Bull. Korean Math. Soc. **51** (2014), No. 3, pp. 883–899
http://dx.doi.org/10.4134/BKMS.2014.51.3.883

GCR-LIGHTLIKE SUBMANIFOLDS OF A SEMI-RIEMANNIAN PRODUCT MANIFOLD

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Abstract. We introduce GCR-lightlike submanifold of a semi-Riemannian product manifold and give an example. We study geodesic GCR-lightlike submanifolds of a semi-Riemannian product manifold and obtain some necessary and sufficient conditions for a GCR-lightlike submanifold to be a GCR-lightlike product. Finally, we discuss minimal GCR-lightlike submanifolds of a semi-Riemannian product manifold.

1. Introduction

The significant applications of CR-structures in relativity [3, 4] and growing importance of lightlike submanifolds in mathematical physics and moreover availability of limited information on theory of lightlike submanifolds, motivated Duggal and Bejancu [5] to introduce CR-lightlike submanifolds of indefinite Kaehler manifolds. Similar to CR-lightlike submanifolds, semi-invariant lightlike submanifolds of a semi-Riemannian product manifold were introduced by Atçeken and Kiliç in [1]. Since CR-lightlike submanifold does not include the complex and totally real cases therefore Duggal and Sahin [7] introduced Screen Cauchy-Riemann (SCR)-lightlike submanifold of indefinite Kaehler manifolds, which contains complex and screen real sub-cases. The SCR-lightlike submanifolds, analogously, Screen Semi-Invariant lightlike submanifolds, of semi-Riemannian product manifolds were introduced by Khursheed et al. [9] and Kiliç et al. [10], respectively. Since there is no inclusion relation between SCR and CR cases therefore Duggal and Sahin [8] introduced Generalized Cauchy-Riemann (GCR)-lightlike submanifold of indefinite Kaehler manifolds which acts as an umbrella of real hypersurfaces, invariant, screen real and CR lightlike submanifolds and further developed by [11, 12, 13, 14].

Received May 23, 2013; Revised November 20, 2013.
2010 *Mathematics Subject Classification.* 53C15, 53C40, 53C50.
Key words and phrases. CR-lightlike submanifolds, SCR-lightlike submanifolds, GCR-lightlike submanifolds, semi-Riemannian product manifold, minimal lightlike submanifolds.

Since the geometry of lightlike submanifolds of semi-Riemannian product manifolds is a topic of chief discussion [16, 17, 18] therefore we introduce GCR-lightlike submanifolds of a semi-Riemannian product manifold. We study geodesic GCR-lightlike submanifolds of a semi-Riemannian product manifold and obtain some necessary and sufficient conditions for a GCR-lightlike submanifold to be a GCR-lightlike product. Finally, we discuss minimal GCR-lightlike submanifolds of a semi-Riemannian product manifold.

2. Lightlike submanifolds

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g be the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\bar{M}$, for detail see [5]. For a degenerate metric g on M, $TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $RadT_xM = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping $RadTM : x \in M \longrightarrow RadT_xM$, defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $RadTM$ is called the radical distribution on M. Screen distribution $S(TM)$ is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM therefore

$$TM = RadTM \perp S(TM) \tag{1}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $RadTM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $RadTM$ in $S(TM^{\perp})^{\perp}$, respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{2}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{3}$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\bar{M}$ along M, on u as $\{\xi_1, \ldots, \xi_r, W_{r+1}, \ldots, W_n, N_1, \ldots, N_r, X_{r+1}, \ldots, X_m\}$, where $\{\xi_1, \ldots, \xi_r\}, \{N_1, \ldots, N_r\}$ are local lightlike bases of $\Gamma(RadTM\mid_u)$, $\Gamma(ltr(TM)\mid_u)$ and $\{W_{r+1}, \ldots, W_n\}, \{X_{r+1}, \ldots, X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp})\mid_u)$ and $\Gamma(S(TM)\mid_u)$ respectively. For this quasi-orthonormal fields of frames, we have:

Theorem 2.1 ([5]). *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $RadTM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)\mid_{\mathrm{u}})$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_{\mathrm{u}}$, where u is a coordinate neighborhood of M such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0 \text{ for any } i, j \in \{1, 2, \ldots, r\}, \tag{4}$$

where $\{\xi_1, \ldots, \xi_r\}$ *is a lightlike basis of* $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ then according to the decomposition (3), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{5}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, A_U is a linear a operator on M and known as shape operator.

According to (2) considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ respectively, then (5) become

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \quad \bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{6}$$

where we put $h^l(X,Y) = L(h(X,Y)), h^s(X,Y) = S(h(X,Y)), D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$.

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued, respectively, therefore they are called the lightlike second fundamental form and the screen second fundamental form on M. In particular

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \ \bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{7}$$

where $X \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Using (6) and (7) we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{8}$$

$$\bar{g}(h^l(X,Y),\xi) + \bar{g}(Y, h^l(X,\xi)) + \bar{g}(Y, \nabla_X \xi) = 0 \tag{9}$$

for any $W \in \Gamma(S(TM^{\perp})), \xi \in \Gamma(Rad(TM))$. Let P be the projection morphism of TM on $S(TM)$ then using (1), we can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$\nabla_X PY = \nabla^*_X PY + h^*(X,PY), \quad \nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{10}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where $\{\nabla^*_X PY, A^*_\xi X\}$ and $\{h^*(X,PY), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(RadTM)$, respectively. ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $RadTM$, respectively. h^* and A^* are $\Gamma(RadTM)$-valued and $\Gamma(S(TM))$-valued bilinear forms and are called as second fundamental forms of distributions $S(TM)$ and $RadTM$, respectively. Using (6) and (10), we obtain

$$\bar{g}(h^l(X,PY),\xi) = g(A^*_\xi X, PY), \quad \bar{g}(h^*(X,PY),N) = \bar{g}(A_N X, PY) \tag{11}$$

for any $X, Y \in \Gamma(TM), \xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$.

3. Semi-Riemannian product manifolds

Let (M_1, g_1) and (M_2, g_2) be two m_1 and m_2 dimensional semi-Riemannian manifolds with constant indexes $q_1 > 0$ and $q_2 > 0$, respectively. Let $\pi : M_1 \times M_2 \to M_1$ and $\sigma : M_1 \times M_2 \to M_2$ be the projections which are given by $\pi(x, y) = x$ and $\sigma(x, y) = y$ for any $(x, y) \in M_1 \times M_2$. We denote the product manifold by $(\bar{M}, \bar{g}) = (M_1 \times M_2, \bar{g})$, where

$$\bar{g}(X, Y) = g_1(\pi_* X, \pi_* Y) + g_2(\sigma_* X, \sigma_* Y),$$

for any $X, Y \in \Gamma(T\bar{M})$, where $*$ means the differential mapping. Then we have

$$\pi_*^2 = \pi_*, \quad \sigma_*^2 = \sigma_*, \quad \pi_* \sigma_* = \sigma_* \pi_* = 0, \quad \pi_* + \sigma_* = I,$$

where I is the identity map of $T(M_1 \times M_2)$. Thus $(\bar{M}, \bar{g})$ is a $(m_1 + m_2)$-dimensional semi-Riemannian manifold with constant index $(q_1 + q_2)$. The semi-Riemannian product manifold $\bar{M} = M_1 \times M_2$ is characterized by M_1 and M_2 which are totally geodesic submanifolds of $\bar{M}$. Now if we put $F = \pi_* - \sigma_*$ then we see that $F^2 = I$ and

$$\bar{g}(FX, Y) = \bar{g}(X, FY), \tag{12}$$

for any $X, Y \in \Gamma(T\bar{M})$, where F is called an almost product structure on $M_1 \times M_2$. If we denote the Levi-Civita connection on $\bar{M}$ by $\bar{\nabla}$, then it can be seen that

$$(\bar{\nabla}_X F)Y = 0, \tag{13}$$

for any $X, Y \in \Gamma(T\bar{M})$, that is, F is parallel with respect to $\bar{\nabla}$.

4. Generalized Cauchy-Riemann lightlike submanifolds

Definition 4.1. Let $(M, g, S(TM))$ be a real lightlike submanifold of a semi-Riemannian product manifold $(\bar{M}, \bar{g})$. Then M is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied

(A) There exist two subbundles D_1 and D_2 of $Rad(TM)$, such that

$$Rad(TM) = D_1 \oplus D_2, \quad FD_1 = D_1, \quad FD_2 \subset S(TM).$$

(B) There exist two subbundles D_0 and D' of $S(TM)$, such that

$$S(TM) = \{FD_2 \oplus D'\} \perp D_0, \quad FD_0 = D_0, \quad FD' = L_1 \perp L_2,$$

where D_0 is a non degenerate distribution on M, L_1 and L_2 are vector subbundles of $ltr(TM)$ and $S(TM)^\perp$, respectively.

Then the tangent bundle TM of M is decomposed as $TM = D \perp D'$ and $D = Rad(TM) \oplus D_0 \oplus FD_2$. M is called a proper GCR-lightlike submanifold if $D_1 \neq \{0\}, D_2 \neq \{0\}, D_0 \neq \{0\}$ and $L_2 \neq \{0\}$, which has the following features:

1. The condition (A) implies that $\dim(\mathrm{Rad}(TM)) \geq 3$.
2. The condition (B) implies that $\dim\,(D) = 2s \geq 6$, $\dim\,(D') \geq 2$ and $\dim\,(D_2) = \dim\,(L_1)$. Thus $\dim\,(M) \geq 8$ and $\dim\,(\bar{M}) \geq 12$.
3. Any proper 8-dimensional GCR-lightlike submanifold is 3-lightlike.

Example. Let $R_4^{12} = R_2^6 \times R_2^6$ be a semi-Riemannian product manifold with the product structure $F(\partial x_i, \partial y_i) = (\partial y_i, \partial x_i)$, where (x^i, y^i) are cartesian coordinates of R_4^{12}. Let M be a submanifold of R_4^{12} given by:

$$x_1 = u_1, \quad x_2 = u_5, \quad x_3 = u_3, \quad x_4 = \sqrt{1-u_4^2}, \quad x_5 = u_6, \quad x_6 = u_2,$$

$$y_1 = u_2, \quad y_2 = u_3, \quad y_3 = u_8, \quad y_4 = u_4, \quad y_5 = u_7, \quad y_6 = u_1.$$

Then TM is spanned by $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8$, where

$$Z_1 = \partial x_1 + \partial y_6, \quad Z_2 = \partial y_1 + \partial x_6, \quad Z_3 = \partial x_3 + \partial y_2,$$

$$Z_4 = -y_4 \partial x_4 + x_4 \partial y_4, \quad Z_5 = \partial x_2, \quad Z_6 = \partial x_5, \quad Z_7 = \partial y_5, \quad Z_8 = \partial y_3.$$

Clearly, M is a 3-lightlike submanifold with $Rad(TM) = Span\{Z_1, Z_2, Z_3\}$ and $FZ_1 = Z_2$, therefore $D_1 = Span\{Z_1, Z_2\}$. Since $FZ_3 = \partial y_3 + \partial x_2 = Z_8 + Z_5 \in \Gamma(S(TM))$, therefore $D_2 = Span\{Z_3\}$. Moreover $FZ_6 = Z_7$ therefore $D_0 = Span\{Z_6, Z_7\}$. The lightlike transversal bundle $ltr(TM)$ is spanned by

$$\{N_1 = \frac{1}{2}(-\partial x_1 + \partial y_6), N_2 = \frac{1}{2}(-\partial y_1 + \partial x_6), N_3 = \frac{1}{2}(-\partial x_3 + \partial y_2)\}.$$

Clearly, $Span\{N_1, N_2\}$ is invariant with respect to F and $FN_3 = -\frac{1}{2}Z_8 + \frac{1}{2}Z_5$. Hence $L_1 = Span\{N_3\}$. By direct calculations, we obtain $S(TM^\perp) = Span\{W = -y_4 \partial y_4 + x_4 \partial x_4\}$. Since $FZ_4 = W$, thus $L_2 = S(TM^\perp)$. Hence $D' = Span\{FN_3, FW = Z_4\}$. Thus, M is a proper GCR-lightlike submanifold of semi-Riemannian product manifold R_4^{12}.

Let Q, P_1 and P_2 be the projections on D, $FL_1 = M_1$ and $FL_2 = M_2$, respectively. Then for any $X \in \Gamma(TM)$, we have $X = QX + P_1X + P_2X$, applying F to both sides, we obtain

$$FX = fX + wP_1X + wP_2X, \tag{14}$$

and we can write the equation (14) as

$$FX = fX + wX, \tag{15}$$

where fX and wX are the tangential and transversal components of FX, respectively. Similarly

$$FV = BV + CV, \tag{16}$$

for any $V \in \Gamma(tr(TM))$, where BV and CV are the sections of TM and $tr(TM)$, respectively. Since F is parallel on M, using (6), (7), (14) and (16), we obtain

$$(\nabla_X f)Y = A_{wP_1Y}X + A_{wP_2Y}X + Bh(X,Y). \tag{17}$$

$$D^s(X, wP_1Y) = -\nabla_X^s wP_2Y + wP_2\nabla_X Y - h^s(X, fY) + Ch^s(X,Y). \tag{18}$$

$$D^l(X, wP_2Y) = -\nabla_X^l wP_1Y + wP_1\nabla_X Y - h^l(X, fY) + Ch^l(X,Y). \tag{19}$$

Theorem 4.2. *Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then the induced connection is a metric connection if and only if the following conditions hold*

$$\nabla_X^{*t} FY - A^*_{FY} X \in \Gamma(FD_2 \oplus D_1), \quad when \quad Y \in \Gamma(D_1),$$
$$\nabla_X^{*} FY + h^*(X, FY) \in \Gamma(FD_2 \oplus D_1), \quad when \quad Y \in \Gamma(D_2),$$
$$and \quad Bh(X, FY) = 0, \quad when \quad Y \in \Gamma(Rad(TM)).$$

Proof. Since F is an almost product structure of $\bar{M}$ therefore we have $\bar{\nabla}_X Y = \bar{\nabla}_X F^2 Y$ for any $Y \in \Gamma(Rad(TM))$ and $X \in \Gamma(TM)$. Then from (13), we obtain $\bar{\nabla}_X Y = F \bar{\nabla}_X FY$ and then using (6) and (16), we obtain

$$\nabla_X Y + h(X, Y) = F(\nabla_X FY + h(X, FY)). \tag{20}$$

Since $Rad(TM) = D_1 \oplus D_2$ therefore using (10), (15) and (16) in (20) and then equating the tangential part for any $Y \in \Gamma(D_1)$, we obtain

$$\nabla_X Y = f(-A^*_{FY} X + \nabla_X^{*t} FY) + Bh(X, FY), \tag{21}$$

and for any $Y \in \Gamma(D_2)$, we obtain

$$\nabla_X Y = f(\nabla_X^* FY + h^*(X.FY)) + Bh(X, FY). \tag{22}$$

Thus from (21), $\nabla_X Y \in \Gamma(Rad(TM))$, if and only if

$$f(-A^*_{FY} X + \nabla_X^{*t} FY) \in \Gamma(FD_2 \oplus D_1) \quad \text{and} \quad Bh(X, FY) = 0. \tag{23}$$

From (22), $\nabla_X Y \in \Gamma(Rad(TM))$, if and only if

$$\nabla_X^* FY + h^*(X, FY) \in \Gamma(FD_2 \oplus D_1) \quad \text{and} \quad Bh(X, FY) = 0. \tag{24}$$

Thus the assertion follows from (23) and (24). □

Theorem 4.3. *Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then*

(i) *The distribution D is integrable, if and only if,*

$$h(X, FY) = h(FX, Y), \quad \forall \quad X, Y \in \Gamma(D).$$

(ii) *The distribution D' is integrable, if and only if,*

$$A_{FZ} V = A_{FV} Z, \quad \forall \quad Z, V \in \Gamma(D').$$

Proof. From (18) and (19), we obtain $w\nabla_X Y = h(X, fY) - Ch(X, Y)$ for any $X, Y \in \Gamma(D)$, which implies that $w[X, Y] = h(X, fY) - h(fX, Y)$, which proves (i).

Next, from (17), we have $f\nabla_Z V = -A_{wV} Z - Bh(Z, V)$ for any $Z, V \in \Gamma(D')$, therefore $f[Z, V] = A_{wZ} V - A_{wV} Z$, which completes the proof. □

Theorem 4.4. *Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then D-defines a totally geodesic foliation in M if and only if $Bh(X, Y) = 0$ for any $X, Y \in \Gamma(D)$.*

Proof. Using the definition of GCR-lightlike submanifolds, D-defines a totally geodesic foliation in M if and only if, $\nabla_X Y \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$, that is, if and only if

$$g(\nabla_X Y, F\xi) = g(\nabla_X Y, FW) = 0,$$

for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(D_2)$ and $W \in \Gamma(L_2)$. From (6) and (13), we obtain

$$\begin{aligned} g(\nabla_X Y, F\xi) &= \bar{g}(\bar{\nabla}_X FY, \xi) \\ &= \bar{g}(h^l(X, FY), \xi),\ \forall X, Y \in \Gamma(D),\ \xi \in \Gamma(D_2). \end{aligned} \tag{25}$$

Similarly, using (6) and (13), we obtain

$$\begin{aligned} g(\nabla_X Y, FW) &= \bar{g}(\bar{\nabla}_X FY, W) \\ &= \bar{g}(h^s(X, FY), W), \forall X, Y \in \Gamma(D),\ W \in \Gamma(L_2). \end{aligned} \tag{26}$$

It follows from (25) and (26) that D defines a totally geodesic foliation in M, if and only if, $h^s(X, FY)$ has no components in L_2 and $h^l(X, FY)$ has no components in L_1 for any $X, Y \in \Gamma(D)$, that is, using (16), $Bh(X, Y) = 0$ for any $X, Y \in \Gamma(D)$. □

Theorem 4.5. *Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then D'-defines a totally geodesic foliation in M, if and only if, $A_{wY} X \in \Gamma(D')$ for any $X, Y \in \Gamma(D')$.*

Proof. From (17), we obtain that $f\nabla_X Y = -A_{wY} X - Bh(X, Y)$ for any $X, Y \in \Gamma(D')$. If D' defines a totally geodesic foliation in M, then $A_{wY} X = -Bh(X, Y)$, which implies that $A_{wY} X \in \Gamma(D')$ for any $X, Y \in \Gamma(D')$. Conversely, let $A_{wY} X \in \Gamma(D')$ for any $X, Y \in \Gamma(D')$, therefore $f\nabla_X Y = 0$, which implies that $\nabla_X Y \in \Gamma(D')$. Hence the result follows. □

Definition 4.6. A GCR-lightlike submanifold of a semi-Riemannian product manifold is called D geodesic (respectively, D' geodesic) GCR-lightlike submanifold if its second fundamental form h satisfies $h(X, Y) = 0$ for any $X, Y \in \Gamma(D)$ (respectively, $X, Y \in \Gamma(D')$).

Theorem 4.7. *Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then the distribution D defines a totally geodesic foliation in $\bar{M}$ if and only if M is D-geodesic.*

Proof. Let D defines a totally geodesic foliation in $\bar{M}$ then $\bar{\nabla}_X Y \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$. Then using (6) for any $\xi \in \Gamma(D_2)$ and $W \in \Gamma(L_2)$, we obtain

$$\bar{g}(h^l(X, Y), \xi) = \bar{g}(\bar{\nabla}_X Y, \xi) = 0, \quad \bar{g}(h^s(X, Y), W) = \bar{g}(\bar{\nabla}_X Y, W) = 0.$$

Hence $h^l(X, Y) = h^s(X, Y) = 0$, which implies that M is D-geodesic.

Conversely, let us assume that M is D-geodesic. Now using (6) and (13), for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(D_2)$ and $W \in \Gamma(L_2)$, we have

$$\bar{g}(\bar{\nabla}_X Y, F\xi) = \bar{g}(\bar{\nabla}_X FY, \xi) = \bar{g}(h^l(X, FY), \xi) = 0,$$

and

$$\bar{g}(\bar{\nabla}_X Y, FW) = \bar{g}(\bar{\nabla}_X FY, W) = \bar{g}(h^s(X, FY), W) = 0.$$

Hence $\bar{\nabla}_X Y \in \Gamma(D)$, which completes the proof. □

Theorem 4.8. *Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then M is D-geodesic, if and only if,*

$$g(A_W X, Y) = \bar{g}(D^l(X, W), Y),$$

and

$$\nabla^*_X F\xi \notin \Gamma(D_0 \perp FL_1), \quad A^*_{\xi'} X \notin \Gamma(FL_1), \quad h^l(X, \xi') \notin \Gamma(L_1),$$

for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(D_2)$, $\xi' \in \Gamma(Rad(TM))$ and $W \in \Gamma(L_2)$.

Proof. Using the definition of GCR-lightlike submanifolds, M is D-geodesic, if and only if,

$$\bar{g}(h^l(X, Y), \xi) = 0,$$

$$\bar{g}(h^s(X, Y), W) = 0$$

for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(D_2)$ and $W \in \Gamma(L_2)$. Thus for any $X, Y \in \Gamma(D)$, first part of assertion follows from (8).

Now, for $X, Y \in \Gamma(D)$ and $\xi \in \Gamma(D_2)$, using (6), (10) and (12), we have

$$\begin{aligned} \bar{g}(h^l(X, Y), \xi) &= \bar{g}(\bar{\nabla}_X Y, \xi) \\ &= -\bar{g}(FY, \bar{\nabla}_X F\xi) \\ &= -g(FY, \nabla_X F\xi) - \bar{g}(FY, h^l(X, F\xi)) \\ &= -g(FY, \nabla^*_X F\xi) - \bar{g}(FY, h^l(X, F\xi)). \end{aligned} \tag{27}$$

Since $Y \in \Gamma(D)$, this implies that $Y \in \Gamma(D_0)$, $Y \in \Gamma(D_1)$, $Y \in \Gamma(D_2)$, or $Y \in \Gamma(FD_2)$. If $Y \in \Gamma(D_0)$ or $Y \in \Gamma(D_2)$, then we have

$$\bar{g}(FY, h^l(X, F\xi)) = 0, \tag{28}$$

and if $Y \in \Gamma(D_1)$ or $Y \in \Gamma(FD_2)$, then we have

$$\bar{g}(FY, h^l(X, F\xi)) = g(A^*_{\xi'} X, F\xi) + \bar{g}(h^l(X, \xi'), F\xi) \tag{29}$$

for any $\xi' = FY \in \Gamma(Rad(TM))$. Now using (28) and (29) in (27), we obtain

$$\bar{g}(h^l(X, Y), \xi) = -g(FY, \nabla^*_X F\xi) - g(A^*_{\xi'} X, F\xi) - \bar{g}(h^l(X, \xi'), F\xi),$$

which proves the second part of the assertion. □

Theorem 4.9. *Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then M is D'-geodesic, if and only if, $A_W X$ and $A^*_\xi X$ have no components in $M_2 \perp FD_2$, for any $X \in \Gamma(D')$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^\perp)$.*

Proof. For any $X, Y \in \Gamma(D')$ and $W \in \Gamma(S(TM^{\perp})$ using (8), we obtain

$$\bar{g}(h^s(X,Y),W) = g(A_W X, Y), \tag{30}$$

and for any $\xi \in \Gamma(Rad(TM))$ using (9) and (10), we obtain

$$\bar{g}(h^l(X,Y),\xi) = g(A^*_\xi X, Y). \tag{31}$$

Hence the assertion follows from (30) and (31). □

Definition 4.10. A GCR-lightlike submanifold of a semi-Riemannian product manifold is called mixed-geodesic GCR-lightlike submanifold if its second fundamental form h satisfies $h(X,Y) = 0$ for any $X \in \Gamma(D)$ and $Y \in \Gamma(D')$.

Theorem 4.11. *Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then M is mixed geodesic, if and only if,*

$$A^*_\xi X \in \Gamma(D_0 \perp FL_1), \quad \textit{and} \quad A_W X \in \Gamma(D_0 \perp Rad(TM) \perp FL_1)$$

for any $X \in \Gamma(D)$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^{\perp}))$.

Proof. Using (9) and (10), for any $X \in \Gamma(D)$, $Y \in \Gamma(D')$ and $\xi \in \Gamma(Rad(TM))$, we obtain

$$\bar{g}(h^l(X,Y),\xi) = g(A^*_\xi X, Y), \tag{32}$$

and for any $W \in \Gamma(S(TM^{\perp}))$ using (8), we obtain

$$\bar{g}(h^s(X,Y),W) = g(A_W X, Y). \tag{33}$$

Hence the result follows from (32) and (33). □

Theorem 4.12. *Let M be a mixed geodesic GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then $A^*_\xi X \in \Gamma(FD_2)$ for any $X \in \Gamma(D')$ and $\xi \in \Gamma(D_2)$.*

Proof. Let $X \in \Gamma(D')$ and $\xi \in \Gamma(D_2)$ then we have

$$h(X, F\xi) = \bar{\nabla}_X F\xi - \nabla_X F\xi = F\nabla_X \xi + Fh(X,\xi) - \nabla_X F\xi.$$

Since M is mixed geodesic, therefore $F\nabla_X\xi = \nabla_X F\xi$. Using (10) and (15), we get

$$-fA^*_\xi X - wA^*_\xi X + F\nabla^{*t}_X \xi = \nabla^*_X F\xi + h^*(X, F\xi).$$

Equating the transversal components, we have $wA^*_\xi X = 0$. Thus

$$A^*_\xi X \in \Gamma(FD_2 \perp D_0).$$

Now, for any $Z \in \Gamma(D_0)$ and $\xi \in \Gamma(D_2)$, we have

$$\bar{g}(A^*_\xi X, Z) = \bar{g}(\nabla_X \xi + \nabla^{*t}_X \xi, Z) = \bar{g}(\bar{\nabla}_X \xi, Z) = -g(\xi, \nabla_X Z + h(X,Z)) = 0.$$

If $A^*_\xi X \in \Gamma(D_0)$, then using the non-degeneracy of D_0 for any $Z \in \Gamma(D_0)$, we must have $\bar{g}(A^*_\xi X, Z) \neq 0$. Therefore $A^*_\xi X \notin \Gamma(D_0)$. Hence the assertion is proved. □

Theorem 4.13. *Let M be a mixed geodesic GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then the transversal section $V \in \Gamma(FD')$ is D-parallel, if and only if, $\nabla_X FV \in \Gamma(D)$ for any $X \in \Gamma(D)$.*

Proof. Let $Y \in \Gamma(D')$ such that $FY = wY = V \in \Gamma(L_1 \perp L_2)$ and $X \in \Gamma(D)$, then using hypothesis that M is a mixed geodesic in (17), we have $f\nabla_X Y = -A_{wY}X = -A_V X$. Now, $\nabla^t_X V = \bar{\nabla}_X V + A_V X = \bar{\nabla}_X FY - f\nabla_X Y$. Since $\bar{\nabla}$ is an almost product structure and M is mixed geodesic therefore we have $\nabla^t_X V = w\nabla_X Y$, that is, $\nabla^t_X V = w\nabla_X FV$, which proves the theorem. □

Theorem 4.14. *Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$ such that $D^s(X,V) \in \Gamma(L_2^\perp)$. Then $A_{FV}X = FA_V X$ for any $X \in \Gamma(D)$ and $V \in \Gamma(L_1^\perp)$.*

Proof. Let $X \in \Gamma(D)$, $Y \in \Gamma(D')$ and $V \in \Gamma(L_1^\perp)$ then we have

$$\begin{aligned} g(A_{FV}X - FA_V X, Y) &= g(A_{FV}X, Y) - g(A_V X, FY) \\ &= -g(\bar{\nabla}_X FV, Y) + g(\bar{\nabla}_X V, FY) \\ &= -g(\bar{\nabla}_X V, FY) + g(\bar{\nabla}_X V, FY) \\ &= 0. \end{aligned} \tag{34}$$

For any $X \in \Gamma(D)$, $Z \in \Gamma(D_0)$ and $V \in \Gamma(L_1^\perp)$, we have

$$\begin{aligned} g(A_{FV}X - FA_V X, Z) &= g(A_{FV}X, Z) - g(A_V X, FZ) \\ &= -g(\bar{\nabla}_X FV, Z) + g(\bar{\nabla}_X V, FZ) \\ &= -g(\bar{\nabla}_X V, FZ) + g(\bar{\nabla}_X V, FZ) \\ &= 0. \end{aligned} \tag{35}$$

For any $X \in \Gamma(D)$, $N \in \Gamma(ltr(TM))$ and $V \in \Gamma(L_1^\perp)$, we have

$$\begin{aligned} g(A_{FV}X - FA_V X, N) &= g(A_{FV}X, N) - g(A_V X, FN) \\ &= -g(\bar{\nabla}_X FV, N) + g(\bar{\nabla}_X V, FN) \\ &= -g(F\bar{\nabla}_X V, N) + g(F\bar{\nabla}_X V, N) \\ &= 0. \end{aligned} \tag{36}$$

For any $X \in \Gamma(D)$, $FN \in \Gamma(FL_1)$ and $V \in \Gamma(L_1^\perp)$, we also have

$$\begin{aligned} g(A_{FV}X - FA_V X, FN) &= g(A_{FV}X, FN) - g(A_V X, N) \\ &= -g(\bar{\nabla}_X FV, FN) + g(\bar{\nabla}_X V, N) \\ &= -g(F\bar{\nabla}_X V, FN) + g(\bar{\nabla}_X V, N) \\ &= -g(\bar{\nabla}_X V, N) + g(\bar{\nabla}_X V, N) \\ &= 0. \end{aligned} \tag{37}$$

Hence the assertion follows from (34)-(37). □

5. GCR-Lightlike product

Definition 5.1 ([15]). A GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$ is called a GCR-lightlike product if both the distributions D and D' define totally geodesic foliations in M.

Theorem 5.2. *Let M be a totally geodesic GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Suppose that there exists a transversal vector bundle of M, which is parallel along D' with respect to the Levi-Civita connection on M, that is, $\bar{\nabla}_X V \in \Gamma(tr(TM))$ for any $V \in \Gamma(tr(TM))$ and $X \in \Gamma(D')$. Then M is a GCR-lightlike product.*

Proof. Since M is a totally geodesic GCR-lightlike submanifold, therefore $Bh(X,Y) = 0$ for any $X, Y \in \Gamma(D)$. Therefore, the distribution D defines a totally geodesic foliation in M. Next, since $\bar{\nabla}_X V \in \Gamma(tr(TM))$ for any $V \in \Gamma(tr(TM))$ and $X \in \Gamma(D')$, therefore using (7), we obtain $A_V X = 0$, then from (17), we get $f\nabla_X Y = 0$ for any $X, Y \in \Gamma(D')$, which implies that $\nabla_X Y \in \Gamma(D')$. Hence the distribution D' defines a totally geodesic foliation in M. Thus M is a GCR-lightlike product. □

Definition 5.3. A lightlike submanifold of a semi-Riemannian manifold is said to be an irrotational submanifold if $\bar{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. Thus M is an irrotational lightlike submanifold, if and only if, $h^l(X,\xi) = 0$, $h^s(X,\xi) = 0$.

Theorem 5.4. *Let M be an irrotational GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then M is a GCR-lightlike product if the following conditions are satisfied*:

(A) $\bar{\nabla}_X U \in \Gamma(S(TM^{\perp}))$ *for any* $X \in \Gamma(TM)$ *and* $U \in \Gamma(tr(TM))$.
(B) $A^*_\xi Y \in \Gamma(FL_2)$ *for any* $Y \in \Gamma(D)$.

Proof. Using (7) with (A), we get $A_W X = 0$, $D^l(X,W) = 0$ and $\nabla^l_X W = 0$ for any $X \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$. Therefore for any $X, Y \in \Gamma(D)$ and $W \in \Gamma(S(TM^{\perp}))$ and using (8), we obtain $\bar{g}(h^s(X,Y), W) = 0$, then non-degeneracy of $S(TM^{\perp})$ implies that $h^s(X,Y) = 0$. Hence, $Bh^s(X,Y) = 0$. Now, let $X, Y \in \Gamma(D)$ and $\xi \in \Gamma(Rad(TM))$, then using (B), we have $\bar{g}(h^l(X,Y),\xi) = -g(\nabla_X \xi, Y) = g(A^*_\xi X, Y) = 0$. Then using (4), we get $h^l(X,Y) = 0$. Hence $Bh^l(X,Y) = 0$. Thus the distribution D defines a totally geodesic foliation in M.

Next, let $X, Y \in \Gamma(D')$, then $FY = wY \in \Gamma(L_1 \perp L_2) \subset tr(TM)$. Using (17), we obtain $f\nabla_X Y = -Bh(X,Y)$, comparing the components along D, we get $f\nabla_X Y = 0$, which implies that $\nabla_X Y \in \Gamma(D')$. Thus the distribution D' defines a totally geodesic foliation in M. Hence M is a GCR-lightlike product. □

Theorem 5.5. *Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then M is a GCR-lightlike product if and only if $(\nabla_X f)Y = 0$ for any $X, Y \in \Gamma(D)$ or $X, Y \in \Gamma(D')$.*

Proof. Let $(\nabla_X f)Y = 0$ for any $X, Y \in \Gamma(D)$ or $X, Y \in \Gamma(D')$. Let $X, Y \in \Gamma(D)$, then $wY = 0$ and (17) gives that $Bh(X, Y) = 0$. Hence using the Theorem (4.4), the distribution D defines a totally geodesic foliation in M. Next, let $X, Y \in \Gamma(D')$. Since $BV \in \Gamma(D')$ for any $V \in \Gamma(tr(TM))$, then (17) implies that $A_{wY}X \in \Gamma(D')$. Hence using Theorem 4.5, the distribution D' defines a totally geodesic foliation in M. Since both the distributions D and D' define totally geodesic foliations in M, hence M is a GCR-lightlike product.

Conversely, let M be a GCR-lightlike product, therefore the distributions D and D' define totally geodesic foliations in M. Using (13), for any $X, Y \in \Gamma(D)$, we have $\bar{\nabla}_X FY = F\bar{\nabla}_X Y$, then comparing the transversal components, we obtain $h(X, FY) = Fh(X, Y)$ and then $(\nabla_X f)Y = \nabla_X fY - f\nabla_X Y = \bar{\nabla}_X FY - h(X, FY) - F\bar{\nabla}_X Y + h(X, FY) = 0$, that is $(\nabla_X f)Y = 0$ for any $X, Y \in \Gamma(D)$. Let D' defines a totally geodesic foliation in M and using (13), we have $\bar{\nabla}_X FY = F\bar{\nabla}_X Y$, then comparing the tangential components on both sides, we obtain $-A_{wY}X = Bh(X, Y)$, then (17) implies that $(\nabla_X f)Y = 0$, which completes the proof. □

Definition 5.6 ([6])**.** A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that, for any $X, Y \in \Gamma(TM)$,

$$h(X, Y) = H\bar{g}(X, Y). \tag{38}$$

Using (7), it is clear that M is a totally umbilical, if and only if, on each coordinate neighborhood u there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^\perp))$ such that

$$h^l(X, Y) = H^l g(X, Y), \quad h^s(X, Y) = H^s g(X, Y), \quad D^l(X, W) = 0 \tag{39}$$

for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^\perp))$. M is called totally geodesic if $H = 0$, that is, if $h(X, Y) = 0$.

Lemma 5.7. *Let M be a totally umbilical GCR-lightlike submanifold of semi-Riemannian product manifold $\bar{M}$. Then the distribution D' defines a totally geodesic foliation in M.*

Proof. Let $X, Y \in \Gamma(D')$ then (17) implies that $f\nabla_X Y = -A_{wY}X - Bh(X, Y)$, then for any $Z \in \Gamma(D_0)$, we have

$$\begin{aligned} g(f\nabla_X Y, Z) &= -g(A_{wY}X, Z) - g(Bh(X, Y), Z) \\ &= \bar{g}(\bar{\nabla}_X wY, Z) = \bar{g}(\bar{\nabla}_X FY, Z) = \bar{g}(\bar{\nabla}_X Y, FZ) = \bar{g}(\bar{\nabla}_X Y, Z') \\ &= -g(Y, \nabla_X Z'), \end{aligned} \tag{40}$$

where $Z' = FZ \in \Gamma(D_0)$. Since $X \in \Gamma(D')$ and $Z \in \Gamma(D_0)$, then from (18) and (19), we have $wP\nabla_X Z = h(X, fZ) - Ch(X, Z) = Hg(X, fZ) - CHg(X, Z) = 0$, therefore $wP\nabla_X Z = 0$, which implies that $\nabla_X Z \in \Gamma(D)$. Thus (40) implies that $g(f\nabla_X Y, Z) = 0$, then the non-degeneracy of D_0 implies that $f\nabla_X Y = 0$. Hence $\nabla_X Y \in \Gamma(D')$ for any $X, Y \in \Gamma(D')$. Thus the result follows. $\square$

Theorem 5.8. *Let M be a totally umbilical GCR-lightlike submanifold of semi-Riemannian product manifold $\bar{M}$. Then M is a GCR-lightlike product if and only if $Bh(X,Y) = 0$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.*

Proof. Let M be a GCR-lightlike product therefore the distributions D and D' define totally geodesic foliations in M. Therefore using Theorem 4.4, we have $Bh(X,Y) = 0$ for any $X, Y \in \Gamma(D)$. Now using the hypothesis for $X \in \Gamma(D')$ and $Y \in \Gamma(D)$, we have $Bh(X,Y) = g(X,Y)BH = 0$. Thus $Bh(X,Y) = 0$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.

Conversely, let $Bh(X,Y) = 0$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$. Now for any $X, Y \in \Gamma(D)$, we have $Bh(X,Y) = 0$, which implies that D defines a totally geodesic foliation in M. Let $X, Y \in \Gamma(D')$, then (17) implies that $A_{wY}X = -f\nabla_X Y - Bh(X,Y)$ and using Lemma 5.7, we obtain $fA_{wY}X + wA_{wY}X = -h(X,Y)$, comparing the tangential components on both sides, we have $fA_{wY}X = 0$, which implies that $A_{wY}X \in \Gamma(D')$. Hence using Theorem 4.5, the distribution D' defines a totally geodesic foliation in M. Hence the result follows. $\square$

Theorem 5.9. *Let M be a GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then M is totally geodesic manifold, if and only if, $Rad(TM)$ and $S(TM^\perp)$ are Killing distributions on $\bar{M}$.*

Proof. For any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, consider

$$\begin{aligned}
\bar{g}(h(X,Y),\xi) &= \bar{g}(\bar{\nabla}_X Y, \xi) = X\bar{g}(Y,\xi) - \bar{g}(\bar{\nabla}_X \xi, Y)\\
&= \bar{g}([\xi, X], Y) - \bar{g}(\bar{\nabla}_\xi X, Y)\\
&= \bar{g}([\xi, X], Y) - \xi\bar{g}(X,Y) + \bar{g}(\bar{\nabla}_\xi Y, X)\\
&= -\xi\bar{g}(X,Y) + \bar{g}([\xi,X],Y) + \bar{g}([\xi,Y],X) - \bar{g}(\bar{\nabla}_Y X, \xi)\\
&= -(L_\xi \bar{g})(X,Y) - \bar{g}(h(X,Y),\xi), \qquad (41)
\end{aligned}$$

which implies that

$$2\bar{g}(h(X,Y),\xi) = -(L_\xi \bar{g})(X,Y) \qquad (42)$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$.

Similarly, for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^\perp))$, we have

$$\begin{aligned}
\bar{g}(h(X,Y),W) &= \bar{g}(\bar{\nabla}_X Y, W) = X\bar{g}(Y,W) - \bar{g}(\bar{\nabla}_X W, Y)\\
&= \bar{g}([W, X], Y) - \bar{g}(\bar{\nabla}_W X, Y)\\
&= \bar{g}([W, X], Y) - W\bar{g}(X,Y) + \bar{g}(\bar{\nabla}_W Y, X)\\
&= -W\bar{g}(X,Y) + \bar{g}([W,X],Y) + \bar{g}([W,Y],X) - \bar{g}(\bar{\nabla}_Y X, W)\\
&= -(L_W \bar{g})(X,Y) - \bar{g}(h(X,Y),W), \qquad (43)
\end{aligned}$$

which implies that

$$2\bar{g}(h(X,Y),W) = -(L_W\bar{g})(X,Y) \tag{44}$$

for any $X,Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$. Thus from (42) and (44), we have $h(X,Y)=0$, if and only if, $(L_\xi\bar{g})(X,Y)=0$ and $(L_W\bar{g})(X,Y)=0$, for any $X,Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Thus the result follows. □

Theorem 5.10. *Let M be a totally umbilical GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. If the induced connection is a metric connection, then $h^*(X,Y)=0$ for any $X,Y \in \Gamma(D_0)$.*

Proof. Let the induced connection ∇ be a metric connection, then from Theorem 2.2 on page 159 of [5], we have $h^l=0$. Hence using hypothesis in (19), we get $wP_1\nabla_X Y=0$, therefore, $\nabla_X Y \in \Gamma(S(TM))$, which implies that $h^*(X,Y)=0$ for any $X,Y \in \Gamma(D_0)$. Thus the result follows. □

6. Minimal GCR-lightlike submanifolds

Definition 6.1 ([2])**.** A lightlike submanifold $(M,g,S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\bar{M},\bar{g})$ is said to be minimal if $h^s=0$ on $Rad(TM)$ and *trace* $h=0$, where trace is written with respect to g restricted to $S(TM)$.

Theorem 6.2. *Let M be a totally umbilical GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then M is minimal, if and only if, M is totally geodesic.*

Proof. Suppose M is minimal then $h^s(X,Y)=0$ for any $X,Y \in \Gamma(Rad(TM))$. Since M is totally umbilical therefore $h^l(X,Y)=H^lg(X,Y)=0$ for any $X,Y \in \Gamma(Rad(TM))$. Now, choose an orthonormal basis $\{e_1,e_2,\ldots,e_{m-r}\}$ of $S(TM)$ then from (39), we obtain

$$trace\ h(e_i,e_i) = \sum_{i=1}^{m-r} \epsilon_i g(e_i,e_i)H^l + \epsilon_i g(e_i,e_i)H^s = (m-r)H^l + (m-r)H^s.$$

Since M is minimal and $ltr(TM) \cap S(TM^{\perp}) = \{0\}$, we get $H^l=0$ and $H^s=0$. Hence M is totally geodesic. Converse follows directly. □

Theorem 6.3. *A totally umbilical proper GCR-lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$ is minimal, if and only if,*

$$trace\ A_{W_p}=0 \quad and \quad trace\ A^*_{\xi_k}=0 \quad on \quad D_0 \perp FL_2$$

for $W_p \in \Gamma(S(TM^{\perp}))$, where $k \in \{1,2,\ldots,r\}$ and $p \in \{1,2,\ldots,n-r\}$.

Proof. Using (38), it is clear that $h^s(X,Y) = 0$ on $Rad(TM)$. Using the definition of a GCR-lightlike submanifold, we have

$$\begin{aligned} trace\ h|_{S(TM)} = & \sum_{i=1}^{a} h(Z_i, Z_i) + \sum_{j=1}^{b} h(F\xi_j, F\xi_j) \\ & + \sum_{j=1}^{b} h(FN_j, N_j) + \sum_{l=1}^{c} h(FW_l, FW_l), \end{aligned}$$

where $a = \dim(D_0)$, $b = \dim(D_2)$ and $c = \dim(L_2)$. Since M is totally umbilical therefore from (38), we have $h(F\xi_j, F\xi_j) = h(FN_j, N_j) = 0$. Thus above equation becomes

$$\begin{aligned} trace\ h|_{S(TM)} = & \sum_{i=1}^{a} h(Z_i, Z_i) + \sum_{l=1}^{c} h(FW_l, FW_l) \\ = & \sum_{i=1}^{a} \frac{1}{r} \sum_{k=1}^{r} \bar{g}(h^l(Z_i, Z_i), \xi_k) N_k \\ & + \sum_{i=1}^{a} \frac{1}{n-r} \sum_{p=1}^{n-r} \bar{g}(h^s(Z_i, Z_i), W_p) W_p \\ & + \sum_{l=1}^{c} \frac{1}{r} \sum_{k=1}^{r} \bar{g}(h^l(FW_l, FW_l), \xi_k) N_k \\ & + \sum_{l=1}^{c} \frac{1}{n-r} \sum_{p=1}^{n-r} \bar{g}(h^s(FW_l, FW_l), W_p) W_p, \end{aligned} \tag{45}$$

where $\{W_1, W_2, \ldots, W_{n-r}\}$ is an orthonormal basis of $S(TM^\perp)$. Using (8) and (11) in (45), we obtain

$$\begin{aligned} trace\ h|_{S(TM)} = & \sum_{i=1}^{a} \frac{1}{r} \sum_{k=1}^{r} \bar{g}(A^*_{\xi_k} Z_i, Z_i) N_k \\ & + \sum_{i=1}^{a} \frac{1}{n-r} \sum_{p=1}^{n-r} \bar{g}(A_{W_p} Z_i, Z_i) W_p \\ & + \sum_{l=1}^{c} \frac{1}{r} \sum_{k=1}^{r} \bar{g}(A^*_{\xi_k} FW_l, FW_l) N_k \\ & + \sum_{l=1}^{c} \frac{1}{n-r} \sum_{p=1}^{n-r} \bar{g}(A_{W_p} FW_l, FW_l) W_p. \end{aligned}$$

Thus $trace\ h|_{S(TM)} = 0$, if and only if, $trace\ A_{W_p} = 0$ and trace $A_{\xi^*_k} = 0$ on $D_0 \perp FL_2$. Hence the result follows. $\square$

Theorem 6.4. *Let M be an irrotational lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then M is minimal, if and only if, trace $A^*_{\xi_k}|_{S(TM)} = 0$ and trace $A_{W_j}|_{S(TM)} = 0$, where $W_j \in \Gamma(S(TM^{\perp})$, $k \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, n-r\}$.*

Proof. M is irrotational implies that $h^s(X, \xi) = 0$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, therefore $h^s = 0$ on $Rad(TM)$. Also

$$\begin{aligned} trace\ h|_{S(TM)} &= \sum_{i=1}^{m-r} \epsilon_i (h^l(e_i, e_i) + h^s(e_i, e_i)) \\ &= \sum_{i=1}^{m-r} \epsilon_i \{ \frac{1}{r} \sum_{k=1}^{r} \bar{g}(h^l(e_i, e_i), \xi_k) N_k + \frac{1}{n-r} \sum_{j=1}^{n-r} \bar{g}(h^s(e_i, e_i), W_j) W_j \} \\ &= \sum_{i=1}^{m-r} \epsilon_i \{ \frac{1}{r} \sum_{k=1}^{r} \bar{g}(A^*_{\xi_k} e_i, e_i) N_k + \frac{1}{n-r} \sum_{j=1}^{n-r} \bar{g}(A_{W_j} e_i, e_i) W_j \}. \end{aligned}$$

Hence theorem follows. $\square$

Acknowledgment. The authors would like to thank the anonymous referee for his/her valuable suggestions that helped them to improve this paper.

References

[1] M. Atçeken and E. Kiliç, *Semi-invariant lightlike submanifolds of a semi-Riemannian product manifold*, Kodai Math. J. **30** (2007), no. 3, 361–378.

[2] C. L. Bejan and K. L. Duggal, *Global lightlike manifolds and harmonicity*, Kodai Math. J. **28** (2005), no. 1, 131–145.

[3] K. L. Duggal, *CR-structures and Lorentzian geometry*, Acta Appl. Math. **7** (1986), no. 3, 211–223.

[4] ———, *Lorentzian geometry of CR submanifolds*, Acta Appl. Math. **17** (1989), no. 2, 171–193.

[5] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of semi-Riemannian Manifolds and Applications*, Vol. 364 of Mathematics and its Applications, Kluwer Academic Publishers, The Netherlands, 1996.

[6] K. L. Duggal and D. H. Jin, *Totally umbilical lightlike submanifolds*, Kodai Math. J. **26** (2003), no. 1, 49–68.

[7] K. L. Duggal and B. Sahin, *Screen Cauchy-Riemann lightlike submanifolds*, Acta Math. Hungar. **106** (2005), no. 1-2, 137–165.

[8] ———, *Generalized Cauchy-Riemann lightlike submanifolds of Kaehler manifolds*, Acta Math. Hungar. **112** (2006), no. 1-2, 107–130.

[9] S. M. Khursheed Haider, Mamta Thakur, and Advin, *Screen Cauchy-Riemann lightlike submanifolds of a semi-Riemannian product manifold*, Int. Electron. J. Geom. **4** (2011), no. 2, 141–154.

[10] E. Kiliç, B. Şahin, and S. Keleş, *Screen semi invariant lightlike submanifolds of a semi-Riemannian product manifolds*, Int. Electron. J. Geom. **4** (2011), 120–135.

[11] R. Kumar, S. Kumar, and R. K. Nagaich, *GCR-lightlike product of indefinite Kaehler manifolds*, ISRN Geometry. **2011** (2011), Article ID 531281, 13 pages.

[12] ———, *Integrability of distributions in GCR-lightlike submanifolds of indefinite Kaehler manifolds*, Commun. Korean Math. Soc. **27** (2012), no. 3, 591–602.

[13] S. Kumar, R. Kumar, and R. K. Nagaich, *Characterization of holomorphic bisectional curvature of GCR-lightlike submanifolds*, Advances in Mathematical Physics. **2012** (2012), Article ID 356263, 18 pages.

[14] ———, *GCR-lightlike submanifolds of indefinite nearly Kaehler manifolds*, Bull. Korean Math. Soc. **50** (2013), no. 4, 1173–1192.

[15] D. N. Kupeli, *Singular Semi-Riemannian Geometry*, vol. 366, Kluwer, Dordrecht, 1996.

[16] B. O'Neill, *Semi-Riemannian Geometry with Applicitaions to Relativity*, Academic Press, New York, 1983.

[17] B. Şahin and M. Atçeken, *Semi-Invariant submanifolds of Riemannian product manifold*, Balkan J. Geom. Appl. **8** (2003), no. 1, 91–100.

[18] X. Senlin and N. Yilong, *Submanifolds of product Riemannian manifold*, Acta Math. Sci. Ser. B Engl. Ed. **20** (2000), no. 2, 213–218.

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