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SOME CLASSIFICATIONS OF RULED SUBMANIFOLDS

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Abstract. Ruled submanifolds in Euclidean space satisfying some algebraic equations concerning the Laplace operator related to the isometric immersion and Gauss map are studied. Cylinders over a finite type curve or generalized helicoids are characterized with such algebraic equations.

1. Introduction

The theory of minimal submanifolds is still a very interesting subject in differential geometry from various points of view. As of minimal surfaces, it is well known that the only minimal ruled surfaces in Euclidean 3-space is part of the plane or the helicoid by the theorem of Catalan. This notion was generalized by the theory of ruled submanifolds in a Riemannian manifold in such a way that they are defined by a foliation of totally geodesic submanifolds of a given Riemannian manifold. In particular, if the ambient manifold is Euclidean, it is more interesting. The minimal ruled submanifold in Euclidean space was independently studied by Lumiste ([11]) and Barbosa et al. ([3]). They showed that a minimal ruled submanifold of Euclidean space is part of the plane or the generalized helicoid up to rigid motion parameterized by

$$\begin{aligned} x(s, t_1, \ldots, t_n) = (&t_1 \cos(a_1 s), t_1 \sin(a_1 s), \ldots, \\ &t_k \cos(a_k s), t_k \sin(a_k s), t_{k+1}, \ldots, t_n, bs), \end{aligned}$$

where $a_1, a_2, \ldots, a_k$ and b are real numbers.

On the other hand, a minimal submanifold M in Euclidean space $\mathbb{E}^m$ with the isometric immersion $x : M \to \mathbb{E}^m$ is characterized by their immersions and the Laplace operator Δ defined on them, namely, $\Delta x = 0$. Generalizing this, Takahashi showed: Let $x : M \to \mathbb{E}^m$ be an isometric immersion of a

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Riemannian manifold M into the Euclidean space $\mathbb{E}^m$. If $\Delta x = \lambda x$ ($\lambda \neq 0$) holds, then M is a minimal submanifold in a hypersphere of Euclidean space ([12]).

Extending this point of view, in the late 1970's by using the spectral decomposition Chen introduced the notion of finite type smooth map on Riemannian manifolds in Euclidean space ([4, 5]). A smooth map Φ on a Riemannian manifold M into an m-dimensional Euclidean space $\mathbb{E}^m$ is said to be of *finite type* if Φ can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is, $\Phi = \Phi_0 + \Sigma_{i=1}^k \Phi_i$, where Φ_0 is a constant map, $\Phi_1, \ldots, \Phi_k$ non-constant maps such that $\Delta\Phi_i = \lambda_i \Phi_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$ ([4, 5]). Furthermore, M is said to be of k-type if all eigenvalues λ_1, λ_2, …, λ_k are mutually different. In particular, null 1-type submanifolds in Euclidean space are minimal and null 1-type smooth vector fields are also said to be harmonic.

In this regards, Dillen ([7]) showed that an n-dimensional ruled submanifold in $\mathbb{E}^m$ of finite type immersion is part of either a cylinder on a curve of finite type or the generalized helicoid.

On the other hand, in [1] Baikoussis proved that an n-dimensional ruled submanifold M in $\mathbb{E}^m$ with finite type Gauss map is part of an n-plane. In the case, the Gauss map G is in fact harmonic, that is, $\Delta G = 0$.

However, if the Laplacian of each component X_A ($A = 1, 2, \ldots, m$) of a certain vector field X in $\mathbb{E}^m$ is a linear function in $X_1, X_2, \ldots, X_m$, it is not of finite type in general. In other words, it has the form

$$\Delta X = AX + \mathbf{b}$$

for some $m \times m$-matrix A and a constant vector $\mathbf{b}$ (cf. [2, 8, 9]).

In this article, we study the ruled submanifold M in the Euclidean space $\mathbb{E}^m$ satisfying the equation

$$\Delta x = Ax + \mathbf{b} \quad \text{and} \quad \Delta G = AG + \mathbf{b}$$

for some $m \times m$-matrix A and a constant vector $\mathbf{b}$, where x is the isometric immersion of M into $\mathbb{E}^m$ and G is the Gauss map defined on M.

All of geometric objects under consideration are smooth and submanifolds are assumed to be connected unless otherwise stated.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold isometrically immersed into an m-dimensional Euclidean space $\mathbb{E}^m$ via the immersion x.

Let $(x_1, x_2, \ldots, x_n)$ be a local coordinate system of M in $\mathbb{E}^m$. For the components g_{ij} of the metric $\langle \cdot, \cdot \rangle$ on M induced from that of $\mathbb{E}^m$, we denote by (g^{ij}) (respectively, $\mathcal{G}$) the inverse matrix (respectively, the determinant) of the matrix (g_{ij}). Then, the Laplacian Δ on M is given by

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}} \sum_{i,j} \frac{\partial}{\partial x_i} (\sqrt{|\mathcal{G}|} g^{ij} \frac{\partial}{\partial x_j}).$$

We now choose an adapted local orthonormal frame $\{e_1, e_2, \dots, e_m\}$ in $\mathbb{E}^m$ such that $e_1, e_2, \dots, e_n$ are tangent to M and $e_{n+1}, e_{n+2}, \dots, e_m$ normal to M. The Gauss map $G : M \to G(n,m) \subset \mathbb{E}^N$ $(N = {}_mC_n)$, $G(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p)$, of x is a smooth map which carries a point p in M to an oriented n-plane in $\mathbb{E}^m$ which is obtained from the parallel translation of the tangent space of M at p to an n-plane passing through the origin in $\mathbb{E}^m$, where $G(n,m)$ is the Grassmannian manifold consisting of all oriented n-planes through the origin of $\mathbb{E}^m$.

An inner product $\ll \cdot, \cdot \gg$ on $G(n,m) \subset \mathbb{E}^N$ is defined by

$$\ll e_{i_1} \wedge \ldots \wedge e_{i_n}, e_{j_1} \wedge \cdots \wedge e_{j_n} \gg = \det(\langle e_{i_l}, e_{j_k} \rangle).$$

Then, $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} \mid 1 \le i_1 < \cdots < i_n \le m\}$ is an orthonormal basis of $\mathbb{E}^N$.

We now consider a ruled submanifold M over the base curve α parameterized by

$$x = x(s, t_1, t_2, \dots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s), \quad s \in I, \ t_i \in I_i,$$

where I_i's are some open intervals for $i = 1, 2, \dots, r$.

Here, we may assume that the base curve α is of unit speed and

$$\langle \alpha'(s), e_i(s) \rangle = 0, \quad \langle e_i(s), e_j(s) \rangle = \delta_{ij} \quad \langle e_i'(s), e_j(s) \rangle = 0$$

for $i, j = 1, 2, \dots, r$.

For each s, let $E(s,r)$ be an open subset of $\mathrm{Span}\{e_1, e_2, \dots, e_r\}$, the linear span of $e_1, e_2, \dots, e_r$. We call $E(s,r)$ the rulings of M. In particular, the ruled submanifold M is said to be *cylindrical* if $E(s,r)$ is parallel along α, or *non-cylindrical* otherwise.

3. Ruled submanifolds satisfying $\Delta x = Ax + \mathbf{b}$

Let M be an $(r+1)$-dimensional ruled submanifold in $\mathbb{E}^m$ with the base curve α. Without loss of generality, we may assume that α is a unit speed curve, that is, $\langle \alpha'(s), \alpha'(s) \rangle = 1$. From now on, the prime $'$ denotes d/ds unless otherwise stated. We may also choose orthonormal vector fields $e_1(s), \dots, e_r(s)$ generating the rulings along α with

$$\langle \alpha'(s), e_i(s) \rangle = 0, \ \langle e_i'(s), e_j(s) \rangle = 0, \quad i, j = 1, 2, \dots, r. \tag{3.1}$$

A parametrization of M is given by

$$x = x(s, t_1, t_2, \dots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s). \tag{3.2}$$

We now consider the case that the ruled submanifold M is cylindrical. Then, we may take the generators $e_1, e_2, \dots, e_r$ of the rulings $E(s,r)$ as constant vector fields.

Suppose M satisfies the equation

$$\Delta x = Ax + \mathbf{b} \tag{3.3}$$

for some $m \times m$-matrix A and a constant vector $\mathbf{b}$. Then, by Cayley-Hamilton's Theorem, there exist some constants $c_0, c_1, \dots, c_m$ such that

$$A^m + c_o A^{m-1} + \cdots + c_{m-1} A + c_m I = 0, \tag{3.4}$$

where I denotes the identity matrix of degree m. Using (3.3), we have

$$\Delta^{m+1} x + c_0 \Delta^m x + \cdots + c_m \Delta x = 0.$$

According to Proposition 4.1 in [6], the base curve α is of finite type. Therefore, we have:

Theorem 3.1. *Let M be a cylindrical ruled submanifold of $\mathbb{E}^m$ satisfying $\Delta x = Ax + \boldsymbol{b}$ for some $m \times m$-matrix A and a constant vector $\boldsymbol{b} \in \mathbb{R}^m$. Then, M is a cylinder over the finite type base curve.*

Next, consider the case that M is non-cylindrical.
If we define a function q on M by

$$q = ||x_s||^2 = 1 + 2\sum_{i=1}^{r} t_i \langle \alpha', e_i' \rangle + \sum_{i,j=1}^{r} t_i t_j \langle e_i', e_j' \rangle, \tag{3.5}$$

then the Laplacian Δ of M is given by

$$\Delta = \frac{1}{2q^2}\frac{\partial q}{\partial s}\frac{\partial}{\partial s} - \frac{1}{q}\frac{\partial^2}{\partial s^2} - \frac{1}{2q}\sum_{i=1}^{r}\frac{\partial q}{\partial t_i}\frac{\partial}{\partial t_i} - \sum_{i=1}^{r}\frac{\partial^2}{\partial t_i^2}.$$

The function q is a polynomial in $t = (t_1, t_2, \dots, t_r)$ with functions of s as coefficients. Since M is non-cylindrical, we may assume that q has degree 2 in t without loss of generality. Then, we can easily have:

Lemma 3.1 ([7])**.** *Let M be a non-cylindrical ruled submanifold in $\mathbb{E}^m$ parametrized by* (3.2)*. If P is a polynomial with functions in s as coefficients and* $\deg(P) = d$*, then*

$$\Delta\left(\frac{P(t)}{q^m}\right) = \frac{\tilde{P}(t)}{q^{m+3}},$$

where $\tilde{P}$ is a polynomial in t with functions in s as coefficients and $\deg(\tilde{P}) \leq d + 4$.

We now suppose that M satisfies (3.3) for a non-trivial matrix A. Then, for some constant $c_0, c_1, \dots, c_m$, we have

$$\Delta^{m+1} x + c_0 \Delta^m x + \cdots + c_m \Delta x = 0.$$

Let x_A be the A-th component of x in $\mathbb{E}^m$, where $A = 1, 2, \dots, m$. Then,

$$\Delta x_A = \frac{Q_A(t)}{q^2}$$

for some polynomial $Q_A(t)$ in $t = (t_1, t_2, \ldots, t_r)$ with $\deg Q_A(t) \leq 5$. By applying Lemma 3.2, we have

$$\Delta^j x_A = \frac{Q_{Aj}(t)}{q^{3j-1}},$$

with $\deg Q_A j \leq 1 + 4j$, $j = 1, 2, \ldots$. If j goes up by one, the degree of numerator of $\Delta^j x_A$ goes up by at most 3 while that of the denominator goes up by 4. Thus, for some positive integer i, $\Delta^{i+1} x + \lambda_1 \Delta^i x + \cdots + \lambda_i \Delta x = 0$ never occurs unless $\Delta x = 0$, that is, M is minimal. Therefore, the matrix A must be zero. Hence, we have:

Theorem 3.2. *Let M be a non-cylindrical ruled submanifold of $\mathbb{E}^m$ satisfying $\Delta x = Ax + \boldsymbol{b}$ for some $m \times m$-matrix A and a constant vector $\boldsymbol{b} \in \mathbb{R}^m$. Then, M is minimal, i.e., M is part of a plane or a generalized helicoid.*

If we consider the result of [7], we have the following characterization of the ruled submanifold of finite type immersion.

Theorem 3.3. *Let $x : M \to \mathbb{E}^m$ be an isometric immersion of ruled submanifold in $\mathbb{E}^m$. Then, x is of finite type if and only if M is part of a cylinder over a finite type curve or x satisfies $\Delta x = Ax + \boldsymbol{b}$ for some $m \times m$-matrix A and a constant vector $\boldsymbol{b} \in \mathbb{R}^m$.*

4. Ruled submanifolds satisfying $\Delta G = AG + \mathbf{b}$

In this section, we always assume that the parametrization (3.2) satisfies the condition (3.1). Then, M has the Gauss map

$$G = \frac{1}{\|x_s\|} x_s \wedge x_{t_1} \wedge \cdots \wedge x_{t_r},$$

or, equivalently

$$G = \frac{1}{q^{1/2}} (\Phi + \sum_{i=1}^{r} t_i \Psi_i), \tag{4.1}$$

where the vectors Φ and Ψ_i are defined by

$$\Phi = \alpha' \wedge e_1 \wedge \cdots \wedge e_r \quad \text{and} \quad \Psi_i = e_i' \wedge e_1 \wedge \cdots \wedge e_r$$

for $i = 1, 2, \ldots, r$.

Now, we prove:

Theorem 4.1. *The only ruled submanifolds of $\mathbb{E}^m$ with $\Delta G = AG + \boldsymbol{b}$ for some matrix A and a vector $\boldsymbol{b}$ are parts of planes or cylinders over a curve of finite type.*

Proof. Let M be a cylindrical $(r+1)$-dimensional ruled submanifold parameterized by (3.2) in $\mathbb{E}^m$ satisfying

$$\Delta G = AG + \mathbf{b} \tag{4.2}$$

for some $m \times m$-matrix A and a constant vector $\mathbf{b}$. We may assume that $e_1, e_2, \ldots, e_r$ generating the rulings are constant vectors.

The Laplacian Δ of M is then naturally expressed by

$$\Delta = -\frac{\partial^2}{\partial s^2} - \sum_{i=1}^{r} \frac{\partial^2}{\partial t_i^2}$$

and the Gauss map G of M is given by

$$G = \alpha' \wedge e_1 \wedge \cdots \wedge e_r.$$

If we denote by Δ' the Laplacian of α, that is $\Delta' = -\frac{\partial^2}{\partial s^2}$, we have the Laplacian ΔG of the Gauss map

$$\Delta G = \Delta' \alpha' \wedge e_1 \wedge \cdots \wedge e_r. \tag{4.3}$$

Since (4.2) holds, we have (3.4) and thus we obtain

$$\Delta^{m+1} G + c_0 \Delta^m G + \cdots + c_m \Delta G = 0, \tag{4.4}$$

or, equivalently,

$$\Delta'^{m+1} \alpha' + c_0 \Delta'^k \alpha' + \cdots + c_m \Delta' \alpha' = 0,$$

which implies that $\alpha'(s) = a_1 + a_2 s + \sum_{i=1}^{k} \{\tilde{b}_i \cos(l_i s) + \tilde{c}_i \sin(l_i s)\}$ for some positive integer k, where $l_1 < l_2 < \cdots < l_k$ are positive real numbers and $a_1, a_2, \tilde{b}_i, \tilde{c}_i$ are vectors in $\mathbb{E}^m$ such that b_i and c_i are not simultaneously zero for each $i = 1, 2, \ldots, k$ (cf. [10]). Since we assume that the base curve α is of unit speed, the coefficient a_2 vanishes. Thus, we have $\alpha(s) = \tilde{a}_0 + \tilde{a}_1 s + + \sum_{j=1}^{k} \{b_i \cos(l_i s) + c_i \sin(l_i s)\}$ where $\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, b_i, c_i$ are vectors in $\mathbb{E}^m$ such that b_i and c_i are not simultaneously zero for each $i = 1, 2, \ldots, k$. This implies that the curve α is of finite type.

We now suppose that a non-cylindrical ruled submanifold M satisfies $\Delta G = AG + \mathbf{b}$ for some $m \times m$-matrix A and a constant vector $\mathbf{b}$. Then we have the Laplacian

$$\Delta = \frac{1}{2q^2} \frac{\partial q}{\partial s} \frac{\partial}{\partial s} - \frac{1}{q} \frac{\partial^2}{\partial s^2} - \frac{1}{2q} \sum_{i=1}^{r} \frac{\partial q}{\partial t_i} \frac{\partial}{\partial t_i} - \sum_{i=1}^{r} \frac{\partial^2}{\partial t_i^2}.$$

Similarly to obtain (3.4), we have

$$\Delta^{m+1} G + c_0 \Delta^m G + \cdots + c_m \Delta G = 0 \tag{4.5}$$

for some constants $c_0, c_1, \ldots, c_m$.

Quite similarly as in [1], we get

$$G = \frac{G_0(t)}{q^{1/2}}, \Delta G = \frac{G_1(t)}{q^{(1/2)+3}}, \ldots, \Delta^j G = \frac{G_j(t)}{q^{(1/2)+3j}},\ j = 0, 1, 2, \ldots, \tag{4.6}$$

where $G_j(t)$ is a polynomial in $t = (t_1, t_2, \ldots, t_r)$ with functions in s as coefficients and $\deg G_j(t) \le 1 + 4j$. As before, if (4.5) holds, there exist no other cases but

$$\Delta G = 0.$$

If we follow along the argument in [1] with harmonic Gauss map, we obtain M is part of an $(r+1)$-plane. □

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