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QUASIPOLAR MATRIX RINGS OVER LOCAL RINGS

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Abstract. A ring R is called quasipolar if for every $a \in R$ there exists $p^2 = p \in R$ such that $p \in \mathrm{comm}^2_R(a)$, $a + p \in U(R)$ and $ap \in R^{\mathrm{qnil}}$. The class of quasipolar rings lies properly between the class of strongly π-regular rings and the class of strongly clean rings. In this paper, we determine when a 2×2 matrix over a local ring is quasipolar. Necessary and sufficient conditions for a 2×2 matrix ring to be quasipolar are obtained.

1. Introduction

Throughout the paper, rings R are associative with unity and modules M are unitary modules. For an element $a \in R$, l_a and r_a denote the abelian group endomorphisms of R given by left and right multiplication by a, respectively. The symbols $U(R)$ and $J(R)$ stand for the group of units and the Jacobson radical of R. Let $M_n(R)$ be the $n \times n$ matrix ring over R and I_n be the $n \times n$ identity matrix of $M_n(R)$. We write $\mathrm{end}(M)$ for the endomorphism ring of a module M.

Recall that a ring R is called *strongly π-regular* if for every $a \in R$, the chain $aR \supseteq a^2R \supseteq \cdots$ terminates (or equivalently, the chain $Ra \supseteq Ra^2 \supseteq \cdots$ terminates [8]). Clearly, one-sided perfect rings are strongly π-regular. In [15], Nicholson introduced the notion of a strongly clean ring. An element of a ring R is called *strongly clean* if it is the sum of an idempotent and a unit which commute, and R is called *strongly clean* if every element of R is strongly clean. Nicholson [15] proved that any strongly π-regular element is strongly clean by establishing the following results: for $\alpha \in \mathrm{end}(M)$, α is strongly π-regular if and only if there exist α-invariant submodules P and Q such that $M = P \oplus Q$, $\alpha|_P$ is an isomorphism and $\alpha|_Q$ is nilpotent; and α is strongly clean if and only if there exist α-invariant submodules P and Q such that $M = P \oplus Q$, $\alpha|_P$ and $(1-\alpha)|_Q$ are isomorphisms. Some other notable results on strongly clean rings can be found in [1, 2, 3, 4, 13, 16, 17], etc.

Following [10], the *commutant* and *double commutant* of an element a of a ring R are defined by $\mathrm{comm}_R(a) = \{x \in R : ax = xa\}$ and $\mathrm{comm}^2_R(a) =$

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$\{x \in R : xy = yx \text{ for all } y \in \text{comm}_R(a)\}$, respectively (if there is no ambiguity, we use $\text{comm}(a)$ and $\text{comm}^2(a)$ for short). Let $R^{\text{qnil}} = \{a \in R : 1 - ax \in U(R) \text{ for all } x \in \text{comm}(a)\}$ be the set of all quasinilpotent elements of R. It is clear that $J(R) \subseteq R^{\text{qnil}}$. Koliha and Patricio called an element a of a ring R *quasipolar* [11] if there exists $p^2 = p \in \text{comm}^2(a)$ such that $a + p \in U(R)$ and $ap \in R^{\text{qnil}}$, where p is called a *spectral idempotent* of a and is denoted by $p = a^\pi$ (the spectral idempotent of an element is unique if it exists). The quasipolar element coincides with the generalized Drazin inverse in any ring [11]. The notion of a quasipolar ring was introduced by Ying and Chen [18]. A ring R is called *quasipolar* if every element of R is quasipolar. It was proved [18] that local rings and strongly π-regular rings are quasipolar, and quasipolar rings are strongly clean.

In 2004, Wang and Chen [16] proved that there exists a commutative local ring R such that $M_2(R)$ is not strongly clean, which answered a question raised by Nicholson in [15]. This motivated many authors studied strong cleanness of matrix rings over local rings ([2, 3, 4, 12, 13, 17]). The quasipolarity of a 2×2 matrix ring over a commutative local ring was considered in [6].

In this paper, we study when a 2×2 matrix ring over a general local ring is quasipolar. Using the decomposition theorem of quasipolar elements provided in [7], we prove that over a local ring R, $A \in M_2(R)$ is quasipolar if and only if either A is invertible or $A \in (M_2(R))^{\text{qnil}}$ or A is similar to a diagonal matrix of the form $\left(\begin{smallmatrix} u & 0 \\ 0 & j \end{smallmatrix}\right)$ where $u \in U(R)$, $j \in J(R)$ and $l_u - r_j$, $l_j - r_u$ are injective. This result is put to use when we establish some criteria for a 2×2 matrix ring over a local ring is quasipolar. Moreover, the relationship of strongly clean matrices and quasipolar matrices over a commutative local ring are discussed.

2. Several lemmas

Let R be a ring. It is well known that $J(M_2(R)) = M_2(J(R))$. Recall that two elements a, b of R are said to be *similar* if $b = u^{-1}au$ for some $u \in U(R)$.

Note that quasipolarity and quasinilpotent property are preserved by isomorphisms. So we have the following results immediately.

Lemma 2.1. *Let R be a ring, $a \in R$ and $u \in U(R)$. Then*

(1) *a is quasipolar if and only if $u^{-1}au$ is quasipolar. In particular, $(u^{-1}au)^\pi = u^{-1}a^\pi u$.*

(2) *a is quasinilpotent if and only if so is $u^{-1}au$.*

The following result is elementary.

Lemma 2.2. *Let R be a ring, $a \in R$. Then*

(1) *a is invertible if and only if a is quasipolar and $a^\pi = 0$.*

(2) *a is quasinilpotent if and only if a is quasipolar and $a^\pi = 1$.*

It is well known that local rings are projective-free (i.e., any projective module over the ring is free of unique rank). According to [5, Proposition 4.5], every idempotent matrix over a projective-free ring is similar to a diagonal matrix. So the following result is obvious.

Lemma 2.3. *Let R be a local ring, and $E \in M_2(R)$ be a non-trivial idempotent. Then E is similar to the diagonal matrix* $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Lemma 2.4. *Let R be a nonzero ring. Then $R^{\mathrm{qnil}} \cap U(R) = \emptyset$. In particular, if R is a local ring, then $R^{\mathrm{qnil}} = J(R)$.*

Proof. Assume on the contrary. Let $a \in R^{\mathrm{qnil}} \cap U(R)$. Then $-a^{-1} \in \mathrm{comm}_R(a)$. However, $a \in R^{\mathrm{qnil}}$ implies that $0 = 1 + (-aa^{-1}) \in U(R)$, a contradiction. Thus $R^{\mathrm{qnil}} \cap U(R) = \emptyset$. Notice that $J(R) \subseteq R^{\mathrm{qnil}}$ for any ring. Hence $R^{\mathrm{qnil}} = J(R)$ if R is local. □

Yang and Zhou [17, Lemma 4] and Li [12, Lemma 2.4] independently proved that over a local ring R, if $A \in M_2(R)$ with neither A nor $I_2 - A$ is a unit, then A is similar to a matrix of the form $\left(\begin{smallmatrix} 0 & j \\ 1 & u \end{smallmatrix}\right)$ where $j \in J(R)$ and $u \in 1 + J(R)$. We have an analogous result.

Lemma 2.5. *Let R be a local ring, and $A \in M_2(R)$ with $A \notin U(M_2(R)) \cup (M_2(R))^{\mathrm{qnil}}$. Then A is similar to $\left(\begin{smallmatrix} 0 & j \\ 1 & u \end{smallmatrix}\right)$ where $j \in J(R)$ and $u \in U(R)$.*

Proof. Let $T = M_2(R)$ and $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in T$. We proceed with the following two cases.

Case 1. One of b or c is a unit. Note that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} d & c \\ b & a \end{smallmatrix}\right)$. Without loss of generality, we assume that $c \in U(R)$. Let $V = \left(\begin{smallmatrix} c^{-1} & ac^{-1} \\ 0 & 1 \end{smallmatrix}\right)$. Then

$$V^{-1}AV = \begin{pmatrix} c & -cac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} c^{-1} & ac^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & cb - cac^{-1}d \\ 1 & cac^{-1} + d \end{pmatrix}.$$

It is easy to see that if $cb - cac^{-1}d \in U(R)$, then $V^{-1}AV \in U(M_2(R))$, which implies A is a unit, a contradiction. Thus, $cb - cac^{-1}d \in J(R)$. We next show that $cac^{-1} + d \in U(R)$. If not, then $cac^{-1} + d \in J(R)$. Given any $Y \in \mathrm{comm}_T(V^{-1}AV)$, by a routine computation, we obtain that the (i,i)-entry and $(1,2)$-entry of $Y(V^{-1}AV)$ are in $J(R)$ where $i = 1, 2$. It follows that $I_2 + YV^{-1}AV$ is invertible in T. This proves that $V^{-1}AV \in T^{\mathrm{qnil}}$, and thus $A \in T^{\mathrm{qnil}}$ by Lemma 2.1(2), which contradicts the assumption. So $cac^{-1} + d \in U(R)$.

Case 2. Neither b nor c is a unit. If $a,\ d \in J(R)$, then $A \in J(T)$, and this contradict $A \notin T^{\mathrm{qnil}}$. If $a,\ d \in U(R)$, then $A \in U(T)$, contradicting the assumption. So one of a and d is in $J(R)$ and the other is in $U(R)$. Without loss of generality, we may assume that $a \in J(R)$ and $d \in U(R)$. Let $U = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Then $U^{-1}AU = \left(\begin{smallmatrix} a+b & b \\ c+d-a-b & d-b \end{smallmatrix}\right)$. Since $a, b, c \in J(R)$ and $d \in U(R)$, $c + d - a - b \in U(R)$. So we are back to Case 1.

Therefore, A is similar to a matrix of the form $\left(\begin{smallmatrix} 0 & j \\ 1 & u \end{smallmatrix}\right)$ where $j \in J(R)$ and $u \in U(R)$. □

3. Quasipolar matrix rings over noncommutative local rings

In this section, we will develop a criterion for a 2×2 matrix ring over a noncommutative local ring to be quasipolar.

Proposition 3.1. *Let R be a local ring, $u \in U(R)$ and $j \in J(R)$. Then*
(1) *If the matrix $A = \left(\begin{smallmatrix} u & 0 \\ 0 & j \end{smallmatrix}\right) \in M_2(R)$ is quasipolar, then $A^\pi = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.*
(2) *If the matrix $A = \left(\begin{smallmatrix} j & 0 \\ 0 & u \end{smallmatrix}\right) \in M_2(R)$ is quasipolar, then $A^\pi = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.*

Proof. In view of Lemma 2.1, it is enough to prove (1).

Write $E = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Clearly, $E^2 = E$, $A + E \in U(M_2(R))$ and $AE = EA \in J(M_2(R))(\subseteq (M_2(R))^{\mathrm{qnil}})$. It follows from $A^\pi \in \mathrm{comm}^2(A)$ that $A^\pi E = EA^\pi$, whence A^π is diagonal. By the uniqueness of the spectral idempotent and Lemma 2.2, we get $A^\pi = E = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. $\square$

Corollary 3.2. *Let R be a local ring, $A \in M_2(R)$ is quasipolar and $A \notin U(M_2(R)) \cup (M_2(R))^{\mathrm{qnil}}$. Then A is diagonal if and only if A^π is an idempotent diagonal matrix.*

Proof. Suppose that A is a diagonal matrix. The hypothesis $A \notin U(M_2(R)) \cup (M_2(R))^{\mathrm{qnil}}$ implies that A is of the form $\left(\begin{smallmatrix} u & 0 \\ 0 & j \end{smallmatrix}\right)$, where $u \in U(R)$ and $j \in J(R)$. By Proposition 3.1, $A^\pi = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Conversely, since A^π is non-trivial and $A^\pi A = AA^\pi$, it follows that A is a diagonal matrix. $\square$

Corollary 3.3. *Let R be a local ring, $u \in U(R)$ and $j \in J(R)$. The following are equivalent*:
(1) *The matrix $\left(\begin{smallmatrix} j & 0 \\ 0 & u \end{smallmatrix}\right) \in M_2(R)$ is quasipolar.*
(2) *The matrix $\left(\begin{smallmatrix} u & 0 \\ 0 & j \end{smallmatrix}\right) \in M_2(R)$ is quasipolar.*
(3) *The endomorphisms $l_u - r_j$ and $l_j - r_u$ are injective.*

Proof. (1) $\Leftrightarrow$ (2). Note that $\left(\begin{smallmatrix} u & 0 \\ 0 & j \end{smallmatrix}\right)$ is similar to $\left(\begin{smallmatrix} j & 0 \\ 0 & u \end{smallmatrix}\right)$. The result follows by Lemma 2.1(1).

(2) $\Rightarrow$ (3). Let $A = \left(\begin{smallmatrix} u & 0 \\ 0 & j \end{smallmatrix}\right)$. By Proposition 3.1, $A^\pi = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. If $(l_u - r_j)(r) = 0$ for some $r \in R$, we let $C_1 = \left(\begin{smallmatrix} 0 & r \\ 0 & 0 \end{smallmatrix}\right)$. Then $AC_1 = C_1 A$. So $A^\pi \in \mathrm{comm}^2(A)$ implies that $A^\pi C_1 = C_1 A^\pi$, and whence $r = 0$. Thus $l_u - r_j$ is injective. If $(l_j - r_u)(s) = 0$, then let $C_2 = \left(\begin{smallmatrix} 0 & 0 \\ s & 0 \end{smallmatrix}\right)$. A similar argument as the above yields $s = 0$, and hence $l_j - r_u$ is injective.

(3) $\Rightarrow$ (2). Write $A = \left(\begin{smallmatrix} u & 0 \\ 0 & j \end{smallmatrix}\right)$ and $E = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $E^2 = E \in \mathrm{comm}(A)$, $A + E \in U(M_2(R))$ and $AE \in J(M_2(R))$. Let $B = (b_{ij}) \in M_2(R)$ with $B \in \mathrm{comm}(A)$. Then we obtain $ub_{12} - b_{12}j = 0$ and $jb_{21} - b_{21}u = 0$. By hypothesis, $b_{12} = b_{21} = 0$. So $EB = BE$. This proves $E \in \mathrm{comm}^2(A)$. Thus A is quasipolar and $A^\pi = E$. $\square$

We now give a characterization of a 2×2 matrix over a local ring to be quasipolar.

Theorem 3.4. *Let R be a local ring. Then $A \in M_2(R)$ is quasipolar if and only if A is invertible or $A \in (M_2(R))^{\mathrm{qnil}}$ or A is similar to a matrix $\left(\begin{smallmatrix} u & 0 \\ 0 & j \end{smallmatrix}\right)$ with $u \in U(R)$, $j \in J(R)$ and $l_u - r_j$, $l_j - r_u$ are injective.*

Proof. Write $T = M_2(R)$

In view of Lemma 2.2, A is quasipolar if $A \in U(T)$ or $A \in T^{\text{qnil}}$. Suppose that $V^{-1}AV = \left(\begin{smallmatrix} u & 0 \\ 0 & j \end{smallmatrix}\right)$ for some $V \in U(T)$. Since $l_u - r_j$ and $l_j - r_u$ are injective, by Corollary 3.3 $V^{-1}AV$ is quasipolar. Thus $A \in T$ is quasipolar by Lemma 2.1(1).

Conversely, assume that $A \in T$ is quasipolar. By Lemma 2.2, we may assume that $A \notin U(T)$ and $A \notin T^{\text{qnil}}$. Let $E = A^\pi$. In view of Lemma 2.3, there exists $V \in U(T)$ such that $V^{-1}EV = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \doteq F$. Then by Lemma 2.1(1), $F = (V^{-1}AV)^\pi$. Note that $F(V^{-1}AV) = (V^{-1}AV)F$. Thus $V^{-1}AV$ is a diagonal matrix. From $F + V^{-1}AV \in U(T)$, we have the $(1,1)$-entry of $V^{-1}AV$ is a unit of R. Note that $V^{-1}AV$ is not invertible. So the $(2,2)$-entry of $V^{-1}AV$ is in $J(R)$, and the rest follows by Corollary 3.3. □

Recall that a local ring R is called *bleached* [1] if $l_u - r_j$ and $l_j - r_u$ of R are surjective for any $j \in J(R)$ and $u \in U(R)$. We call a local ring *co-bleached* if for any $j \in J(R)$ and $u \in U(R)$, both $l_u - r_j$ and $l_j - r_u$ of R are injective. From Corollary 3.3, we know that $M_2(R)$ is not quasipolar if R is not a co-bleached local ring.

Proposition 3.5. *Let R be a co-bleached local ring and $A \in M_2(R)$. Then the following are equivalent*:

(1) *A is quasipolar in $M_2(R)$.*

(2) *There exists $P^2 = P \in M_2(R)$ such that $P \in \text{comm}(A)$, $A + P \in U(M_2(R))$ and $AP \in (M_2(R))^{\text{qnil}}$. In this case, $P = A^\pi$.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1). It suffices to show that $P \in \text{comm}^2(A)$. If A is a unit or $A \in (M_2(R))^{\text{qnil}}$, then we are done by Lemma 2.2. Otherwise, P is non-trivial. In view of Lemma 2.3, there exists $V \in U(M_2(R))$ such that $V^{-1}PV = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. From $PA = AP$, one has $(V^{-1}PV)(V^{-1}AV) = (V^{-1}AV)(V^{-1}PV)$. So $V^{-1}AV$ is a diagonal matrix with one entry in $U(R)$ and the other in $J(R)$. We can assume that $V^{-1}AV = \left(\begin{smallmatrix} u & 0 \\ 0 & j \end{smallmatrix}\right)$, where $u \in U(R)$ and $j \in J(R)$.

Let $B \in M_2(R)$ with $B \in \text{comm}(A)$. Write $V^{-1}BV = (b_{ij})$. Then

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & j \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & j \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

It follows that $ub_{12} - b_{12}j = 0$ and $jb_{21} - b_{21}u = 0$. Since R is co-bleached, we have $b_{12} = b_{21} = 0$. Thus $(V^{-1}BV)(V^{-1}PV) = (V^{-1}PV)(V^{-1}BV)$, which implies that $BP = PB$. Hence $P \in \text{comm}^2(A)$, and so $P = A^\pi$. □

Combining Proposition 3.5 with [7, Theorem 4.8], we have the following result immediately.

Corollary 3.6. *Let R be a co-bleached local ring, $M =_R (R \oplus R)$ and $\alpha \in E = \text{end}(M)$. The following are equivalent*:

(1) *α is quasipolar in E.*

(2) *$M = P \oplus Q$ where P and Q are α-invariant, $\alpha|_P$ is an isomorphism of* $\mathrm{end}(P)$ *and $\alpha|_Q$ is quasinilpotent in* $\mathrm{end}(Q)$.

The polynomial ring over a ring R in the indeterminate x is denoted by $R[x]$. For a monic polynomial $h(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in R[x]$, the matrix $C_h = \begin{pmatrix} 0 & -a_0 \\ I_{n-1} & \alpha \end{pmatrix}$ is called the *companion matrix* of $h(x)$, where $\alpha = (-a_1, -a_2, \ldots, -a_{n-1})^T$. A square matrix A over R is called a *companion matrix* if $A = C_h$ for a monic polynomial $h(x)$ over R.

Theorem 3.7. *Let R be a co-bleached local ring. The following are equivalent*:

(1) *$M_2(R)$ is quasipolar.*

(2) *For any monic polynomial $h(x)$ of degree* 2*, the companion matrix $C_h \in M_2(R)$ is quasipolar.*

(3) *For any $u \in U(R)$ and $j \in J(R)$, $\left(\begin{smallmatrix} 0 & j \\ 1 & u \end{smallmatrix}\right)$ is quasipolar.*

(4) *For any $A \in M_2(R)$, either A is invertible or $A \in (M_2(R))^{\mathrm{qnil}}$ or A is similar to a diagonal matrix.*

(5) *For any $u \in U(R)$ and $j \in J(R)$, either $\left(\begin{smallmatrix} 0 & j \\ 1 & u \end{smallmatrix}\right) \in M_2(R))^{\mathrm{qnil}}$ or the equation $x^2 - ux - j = 0$ has a solution in $U(R)$ and a solution in $J(R)$.*

Proof. Write $T = M_2(R)$.

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4). Let $A \in T$ be such that $A \notin U(T)$ and $A \notin T^{\mathrm{qnil}}$. By Lemma 2.5, there exists $V \in U(T)$ satisfying $V^{-1}AV = \left(\begin{smallmatrix} 0 & j \\ 1 & u \end{smallmatrix}\right)$ with $j \in J(R)$ and $u \in U(R)$. By Lemma 2.1(1), A is quasipolar. Thus A is similar to a diagonal matrix by Theorem 3.4.

(4) $\Rightarrow$ (1). This follows from Theorem 3.4 since R is co-bleached.

(3) $\Rightarrow$ (5). We may assume that $A \notin T^{\mathrm{qnil}}$. In view of Theorem 3.4, there exists $V \in U(T)$ such that $V^{-1}AV = \left(\begin{smallmatrix} \mu & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ where $\mu \in U(R)$ and $\lambda \in J(R)$. Write $V = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$. Then $AV = V\left(\begin{smallmatrix} \mu & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ implies the following equations:

$$\text{(i)}\quad jz = x\mu, \qquad\qquad \text{(ii)}\quad jw = y\lambda,$$
$$\text{(iii)}\quad x + uz = z\mu, \qquad\qquad \text{(iv)}\quad y + uw = w\lambda.$$

By Eq.(i), $x \in J(R)$. So both y and z are in $U(R)$ since V is invertible. By Eq.(iv), $w \in U(R)$ as R is local. Based on Eqs.(iii) and (iv), put

$$\mu' = z\mu z^{-1} = xz^{-1} + u \in U(R) \ \text{ and } \lambda' = w\lambda w^{-1} = yw^{-1} + u \in J(R).$$

Combining Eq.(i) with Eq.(iii), we obtain

$$(\mu')^2 - u\mu' = (xz^{-1} + u)^2 - u(xz^{-1} + u) = xz^{-1}(x + uz)z^{-1} = j,$$

and by Eqs.(ii) and (iv), one has

$$(\lambda')^2 - u\lambda' = (yw^{-1} + u)^2 - u(yw^{-1} + u) = yw^{-1}(y + uw)w^{-1} = j.$$

Hence $x^2 - ux - j = 0$ has a solution $\mu' \in U(R)$ and a solution $\lambda' \in J(R)$.

(5) $\Rightarrow$ (3). Assume that $\lambda_1 \in U(R)$ and $\lambda_2 \in J(R)$ are two solutions of $x^2 - ux - j = 0$. Let $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ j & u \end{smallmatrix}\right) \in T$. Then α can be viewed as the R-homomorphism

of $(R \oplus R)_R$. Consider the column vectors $v_1 = (1, \lambda_1)^T$, $v_2 = (1, \lambda_2)^T$, and let $P = v_1 R$, $Q = v_2 R$. Then we have $(R \oplus R)_R = P \oplus Q$ (indeed, for any $r = (r_1, r_2)^T$, $r = v_1 \cdot (\lambda_2 - \lambda_1)^{-1}(\lambda_2 r_1 - r_2) + v_2 \cdot (\lambda_2 - \lambda_1)^{-1}(r_2 - \lambda_1 r_1) \in P + Q$ and $P \cap Q = 0$). Since $\lambda_i^2 - u\lambda_i - j = 0$ for $i = 1, 2$, we get

$$\alpha(v_i) = (\lambda_i, j + u\lambda_i)^T = (1, \lambda_i)^T \lambda_i = v_i \lambda_i.$$

So P and Q are both α-invariant, and α acts as an isomorphism on P and $\alpha|_Q \in J(\mathrm{end}(Q))$. Hence $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ j & u \end{smallmatrix}\right) \in T$ is quasipolar by Corollary 3.6. Note that $\left(\begin{smallmatrix} 0 & j \\ 1 & u \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & u \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 0 & 1 \\ j & u \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 1 \\ 1 & u \end{smallmatrix}\right)$. In view of Lemma 2.1(1), $\left(\begin{smallmatrix} 0 & j \\ 1 & u \end{smallmatrix}\right)$ is quasipolar. $\square$

4. Special cases

Commutative local rings are well-known examples of bleached and co-bleached local rings. In this section, over a commutative local ring, we investigate the quasipolarity and strong cleanness of a 2×2 matrix ring. For a commutative ring R, the notations $\mathrm{det}A$ and $\mathrm{tr}A$ denote the determinant and the trace of a square matrix A over R, respectively.

Lemma 4.1. *Let R be a commutative local ring and $A \in M_2(R)$. Then the following are equivalent*:

(1) $A \in (M_2(R))^{\mathrm{qnil}}$.

(2) $\mathrm{det}A \in J(R)$ *and* $\mathrm{tr}A \in J(R)$.

(3) $A^2 \in J(M_2(R))$.

Proof. (1) $\Rightarrow$ (2). Since R is local, by Lemma 2.4 $\mathrm{det}A \in J(R)$. Suppose that $\mathrm{tr}A \in U(R)$. Let $Y = \left(\begin{smallmatrix} -(\mathrm{tr}A)^{-1} & 0 \\ 0 & -(\mathrm{tr}A)^{-1} \end{smallmatrix}\right)$. Then $Y \in \mathrm{comm}(A)$. But $\det(I_2 + AY) = (\mathrm{tr}A)^{-2}\mathrm{det}A \in J(R)$ implies that $I_2 + AY \notin U(M_2(R))$, this causes a contradiction since $A \in (M_2(R))^{\mathrm{qnil}}$. Thus $\mathrm{tr}A \in J(R)$.

(2) $\Rightarrow$ (3). As both $\mathrm{tr}A$ and $\mathrm{det}A$ belong to $J(R)$, by Caylay-Hamilton Theorem, we have $A^2 = \mathrm{tr}A \cdot A - \mathrm{det}A \cdot I_2 \in J(M_2(R))$.

(3) $\Rightarrow$ (1). It is not difficult to check that any element of a ring nilpotent modulo its Jacobson radical, is quasinilpotent. So the result follows. $\square$

Based on Lemma 2.2(2) and Lemma 4.1, we have the following result.

Corollary 4.2 ([6, Theorem 2.6]). *Let R be a commutative local ring and let $A \in M_2(R)$ with* $\mathrm{det}A \in J(R)$. *Then* $\mathrm{tr}A \in J(R)$ *if and only if A is quasipolar and $A^{\pi} = I_2$.*

Theorem 4.3. *Let R be a commutative local ring. The following are equivalent*:

(1) $M_2(R)$ *is quasipolar.*

(2) *For any $j \in J(R)$ and $u \in U(R)$, the equation $x^2 - ux + j = 0$ is solvable in R.*

(3) *For any $j \in J(R)$, the equation $x^2 - x + j = 0$ is solvable in R.*

(4) *For any $A \in M_2(R)$ with* $\mathrm{det}A \in J(R)$ *and $A^2 \notin J(M_2(R))$, the equation* $x^2 - (\mathrm{tr}A)x + \mathrm{det}A = 0$ *is solvable.*

Proof. Note that for any $u \in U(R)$ and $j \in J(R)$, $\left(\begin{smallmatrix} 0 & j \\ 1 & u \end{smallmatrix}\right) \notin (M_2(R))^{\mathrm{qnil}}$ by Lemma 4.1. Then (1) $\Leftrightarrow$ (2) by Theorem 3.7, and (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (2). For any $j \in J(R)$ and $u \in U(R)$, consider the equation $z^2 - z + \frac{j}{u^2} = 0$. Note that $\frac{j}{u^2} \in J(R)$. Assume that $z_0 \in R$ is a solution of the above equation. It is easy to see that uz_0 is a root of the equation $x^2 - ux + j = 0$.

(1) $\Leftrightarrow$ (4). Let $A \in M_2(R)$ with $\det A \in J(R)$. By Lemma 4.1, $A \notin (M_2(R))^{\mathrm{qnil}}$ if and only if $\mathrm{tr}A \in U(R)$ if and only if $A^2 \notin J(M_2(R))$. So the result follows from [6, Proposition 2.8] and Theorem 3.7. $\square$

According to [13, Theorem 2.8], over a commutative local ring R, $M_2(R)$ is strongly clean if and only if for any $j \in J(R)$ and $u \in U(R)$, the equation $x^2 - ux + j = 0$ is solvable in R. So we get the following result immediately.

Corollary 4.4. *Let R be a commutative local ring. Then $M_2(R)$ is strongly clean if and only if $M_2(R)$ is quasipolar.*

For a commutative local ring R, there exists a strongly clean matrix $A \in M_2(R)$ which is not quasipolar (see [6, Example 3.2]). Now combining the above results with results in [6, 13], we observe the following facts.

Remark 4.5. Let R be a commutative local ring and $A \in M_2(R)$. The following hold:

(I) If $\det A \in U(R)$, then A is strongly clean and quasipolar.

(II) $\det A \in J(R)$.

(1) $\det(A - I_2) \in U(R)$:

(i) If $\mathrm{tr}A \in J(R)$, then A is strongly clean and quasipolar.

(ii) If $\mathrm{tr}A \in U(R)$, then A is strongly clean, and A is quasipolar if and only if the equation $x^2 - (\mathrm{tr}A)x + \det A = 0$ is solvable in R.

(2) $\det(A - I_2) \in J(R)$, then A is strongly clean if and only if A is quasipolar if and only if the equation $x^2 - (\mathrm{tr}A)x + \det A = 0$ is solvable in R.

Proposition 4.6. (1) *A direct product $\prod_i R_i$ is quasipolar if and only if each ring R_i is quasipolar.*

(2) *If R is local ring and C_2 is the group of order* 2, *then RC_2 is quasipolar.*

Proof. (1) This is obvious.

(2) If $2 \in J(R)$, then RC_2 is local by [14, Theorem] since C_2 is a 2-group. If $2 \in U(R)$, then $RC_2 \cong R \oplus R$ by [9, Proposition 3]. Since R is a quasipolar ring, so is $R \oplus R$ by (1). Hence RC_2 is quasipolar for all cases. $\square$

Proposition 4.7. *Let R be a commutative local ring. The following are equivalent*:

(1) *$M_2(R)$ is quasipolar.*

(2) *$M_2(R[[x]])$ is quasipolar.*

(3) *For any $n \geq 1$, $M_2(R[x]/(x^n))$ is quasipolar.*

(4) *$M_2(RC_2)$ is quasipolar.*

Proof. Note that R, $R[[x]]$ and $R[x]/(x^n)$ are all commutative local rings. According to [3, Theorem 9], if one of $M_2(R)$, $M_2(R[[x]]$ and $M_2(R[x]/(x^n))$ is strongly clean, then they all do. By Corollary 4.4, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

Next we show that (1) $\Leftrightarrow$ (4). If $2 \in J(R)$, then by [3, Theorem 12], $M_2(R)$ is strongly clean if and only if so is $M_2(RC_2)$. Due to Corollary 4.4, the result holds for this case. If $2 \in U(R)$, then $RC_2 \cong R \oplus R$. Thus, $M_2(RC_2) \cong M_2(R) \oplus M_2(R)$. By Proposition 4.6(1), we are done. $\square$

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