archimedes ([10]). In [4] and [5], they also proved the higher dimensional analogues of some results in [6].

For a few characterizations of parabolas or conic sections by some properties of tangent lines, see [3] and [7].

Among the graphs of functions, B. Richmond and T. Richmond established a dozen characterizations of parabolas using elementary techniques ([9]). In [9], parabola means the graph of a quadratic polynomial in one variable.

Finally, we pose a question as follows.

Question 3. Let X be a strictly convex $C^{(3)}$ plane curve. Suppose that for each $P \in X$ there exists a positive number $\epsilon = \epsilon(P) > 0$ such that $U_P(h) = T_P(h)/2$ for all h with $0 < h < \epsilon(P)$. Then, is it an open part of a parabola?

Throughout this article, all curves are of class $C^{(3)}$ and connected, unless otherwise mentioned.

2. Preliminaries and Theorem 1

In order to prove Theorem 1, we need the following lemma ([6]).

Lemma 4. *Suppose that X is a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ with the unit normal N pointing to the convex side. Then we have*

$$\lim_{h\to 0} \frac{1}{\sqrt{h}} L_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}}, \tag{2.1}$$

where $\kappa(P)$ is the curvature of X at P with respect to the unit normal N.

First of all, we give a proof of (1.1) in Theorem 1. Since $T_P(h) = \frac{h}{2} L_P(h)$, it follows from Lemma 4 that the following holds:

$$\lim_{h\to 0} \frac{1}{h\sqrt{h}} T_P(h) = \frac{\sqrt{2}}{\sqrt{\kappa(P)}}. \tag{1.1}$$

In order to prove (1.2) in Theorem 1, we fix an arbitrary point P on X. Then, we may take a coordinate system (x, y) of $\mathbb{R}^2$: P is taken to be the origin $(0, 0)$ and x-axis is the tangent line ℓ of X at P. Furthermore, we may regard X to be locally the graph of a non-negative strictly convex function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = f'(0) = 0$. Then N is the upward unit normal.

Since the curve X is of class $C^{(3)}$, the Taylor's formula of $f(x)$ is given by

$$f(x) = ax^2 + g(x), \tag{2.2}$$

where $2a = f''(0)$ and $g(x)$ is an $O(|x|^3)$ function. From $\kappa(P) = f''(0) > 0$, we see that a is positive.

For a sufficiently small $h > 0$, we denote by $A_1(s, f(s))$ and $A_2(t, f(t))$ the points where the line $m : y = h$ meets the curve X with $s < 0 < t$. Then $f(s) = f(t) = h$ and we get $B_1(s - h/f'(s), 0)$, $B_2(t - h/f'(t), 0)$ and $B(x_0, y_0)$ with

$$x_0 = \frac{tf'(t) - sf'(s)}{f'(t) - f'(s)} \tag{2.3}$$

and

$$y_0 = \frac{(t-s)f'(t)f'(s) + h(f'(t) - f'(s))}{f'(t) - f'(s)}. \tag{2.4}$$

Noting that $L_P(h) = t - s$, one obtains

$$\begin{aligned} 2U_P(h) &= \{t - s - \frac{h}{f'(t)} + \frac{h}{f'(s)}\}(-y_0) \\ &= h^2 \frac{(f'(t) - f'(s))}{-f'(s)f'(t)} - 2hL_P(h) + \frac{-f'(s)f'(t)}{f'(t) - f'(s)} L_P(h)^2. \end{aligned} \tag{2.5}$$

It follows from (2.5) that

$$2\frac{U_P(h)}{h\sqrt{h}} = \alpha_P(h) - 2\frac{L_P(h)}{\sqrt{h}} + \frac{1}{\alpha_P(h)}(\frac{L_P(h)}{\sqrt{h}})^2, \tag{2.6}$$

where we denote

$$\alpha_P(h) = \sqrt{h}\frac{(f'(t) - f'(s))}{-f'(s)f'(t)}. \tag{2.7}$$

Finally, we prove a lemma, which together with (2.6) and Lemma 4, completes the proof of (2) in Theorem 1.

Lemma 5. *We have the following.*

$$\lim_{h \to 0} \alpha_P(h) = \frac{\sqrt{2}}{\sqrt{\kappa(P)}}. \tag{2.8}$$

Proof. Note that

$$\alpha_P(h) = \frac{\beta_P(h)}{\gamma_P(h)}, \tag{2.9}$$

where we denote

$$\beta_P(h) = \frac{f'(t) - f'(s)}{t - s} \tag{2.10}$$

and

$$\gamma_P(h) = \frac{-f'(s)f'(t)}{\sqrt{h}(t - s)}. \tag{2.11}$$

Applying mean value theorem to the derivative $f'(x)$ of $f(x)$ shows that as h tends to 0, $\beta_P(h)$ goes to $f''(0) = \kappa(P)$. To get the limit of $\gamma_P(h)$, we put

$$\delta_P(h) = \frac{f'(s)f'(t)}{st} \tag{2.12}$$

and

$$\eta_P(h) = \frac{-st}{\sqrt{h}(t-s)}. \tag{2.13}$$

Then, we have

$$\gamma_P(h) = \delta_P(h)\eta_P(h). \tag{2.14}$$

Note that

$$\lim_{h\to 0} \delta_P(h) = \kappa(P)^2. \tag{2.15}$$

If we use $L_P(h) = t - s$, $\eta_P(h)$ can be written as

$$\eta_P(h) = (-\frac{st}{h})/(\frac{L_P(h)}{\sqrt{h}}) \tag{2.16}$$

and the numerator of (2.16) can be decomposed as

$$-\frac{st}{h} = (\frac{L_P(h)}{\sqrt{h}} - \frac{t}{\sqrt{h}})\frac{t}{\sqrt{h}}. \tag{2.17}$$

Now, to obtain the limit of $\frac{t}{\sqrt{h}}$, we use (2.2). Recalling that $\kappa(P) = f''(0) = 2a$, we have

$$\frac{t}{\sqrt{h}} = \frac{t}{\sqrt{at^2 + g(t)}}. \tag{2.18}$$

Since $g(x)$ is an $O(|x|^3)$ function, (2.18) implies that $\lim_{h\to 0} t/\sqrt{h} = 1/\sqrt{a}$. Hence, together with (2.17), Lemma 4 shows that $\lim_{h\to 0}(-st)/h = 1/a$, and hence from (2.16) we get

$$\lim_{h\to 0} \eta_P(h) = \frac{1}{2\sqrt{a}}. \tag{2.19}$$

Thus, it follows from (2.14) and (2.15) that

$$\lim_{h\to 0} \gamma_P(h) = 2a\sqrt{a}. \tag{2.20}$$

Using $\kappa(P) = 2a$, together with (2.9) and (2.10), (2.20) completes the proof of Lemma 5. □

3. Proof of Theorem 2

In this section, we prove Theorem 2.

Suppose that $X = X(s)$ denote a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ which satisfies for all s and sufficiently small $h_i, i = 1, 2$,

$$U(s, h_1, h_2) = \lambda(s)T(s, h_1, h_2). \tag{1.3}$$

Then, in particular, for all $P = X(s)$ and sufficiently small $h > 0$ the curve X satisfies

$$U_P(h) = \lambda(P)T_P(h). \tag{3.1}$$

Hence, Theorem 1 implies that $\lambda(P) = \frac{1}{2}$.

In order to prove the remaining part of Theorem 2, first, we fix an arbitrary point A on X. As in Section 1, we take a coordinate system (x, y) of $\mathbb{R}^2$: A is taken to be the origin $(0,0)$ and x-axis is the tangent line ℓ of X at A. Furthermore, we may regard X to be locally the graph of a non-negative strictly convex function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = f'(0) = 0$ and $2a = f''(0) > 0$.

For sufficiently small $|s|$ and $|t|$ with $0 < s < t$ or $t < s < 0$, we let $A_1 = (s, f(s)), A_2 = (t, f(t))$ be two neighboring points of A on X. Then, the area $T(s,t)$ of the triangle $\triangle AA_1A_2$ is given by

$$2\epsilon T(s,t) = (sf(t) - tf(s)), \tag{3.2}$$

where $\epsilon = 1$ if $0 < s < t$ and $\epsilon = -1$ if $t < s < 0$.

Denote by ℓ, ℓ_1, ℓ_2 the tangent lines passing through the points A, A_1, A_2 and by B, B_1, B_2 the intersection points $\ell_1 \cap \ell_2$, $\ell \cap \ell_1$, $\ell \cap \ell_2$, respectively. Then we have $B_1(s - f(s)/f'(s), 0)$, $B_2(t - f(t)/f'(t), 0)$ and $B(x_0, y_0)$ with

$$y_0 = \frac{(t-s)f'(t)f'(s) + f(s)f'(t) - f'(s)f(t)}{f'(t) - f'(s)}. \tag{3.3}$$

Hence the area $U(s,t)$ of the triangle $\triangle BB_1B_2$ is given by

$$\begin{aligned} 2\epsilon U(s,t) &= \{t - s - \frac{f(t)}{f'(t)} + \frac{f(s)}{f'(s)}\}(y_0) \\ &= \frac{\{(t-s)f'(t)f'(s) + f(s)f'(t) - f'(s)f(t)\}^2}{f'(s)f'(t)(f'(t) - f'(s))}. \end{aligned} \tag{3.4}$$

Second, we prove:

Lemma 6. *The function f satisfies the following*:

$$f(t)f'(t)^2 = 4a(tf'(t) - f(t))^2, \tag{3.5}$$

where a is given by $f''(0) = 2a$.

Proof. Since the curve X satisfies (1.3) with $\lambda = 1/2$, we get $2U(s,t) = T(s,t)$. By letting $s \to 0$, from (3.2) we get

$$\epsilon \lim_{s \to 0} \frac{T(s,t)}{s} = \frac{f(t)}{2}, \tag{3.6}$$

where we use $f'(0) = 0$. From (3.4) we also get

$$\begin{aligned} 2\epsilon \lim_{s \to 0} \frac{U(s,t)}{s} &= \frac{\{f''(0)tf'(t) - f''(0)f(t)\}^2}{f''(0)f'(t)^2} \\ &= 2a\frac{\{tf'(t) - f(t)\}^2}{f'(t)^2}, \end{aligned} \tag{3.7}$$

where we use $f'(0) = 0$ and $f''(0) = 2a > 0$. Together with (3.6), (3.7) completes the proof. □

Third, we prove:

Lemma 7. *The function f satisfies the following*:

$$2f(t)^2 f''(t) = f'(t)^2\{tf'(t) - f(t)\}. \tag{3.8}$$

Proof. By letting $s \to t$, we get from (3.2)

$$\begin{aligned} \epsilon \lim_{s\to t} \frac{T(s,t)}{s-t} &= \frac{1}{2} \lim_{s\to t} \frac{sf(t) - tf(s)}{s-t} \\ &= \frac{1}{2}(f(t) - tf'(t)). \end{aligned} \tag{3.9}$$

On the other hand, from (3.4) we get

$$\begin{aligned} 2\epsilon \lim_{s\to t} \frac{U(s,t)}{s-t} &= \lim_{s\to t} \frac{\{(t-s)f'(t)f'(s) + f(s)f'(t) - f'(s)f(t)\}^2}{(s-t)f'(s)f'(t)(f'(t) - f'(s))} \\ &= -\frac{f(t)^2 f''(t)}{f'(t)^2}. \end{aligned} \tag{3.10}$$

Since $T = 2U$, together with (3.9), (3.10) completes the proof. □

By eliminating $tf'(t) - f(t)$ from (3.5) and (3.8), we get

$$f''(t) = \frac{1}{4\sqrt{a}} \frac{f'(t)^3}{f(t)^{3/2}}. \tag{3.11}$$

Letting $y = f(t)$, a standard method of ordinary differential equations using the substitution $w = dy/dt$ and $y''(t) = w(dw/dy)$ leads to

$$dt = (\frac{1}{2\sqrt{ay}} + c)dy, \tag{3.12}$$

where c is a constant. Since $f(0) = 0$, we obtain from (3.12)

$$t = \frac{1}{\sqrt{a}}(\sqrt{y} + cy). \tag{3.13}$$

After replacing t by x, we have for $y = f(x)$

$$y = \begin{cases} \frac{2\sqrt{a}cx+1-\sqrt{4\sqrt{a}cx+1}}{2c^2}, & \text{if } c \neq 0, \\ ax^2, & \text{if } c = 0. \end{cases} \tag{3.14}$$

Note that

$$f(0) = f'(0) = 0, \ f''(0) = 2a \quad \text{and} \quad f'''(0) = -12\sqrt{a}ac \quad \text{or} \quad 0. \tag{3.15}$$

It follows from (3.14) that the curve X around an arbitrary point A is an open part of the parabola defined by

$$ax^2 - 2\sqrt{a}cxy + c^2y^2 - y = 0. \tag{3.16}$$

Finally using (3.15), in the same manner as in [6], we can show that the curve X is globally an open part of a parabola. This completes the proof of Theorem 2.

4. Corollaries and examples

In this section, we give some corollaries and examples.

Suppose that $X = X(s)$ is a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$ which satisfies for all s and sufficiently small $h_i, i = 1, 2,$

$$U(s, h_1, h_2) = \lambda(s)T(s, h_1, h_2)^{\mu(s)}, \tag{4.1}$$

where $\lambda(s)$ and $\mu(s)$ are some functions. Then, in particular, for all $P = X(s)$ and sufficiently small $h > 0$ the curve X satisfies

$$U_P(h) = \lambda(P)T_P(h)^{\mu(P)}. \tag{4.2}$$

Using Theorem 1, by letting $h \to 0$ we see that $\mu(P) = 1$. Hence, Theorem 1 again implies that $\lambda(P) = \frac{1}{2}$.

Thus, from Theorem 2 we get:

Corollary 8. *Let X denote a strictly convex curve in the plane $\mathbb{R}^2$. Then, the following are equivalent.*

1) *X satisfies* (4.1) *for some functions $\lambda(s)$ and $\mu(s)$.*
2) *X satisfies* (4.1) *with $\lambda = \frac{1}{2}$ and $\mu = 1$.*
3) *X is an open part of a parabola.*

Finally, we give an example of a convex curve which satisfies

$$U_P(h) = \frac{1}{2}T_P(h) \tag{1.5}$$

for sufficiently small $h > 0$ at every point $P \in X$, but it is not a parabola. Note that the example is not of class $C^{(2)}$, and hence it is not strictly convex either.

Example 9. Consider the graph X of a function $f : \mathbb{R} \to \mathbb{R}$ which is given by for some positive distinct constants a and b

$$f(x) = \begin{cases} ax^2, & \text{if } x < 0, \\ bx^2, & \text{if } x \geq 0. \end{cases} \tag{4.3}$$

It is straightforward to show that if P is the origin, then for all h we have

$$U_P(h) = \frac{1}{2}T_P(h). \tag{4.4}$$

Hence X satisfies $U_P(h) = T_P(h)/2$ at the origin for all $h > 0$. If $P \in X$ is not the origin, then there exists a positive number $\varepsilon(P)$ such that for every positive number h with $h < \varepsilon(P)$, X satisfies $U_P(h) = T_P(h)/2$.

Thus, X satisfies $U_P(h) = T_P(h)/2$ for sufficiently small $h > 0$ at every point $P \in X$. But it is not a parabola.

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