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HARMONIC MAPPINGS RELATED TO FUNCTIONS WITH BOUNDED BOUNDARY ROTATION AND NORM OF THE PRE-SCHWARZIAN DERIVATIVE

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Abstract. Let $\mathcal{S}_{\mathcal{H}}^0$ be the class of normalized univalent harmonic mappings in the unit disk. A subclass $\mathcal{V}^{\mathcal{H}}(k)$ of $\mathcal{S}_{\mathcal{H}}^0$, whose analytic part is function with bounded boundary rotation, is introduced. Some bounds for functionals, specially harmonic pre-Schwarzian derivative, described in $\mathcal{V}^{\mathcal{H}}(k)$ are given.

1. Introduction

A harmonic mapping f of the simply connected region Ω is a complex-valued function of the form

$$f = h + \overline{g}, \tag{1.1}$$

where h and g are analytic in Ω, with $g(z_0) = 0$ for some prescribed point $z_0 \in \Omega$. We call h and g analytic and co-analytic parts of f, respectively. If f is (locally) injective, then f is called (locally) univalent. The Jacobian and second complex dilatation of f are given by $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 = |h'(z)|^2 - |g'(z)|^2$ and $\omega(z) = g'(z)/h'(z)$ $(z \in \Omega)$, respectively. A result of Lewy [18] states that f is locally univalent if and only if its Jacobian is never zero, and is sense-preserving if the Jacobian is positive. The sense-preserving case implies $|\omega(z)| < 1$ in $\mathbb{D}$.

Throughout this paper we will assume that f is locally univalent, sense-preserving, and $\Omega = \mathbb{D} \subset \mathbb{C}$, with $z_0 = 0$, where $\mathbb{D}$ is the open unit disk on the complex plane. Following Clunie and Sheil-Small notation [6], the class of all sense-preserving univalent harmonic mappings of $\mathbb{D}$ with $h(0) = g(0) = h'(0) - 1 = 0$ we denote $\mathcal{S}_{\mathcal{H}}$, and its subclass for which $g'(0) = 0$ by $\mathcal{S}_{\mathcal{H}}^0$. Several fundamental information about harmonic mappings in the plane can be found in e.g. [8]. We note that each f satisfying (1.1) in $\mathbb{D}$ is uniquely determined by

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coefficients of the following power series expansions

$$h(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad (z \in \mathbb{D}), \tag{1.2}$$

where $a_n \in \mathbb{C}$, $n = 0, 1, 2, \dots$ and $b_n \in \mathbb{C}$, $n = 1, 2, 3, \dots$. Moreover, when $f \in \mathcal{S}_{\mathcal{H}}$ we have $a_0 = 0, a_1 = 1$. In the sequel, we assume also $g'(0) = b_1$ with $|b_1| = \alpha$. Taking into account the condition $|\omega(z)| < 1$, immediately obtains $0 \le \alpha < 1$. Let

$$\omega(z) = c_0 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{D}). \tag{1.3}$$

According to the relation $\omega = g'/h'$ we have $c_0 = b_1$ with $|b_1| = \alpha$. Therefore, we have

$$\frac{|r-\alpha|}{1-\alpha r} \le |\omega(z)| \le \frac{r+\alpha}{1+\alpha r}, \tag{1.4}$$

and

$$|c_n| \le 1 - |c_0|^2, \quad |\omega'(z)| \le \frac{1-|\omega(z)|^2}{1-|z|^2} \quad (z \in \mathbb{D}). \tag{1.5}$$

see e.g. [4, p. 30, 53].

Let $\mathcal{V}(k)$ denote the class of bounded boundary rotation, i.e., a class of normalized functions f such that

$$\int_0^{2\pi} \left| \mathrm{Re} \left(1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta}} \right) \right| d\theta \le k\pi, \tag{1.6}$$

see [17]. We note that $\mathcal{V}(k_1) \subset \mathcal{V}(k_2)$ for $k_1 < k_2$. We assume $k \ge 2$ and if $k \le 4$, then f is close-to-convex (the converse is not true). Let

$$\begin{aligned} F_k(z) &= \frac{1}{k} \left[\left(\frac{1+z}{1-z} \right)^{k/2} - 1 \right] = \sum_{n=1}^{\infty} B_n(k) z^n \\ &= z + \frac{k}{2} z^2 + \frac{k^2+2}{6} z^3 + \cdots, \end{aligned} \tag{1.7}$$

$z \in \mathbb{D}$. Then the following estimates holds.

Theorem 1.1 ([11], t.II, pp. 16–25)**.** *If $f \in \mathcal{V}(k)$, then*

$$|f(z)| \le F_k(r), \quad F_k'(-r) \le |f'(z)| \le F_k'(r), \tag{1.8}$$

and

$$|a_n| \le B_n(k). \tag{1.9}$$

In the present paper we introduce the concept of planar harmonic mappings with the analytic part being a function with bounded boundary rotation.

Definition 1.2. By $\mathcal{V}^{\mathcal{H}}(k)$ we denote the subclass of $\mathcal{S}_{\mathcal{H}}$ consisting of all harmonic mappings of the form $f = h + \bar{g}$ for which $h \in \mathcal{V}(k)$, with normalization $h(0) = g(0) = h'(0) - 1 = 0$ and $g'(0) = b_1, |b_1| = \alpha$.

The classes of functions $f = h + \bar{g}$ with fixed analytic part were studied previously in the literature. We remain, for example, papers [13], [14], [15]; in [14] authors studied properties of a subset $\bar{\mathcal{S}}_{\mathcal{H}}^{\alpha}$ of $\mathcal{S}_{\mathcal{H}}$ consisting of all univalent anti-analytic perturbations of the identity whereas in [15] the class $\widehat{\mathcal{S}}^{\alpha}$ of all $f \in \mathcal{S}_{\mathcal{H}}$, such that h is convex, normalized univalent functions.

2. Coefficient and distortion results

Theorem 2.1. *Let f be of the form* (1.1) *with the Taylor expansions* (1.2) *and $f \in \mathcal{V}^{\mathcal{H}}(k)$. Then*

$$|b_n| \le \alpha B_n(k) + \frac{1-\alpha^2}{n} \sum_{p=1}^{n-1} pB_p(k), \tag{2.1}$$

where $B_n(k)$ are given by (1.7)*. Specially, we have*

$$|b_2| \le \frac{1-\alpha^2+\alpha k}{2}, \quad |b_3| \le \frac{(1-\alpha^2)(1+k)}{2} + \frac{\alpha(k^2+2)}{6}. \tag{2.2}$$

The result is sharp only for the case $n = 2$.

Proof. By the relation $g' = \omega h'$ we have

$$nb_n = \sum_{p=0}^{n-1} (p+1)a_{p+1}c_{n-p-1} = a_n c_0 + \sum_{p=1}^{n-1} pa_p c_{n-p}. \tag{2.3}$$

Observing that $c_0 = b_1$ so that $|c_0| = |b_1| = \alpha$, and making use of (1.5), and $|a_n| \le B_n(k)$ the assertion immediately follows. We note, that the bounds in (2.1) is sharp only for the case $n = 2$. Indeed, defining $\omega(z) = \alpha + (1-\alpha)z$, $h(z) = F_k(z)$, and applying the relation $g' = \omega h'$ with $g(0) = 0$ by integration we have

$$g(z) = \alpha F_k(z) + (1-\alpha)zF_k(z) - (1-\alpha)\int_0^z F_k(w)dw.$$

In a such case $b_2 = g''(0)/2 = (1-\alpha^2+\alpha k)/2$, that realizes equality for b_2 in (2.2). □

Remark 2.1. The reasoning used in a proof of Theorem 2.1 may be applied to the bounds of coefficients of any harmonic functions $f = h + \bar{g}$, with an assumption $|g'(0)| = \alpha$ and such that the coefficients of the analytic part h satisfy $|a_n| \le B_n$ for $n \ge 1$ (here $h(z) = z + a_2 z^2 + \cdots$). Such approach is also presented in [12].

Theorem 2.2. *Let $f \in \mathcal{V}^{\mathcal{H}}(k)$. Then*

$$\frac{|r-\alpha|}{(1-\alpha r)(1-r^2)}\left(\frac{1-r}{1+r}\right)^{k/2} \le |g'(z)| \le \frac{(r+\alpha)}{(1+\alpha r)(1-r^2)}\left(\frac{1+r}{1-r}\right)^{k/2}, \tag{2.4}$$

$$(2.5)\qquad |g(z)| \leq \frac{\alpha+r}{k(1+\alpha r)}\left(\frac{1-r}{1+r}\right)^{k/2} - \frac{1-\alpha^2}{k}\int_0^r \left(\frac{1+t}{1-t}\right)^{k/2}\frac{dr}{(1-\alpha t)^2},$$

$$(2.6)\qquad |g(z)| \geq \left|\frac{\alpha-r}{k(1-\alpha r)}\left(\frac{1-r}{1+r}\right)^{k/2} + \frac{1-\alpha^2}{k}\int_0^r \left(\frac{1-t}{1+t}\right)^{k/2}\frac{dt}{(1-\alpha t)^2}\right|,$$

(2.7)

$$|f(z)| \leq F_k(r) + \frac{\alpha+r}{k(1+\alpha r)}\left(\frac{1-r}{1+r}\right)^{k/2} - \frac{1-\alpha^2}{k}\int_0^r \left(\frac{1+t}{1-t}\right)^{k/2}\frac{dr}{(1-\alpha t)^2}.$$

Proof. By the relation $g' = \omega h'$ we have $|g'(z)| = |\omega(z)||h'(z)|$. The assertion (2.4) now follows by (1.4) and (1.8).

We note that if φ is univalent, and $m'(r) \leq |\varphi'(z)| \leq M'(r)$ $(0 \leq |z| = r < 1)$ then $\int_0^r m'(r)dr \leq |\varphi(z)| \leq \int_0^r M'(r)dr$. Applying this together with (2.4) we obtain (2.5) and (2.6). □

Remark 2.2. The bounds in (2.5), (2.6) and (2.7) may be represented by the Appell hypergeometric function $F_1(a; b_1, b_2; c; x, y)$, of two real variables x and y, when we apply the following

$$\int_0^r \left(\frac{1+t}{1-t}\right)^{k/2}\frac{dt}{(1-\alpha t)^2}$$
$$= \frac{(-1)^{k/2}}{\alpha}\left[\frac{1}{1-\alpha r}F_1\left(1; \frac{k}{2}, \frac{-k}{2}, 2; \frac{1-\alpha}{1-\alpha r}, \frac{1+\alpha}{1-\alpha r}\right) - F_1\left(1; \frac{k}{2}, \frac{-k}{2}, 2; 1-\alpha, 1+\alpha\right)\right].$$

Remark 2.3. For the special case $k = 1$ we recover from Theorem 2.2 the distortion theorem for the harmonic mappings with analytic part being convex and univalent function (compare [15]).

3. Pre-Schwarzian derivative

The Schwarzian S_f and pre-Schwarzian T_f derivatives of a holomorphic and locally univalent function f is defined by

$$(3.1)\qquad S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = (T_f)' - (T_f)^2/2.$$

The Schwarzian derivative is a basic tool in complex analysis; it measures the deviation of f from a Möbius transformation. The hyperbolic sup-norm of S_f (T_f, respectively) is introduced as follows

$$||S_f|| = \sup_{z\in\mathbb{D}}(1-|z|^2)^2|S_f(z)|, \quad ||T_f|| = \sup_{z\in\mathbb{D}}(1-|z|^2)|T_f(z)|.$$

Both, the Schwarzian and pre-Schwarzian play a central role in the theory of Teichmüller spaces, inner radius of univalence, quasiconformal extension, etc. We quote here one of the most important results.

Theorem 3.1 (Ahlfors-Weill [1], see also Krauss [16], Nehari [19], Duren and Lehto [9])**.** *Let f be the function normalized and analytic in the unit disk. If f is univalent, then $\|S_f\| \leq 6$. Conversely, if $||S_f|| \leq 2$, then f is univalent. Moreover, let $0 \leq k < 1$. If f extends to a k-quasiconformal mapping of the Riemann sphere $\bar{\mathbb{C}}$ then $||S_f|| \leq 6k$. Conversely, if $\|S_f\| \leq 2k$, then f extends to a k-quasiconformal mapping of $\bar{\mathbb{C}}$.*

The first problem in the theory of locally univalent harmonic mappings, is to find a suitable definition of the Schwarzian derivative (the pre-Schwarzian derivative, respectively). A natural definition, using the differential geometry of associated minimal surface, has been proposed by Chuaqui, Duren and Osgood [5], and described by the formula

$$S_f = 2\frac{\partial^2(\log\lambda)}{\partial z^2} - 2\left(\frac{\partial(\log\lambda)}{\partial z}\right)^2, \tag{3.2}$$

where $\lambda = |h'| + |g'|$. In the case, when f is analytic, $\lambda = |f'|$, so that $\log\lambda = \log f'/2 + \log \bar{f}'/2$, therefore (3.2) agrees with the classical formula (3.1). In connection with harmonic Schwarzian derivative we define harmonic pre-Schwarzian as

$$T_f = \frac{2\partial(\log\lambda)}{\partial z} \tag{3.3}$$

which, in the analytic case becomes f''/f', as in (3.1).

We observe that, if $g' = \omega h'$, then [8, p. 191]

$$S_f = S_h + \frac{2\bar{\omega}}{1+|\omega|^2}\left(\omega'' - \frac{\omega' h''}{h'}\right) - 4\left(\frac{\omega'\bar{\omega}}{1+|\omega|^2}\right)^2, \tag{3.4}$$

and

$$T_f = \frac{2\partial(\log\lambda)}{\partial z} = \frac{h''}{h'} + \frac{2\omega'\bar{\omega}}{1+|\omega|^2} = T_h + \frac{2\omega'\bar{\omega}}{1+|\omega|^2}. \tag{3.5}$$

Also, note that

$$S_{f\circ\varphi} = (S_f \circ \varphi)\,\varphi'^2 + S_\varphi, \quad T_{f\circ\varphi} = T_f \circ \varphi + T_\varphi.$$

The above formulas are generalization of the classical transformation formula for Schwarzian and pre-Schwarzian under composition.

In this section we find bounds of the norm of pre-Schwarzian derivative for co-analytic part of harmonic mapping from $\mathcal{S}_{\mathcal{H}}^{\alpha}$ and $\mathcal{V}^{\mathcal{H}}(k)$.

Before we prove the next theorems we remain some fact about the cardinals of polynomials roots. The best known is the classical rule of Descartes-Harriot, but it is not sufficient for computing the number of roots over a given interval. This problem was solved by Sturm, however less known but efficient method was presented by Vincent [20], using continued fractions. The modified Vincent's theorem e.g. its bisection version due to Alesina and Galuzzi [2] was presented after almost 200 years, in 2000. This method was next implemented by Vincent-Akritas-Strzeboński [3] and the continued fractions method for the determining the real zeros turns out to be the fastest method derived from Vincent's theorem.

Theorem 3.2 (Vincent, [20], bisection version (2000), [2], [3])**.** *Let $p(x)$ be a polynomial of degree n. There exists a positive quantity δ so that for every pair of positive rational numbers a, b with $|b-a| < \delta$ every transformed polynomial of the form*

$$V(x) = (1+x)^n p\left(\frac{a+bx}{1+x}\right)$$

has exactly 0 or 1 variations in the sequence of its coefficients. The second case is possible if and only if $p(x)$ has a simple root within (a, b). Moreover, the number of the sign variation is the maximal number of roots in (a, b).

Theorem 3.3. *The norm of the harmonic pre-Schwarzian derivative in the class $\mathcal{S}_{\mathcal{H}}^{\alpha}$ is bounded by*

$$||T_f|| \leq \; 2(1-\alpha^2)\frac{(1-r_0^2)(\alpha+r_0)}{(1+\alpha r_0)[(1+\alpha^2)(1+r_0^2)-4\alpha r_0]}, \tag{3.6}$$

where r_0 is the only root from the interval $(0,1)$ of the equation

$$r_0^4(\alpha^4+4\alpha^2-1)+4\alpha r_0^3(1-\alpha^2)-4r_0^2(\alpha^4+1)+4r_0\alpha(\alpha^2-1)+1+4\alpha^2-\alpha^4 = 0. \tag{3.7}$$

Proof. Since $h(z) \equiv z$ then $T_h \equiv 0$ so that

$$T_f = \frac{2\omega'\bar{\omega}}{1+|\omega|^2}.$$

Making use estimates (1.4), (1.5), we obtain for $|z| = r < 1$

$$|T_f| \leq \frac{2(1-\alpha^2)(\alpha+r)}{(1+\alpha r)[(1+\alpha^2)(1+r^2)-4\alpha r]}, \tag{3.8}$$

so that

$$||T_f|| \leq 2(1-\alpha^2) \sup_{0<r<1} \frac{(1-r^2)(\alpha+r)}{(1+\alpha r)[(1+\alpha^2)(1+r^2)-4\alpha r]}. \tag{3.9}$$

The derivative of the function

$$G(r) := \frac{(1-r^2)(\alpha+r)}{(1+\alpha r)[(1+\alpha^2)(1+r^2)-4\alpha r]}$$

is zero, if the function H, given by

$$H(r) := r^4(\alpha^4+4\alpha^2-1)+4\alpha r^3(1-\alpha^2)-4r^2(\alpha^4+1)+4r\alpha(\alpha^2-1)+1+4\alpha^2-\alpha^4$$

takes its zero for $0 < r < 1$. Note, that $H(0) = 1+4\alpha^2-\alpha^4 > 0$ and $H(1^-) = -4(\alpha^2-1)^2 < 0$ then there exists $r_0 \in (0,1)$ such, that $H(r_0) = 0$. We prove the such root of H is unique on $(0,1)$. It is enough to prove that the derivative

$$H'(r) = 4r^3(\alpha^4+4\alpha^2-1)+12\alpha r^2(1-\alpha^2)-8r(\alpha^4+1)+4\alpha(\alpha^2-1)$$

is negative for $0 < r < 1$ and $0 < \alpha < 1$. Fix now r, and let $L(\alpha) = H'(r)$ for $0 < \alpha < 1$ and $0 < r < 1$, so that

$$L(\alpha) = (-8r + 4r^3)\alpha^4 + (4 - 12r^2)\alpha^3 + 16r^3\alpha^2 + (-4 + 12r^2)\alpha + (-8r - 4r^3).$$

We note that $L(0) = -8r - 4r^3$ and $L(1) = 16r(r^2 - 1) < 0$. Next, define

$$V(x) = (1 + x)^4 L\left(\frac{x}{1 + x}\right).$$

Then, we have

$$\begin{aligned} V(x) = & -4r(2 + r^2) - 4(1 + 8r - 3r^2 + 4r^3)x \\ & -4(1 + 8r - 3r^2 + 4r^3)x^2 \\ & -8(1 - r)(1 + 5r + 2r^2)x^3 - 16r(1 - r^2)x^4. \end{aligned}$$

It is easy to check that the sign of the sequence of coefficient of $V(x)$ has the form $(-, -, -, -, -)$. Therefore there is no sign variation on $(0, 1)$ for every $r \in (0, 1)$. By the Vincent theorem we then conclude that there is no zeros of polynomial $L(\alpha)$ in the interval $(0, 1)$. Since the function L start from the negative value $L(0)$, therefore it must be negative in the entire interval, that implies the negativity of H'.

Hence, H is decreasing for $0 < r < 1$ and $0 < \alpha < 1$, and the equation (3.7) has the only root on the interval $(0, 1)$, which is the only maximum of G on $(0, 1)$. □

Theorem 3.4. *The norm of pre-Schwarzian derivative in the class* $\mathcal{V}^{\mathcal{H}}(k)$ *is bounded by*

$$||T_f|| \leq k + 2r_0 + \frac{2(1 - \alpha^2)(\alpha + r_0)}{(1 + \alpha r_0)[(1 + \alpha^2)(1 + r_0^2) - 4\alpha r_0]}, \tag{3.10}$$

where r_0 *is the only root of the equation*

$$\begin{aligned} & 2 + 5\alpha^2 - 4\alpha^4 + \alpha^6 - 2r\alpha(5 - 2\alpha^2 + \alpha^4) + r^2(-2 + 9\alpha^2 - 16\alpha^4 + 5\alpha^6) \\ & +16r^3\alpha^3 + r^4\alpha^2(-7 + 2\alpha^2 + \alpha^4) + r^5(2\alpha - 4\alpha^3 - 6\alpha^5) + r^6(\alpha + \alpha^3)^2 \end{aligned} \tag{3.11}$$

on the interval $(0, 1)$.

Proof. Since $h \in \mathcal{V}(k)$ then $|T_h| \leq (k + 2r)/(1 - r^2), |z| = r < 1$, and

$$|T_f| = \left|T_h + \frac{2\omega'\bar{\omega}}{1 + |\omega|^2}\right| \leq |T_h| + \left|\frac{2\omega'\bar{\omega}}{1 + |\omega|^2}\right|.$$

Therefore

$$||T_f|| \leq \sup_{0<r<1} (1 - r^2)\left[\frac{k + 2r}{1 - r^2} + \frac{2(1 - \alpha^2)(\alpha + r)}{(1 + \alpha r)[(1 + \alpha^2)(1 + r^2) - 4\alpha r]}\right].$$

The derivative of the right hand function is equal to zero, if

$$\begin{aligned} & 2 + 5\alpha^2 - 4\alpha^4 + \alpha^6 - 2r\alpha(5 - 2\alpha^2 + \alpha^4) + r^2(-2 + 9\alpha^2 - 16\alpha^4 + 5\alpha^6) \\ & +16r^3\alpha^3 + r^4\alpha^2(-7 + 2\alpha^2 + \alpha^4) + r^5(2\alpha - 4\alpha^3 - 6\alpha^5) + r^6(\alpha + \alpha^3)^2 = 0 \end{aligned}$$

for $r \in (0,1)$. Denote the last polynomial by $P(r)$. Then

$$P(0) = 2 + 5\alpha^2 - 4\alpha^4 + \alpha^6 = 2 + 4\alpha^2(1-\alpha^2) + \alpha^2 + \alpha^6 > 0,$$

and

$$P(1) = -8\alpha + 8\alpha^2 + 16\alpha^3 - 16\alpha^4 - 8\alpha^5 + 8\alpha^6 = -8\alpha(1-\alpha)(1-\alpha^2)^2 < 0,$$

so that there exists $r_0 \in (0,1)$, such that $P(r_0) = 0$. It suffices to prove that r_0 is unique. To claim this we prove that $P' < 0$ for $r \in (0,1)$. Let $r \in (0,1)$ be now fixed, and denote by $Q(\alpha)$ the derivative $P'(r)$, that is

$$\begin{aligned} Q(\alpha) = & -4r - 10(1-r^4)\alpha + 2r(9 - 14r^2 + 3r^4)\alpha^2 \\ & + (4 + 48r^2 - 20r^4)\alpha^3 + (-32r + 8r^3 + 12r^5)\alpha^4 \\ & + (-2 - 30r^4)\alpha^5 + (10r + 4r^3 + 6r^5)\alpha^6. \end{aligned}$$

We have $Q(0) = -4r < 0$ and $Q(1) = 8(r-1)^3(1 + 4r + 3r^2) < 0$.

Let S_0 be the number of sign variation in the sequence of coefficients of the polynomial $Q(\alpha)$. Denoting the coefficients of $Q(\alpha)$ by $a_0, \ldots, a_6$ we have

$$\begin{aligned} a_0 &= -4r < 0, \\ a_1 &= -10 + 10r^4 < 0, \\ a_2 &= 2r(9 - 14r^2 + 3r^4) < 0, \text{ or } > 0, \\ a_3 &= 4 + 48r^2 - 20r^4 > 0, \\ a_4 &= 4r(-8 + 2r^2 + 3r^4) < 0, \\ a_5 &= -2(1 + 15r^4) < 0, \\ a_6 &= 10r + 4r^3 + 6r^5 > 0. \end{aligned}$$

Hence the sequence of the sign of the coefficients $a_i (i = 0, \ldots, 6)$ is $(-,-,\pm,+,-,-,+)$, so that the number of sign changes is $S_0 = 3$ for any $r \in (0,1)$. It means, by the classical rule of Descartes-Harriot, that there are 3 or 1 positive roots of $Q(\alpha)$. In order to show that there is no zero in $(0,1)$ we use the Vincent Theorem. The function

$$V(\alpha) = (1+\alpha)^6 Q\left(\frac{\alpha}{1+\alpha}\right)$$

has the following coefficients

$$\begin{aligned} b_0 &= -4r < 0, \\ b_1 &= -10(1-r^4) - 24r < 0, \\ b_2 &= -2[(1-r^4)(25+3r) + 18r + 14r^3] < 0, \\ b_3 &= -8[3(1-r^2)(2+r^3) + 10r^3(1-r1) + 6 + r + r^3] < 0, \\ b_4 &= 8(r-1)(2 + 9r + 9(1-r^2) + 11r^3 + 6r^4) < 0, \\ b_5 &= -8(1-r)[1 + 4(1-r^2) + 7r(1-r) + r^3 + 6r^4] < 0, \end{aligned}$$

$$b_6 = 8(r-1)^3(1+r)(1+3r) < 0,$$

that form the following sequence of sign $(-,-,-,-,-,-)$ with no sign variations. Thus, by the Vincent Theorem, there are no zeros at $(0,1)$ for any $r \in (0,1)$. It means that $Q(\alpha) < 0$ for every $\alpha \in (0,1)$ and $r \in (0,1)$, equivalently $P'(r) < 0$ in $(0,1)$ that ends the proof. □

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