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APPROXIMATION METHODS FOR A COMMON MINIMUM-NORM POINT OF A SOLUTION OF VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS IN BANACH SPACES

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Abstract. We introduce an iterative process which converges strongly to a common minimum-norm point of solutions of variational inequality problem for a monotone mapping and fixed points of a finite family of relatively nonexpansive mappings in Banach spaces. Our theorems improve most of the results that have been proved for this important class of nonlinear operators.

1. Introduction

Let E be a real Banach space with dual E^*. We denote by J the normalized duality mapping from E into 2^{E^*} defined for each $x \in E$ by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 = ||f^*||^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between members of E and E^*. It is well known that E is smooth if and only if J is single-valued and if E is uniformly smooth, then J is uniformly continuous on bounded subsets of E. Moreover, if E is a reflexive and strictly convex real Banach space with a strictly convex dual, then J^{-1} is single valued, one-to-one, surjective, and it is the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$ (see [17]). If $E = H$, a real Hilbert space, then the duality mapping becomes the identity map on H.

Let E be a smooth real Banach space with dual E^*. Let the Lyapunov functional $\phi : E \times E \to \mathbb{R}$, introduced by Alber [1], be defined by

$$\phi(y, x) = ||y||^2 - 2\langle y, Jx \rangle + ||x||^2 \quad \text{for } x, y \in E, \tag{1.1}$$

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where J is the normalized duality mapping from E into 2^{E^*}. It is obvious from the definition of the function ϕ that

$$(||x|| - ||y||)^2 \leq \phi(x,y) \leq (||x|| + ||y||)^2 \quad \text{for } x,y \in E. \tag{1.2}$$

We observe that in a Hilbert space H, (1.1) reduces to $\phi(x,y) = ||x-y||^2$ for $x,y \in H$.

Let E be a reflexive, strictly convex and smooth Banach space and let C be a nonempty, closed and convex subset of E. The *generalized projection mapping*, introduced by Alber [1], is a mapping $\Pi_C : E \to C$ that assigns an arbitrary point $x \in E$ to the minimizer, $\bar{x}$, of $\phi(\cdot, x)$ over C, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min\{\phi(y,x), y \in C\}. \tag{1.3}$$

If E is a Hilbert space, then $\Pi_C = P_C$ is the metric projection of H onto C.

In fact, we have the following result.

Lemma 1.1 ([1]). *Let C be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then there exists a unique element $x_0 \in C$ such that $\phi(x_0, x) = \min\{\phi(z,x) : z \in C\}$.*

Let C be a nonempty subset of a real Banach space E with dual E^*. A mapping $A : C \to E^*$ is said to be *monotone* if for each $x,y \in C$, the following inequality holds:

$$\langle x-y, Ax-Ay\rangle \geq 0. \tag{1.4}$$

A is said to be γ*-inverse strongly monotone* if there exists a positive real number γ such that

$$\langle x-y, Ax-Ay\rangle \geq \gamma||Ax-Ay||^2 \quad \text{for all } x,y \in C. \tag{1.5}$$

If A is γ-inverse strongly monotone, then it is *Lipschitz continuous* with constant $\frac{1}{\gamma}$, i.e., $||Ax-Ay|| \leq \frac{1}{\gamma}||x-y||$ for all $x,y \in C$, and it is called *strongly monotone* if there exists $k > 0$ such that

$$\langle x-y, Ax-Ay\rangle \geq k||x-y||^2 \quad \text{for all } x,y \in C. \tag{1.6}$$

Clearly, the class of monotone mappings includes the class of strongly monotone and the class of γ-inverse strongly monotone mappings.

Suppose that A is a monotone mapping from C into E^*. The variational inequality problem is formulated as finding

$$\text{a point } u \in C \text{ such that } \langle v-u, Au\rangle \geq 0 \quad \text{for all } v \in C. \tag{1.7}$$

The set of solutions of the variational inequality problem is denoted by $VI(C,A)$.

Variational inequality problems are related to the convex minimization problems, the zero of monotone mappings and the complementarity problems. Consequently, many researchers (see, eg, [4, 8, 9, 11, 20, 21]) have made efforts to obtain iterative methods for approximating solutions of variational inequality problems in the setting of Hilbert spaces or Banach spaces.

If $E = H$, a real Hilbert space, Iiduka, Takahashi and Toyoda [6] introduced the following projection algorithm:

$$x_0 = w \in C,\ x_{n+1} = P_C(x_n - \alpha_n A x_n) \text{ for any } n \geq 0, \tag{1.8}$$

where P_C is the metric projection from H onto C and $\{\alpha_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.8) *converges weakly* to an element of $VI(C, A)$ provided that A is a γ-inverse strongly monotone mapping.

When E is a 2-uniformly convex and uniformly smooth Banach space, Iiduka and Takahashi [5] introduced the following iteration scheme for finding a solution of the variational inequality problem for a γ-inverse strongly monotone mapping A:

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \alpha_n A x_n) \text{ for any } n \geq 0, \tag{1.9}$$

where Π_C is the generalized projection from E onto C, J is the normalized duality mapping from E into E^* and $\{\alpha_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.9) *converges weakly* to an element of $VI(C, A)$ provided that $VI(C, A) \neq \emptyset$ and A satisfies $||Ax|| \leq ||Ax - Ap||$ for all $x \in C$ and $p \in VI(C, A)$.

We note that the convergence results obtained above are weak convergence. To obtain strong convergence, Iiduka and Takahashi [4], studied the following iterative scheme, in a 2-uniformly convex and uniformly smooth Banach space E, for a variational inequality problem for a γ-inverse strongly monotone mapping A:

$$\begin{cases} x_0 = w \in K, \text{ chosen arbitrary,} \\ y_n = \Pi_C J^{-1}(Jx_n - \alpha_n A x_n) \\ C_n = \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in E : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), n \geq 1, \text{ for } n \geq 0, \end{cases} \tag{1.10}$$

where $\Pi_{C_n \cap Q_n}$ is the generalized projection from E onto $C_n \cap Q_n$, J is the normalized duality mapping from E into E^* and $\{\alpha_n\}$ is a positive real sequence satisfying certain conditions. They proved that the sequence $\{x_n\}$ converges strongly to an element of $A^{-1}(0)$.

Remark 1.2. We remark that the computation of x_{n+1} in Algorithms (1.10) requires the computations of C_n, Q_n and $C_n \cap Q_n$ for each $n \geq 1$.

Remark 1.3. We note that, as it is mentioned in [24], if C is a subset of a real Banach space E and $A : C \to E^*$ is a monotone mapping satisfying $||Ax|| \leq ||Ax - Ap||$, $\forall x \in C$ and $p \in VI(C, A)$, then $VI(C, A) = A^{-1}(0) = \{p \in C : Ap = 0\}$.

Let T be a mapping from C into itself. We denote by $F(T)$ the fixed points set of T. A point p in C is said to be an *asymptotic fixed point of* T (see [14]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that

$\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is said to be *nonexpansive* if $||Tx - Ty|| \leq ||x - y||$ for each $x, y \in C$, and is called *relatively nonexpansive* if (R1) $F(T) \neq \emptyset$; (R2) $\phi(p, Tx) \leq \phi(p, x)$ for $x \in C$ and (R3) $F(T) = \hat{F}(T)$. If E is a uniformly smooth and uniformly convex real Banach space, and $A \subset E \times E^*$ is a maximal monotone mapping with $A^{-1}(0) \neq \emptyset$, then the resolvent $Q_r := (J + rA)^{-1}J$, for $r > 0$, is relatively nonexpansive (see [13]).

If $E = H$, a real Hilbert space, then the class of relatively nonexpansive mappings contains the class of nonexpansive mappings with $F(T) \neq \emptyset$ (see, eg, [25]).

In [3], Iduka and Takahashi studied the following iterative scheme for a common point of fixed point set of nonexpansive mapping and solution set of a variational inequality problem for a γ-inverse strongly monotone mapping A in a Hilbert space H:

$$\begin{cases} x_0 = w \in C \\ x_{n+1} = \alpha_n w + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n), \ n \geq 0, \end{cases} \tag{1.11}$$

where $\{\alpha_n\}$ is a sequences satisfying certain conditions. They proved that the sequence $\{x_n\}$ converges strongly to an element of $F := F(S) \cap VI(C, A)$ provided that $F \neq \emptyset$.

In addition, many authors have considered the problem of finding a common element of the fixed point set of relatively nonexpansive mapping and the solution set of a variational inequality problem for γ-inverse monotone mapping A (see, e.g., [10, 16, 18, 20, 21, 23]) which is nearest to the initial point $x_0 = w$.

However, we notice that it is quite often to seek a minimum-norm solution of a given nonlinear problem. A point $\bar{x} \in C$, where C is a nonempty, closed and convex subset of a real Hilbert space H, is called a *minimum-norm solution* of a nonlinear problem with solution $F \neq \emptyset$ if and only if there exists $\bar{x} \in F$ satisfying the property that

$$||\bar{x}|| = \min\{||x|| : x \in F\}, \tag{1.12}$$

that is, $\bar{x}$ is the nearest point projection of the origin onto F.

A typical example is the least-squares solution to the constrained linear inverse problem

$$\begin{cases} Ax = b, \\ x \in C, \end{cases} \tag{1.13}$$

where A is a bounded linear operator from H into another real Hilbert space H_1 and b is a given point in H_1. The least-squares solution to (1.13) is the solution of the following minimization problem with the minimum equal to zero:

$$\min_{x \in C} \frac{1}{2} ||Ax - b||^2. \tag{1.14}$$

Let Ω denote the (closed convex) solution set of (1.13) (or equivalently (1.14)). Then, in this case, Ω has a unique element $\bar{x}$ if and only if it is a solution of the following variational inequality:

(1.15) $$\bar{x} \in C \text{ such that } \langle A^*(A\bar{x} - b), x - \bar{x}\rangle \geq 0,\ x \in C,$$

where A^* is the adjoint of A. In addition, we observe that inequality (1.15) can be rewritten as

(1.16) $$\bar{x} \in C,\ \langle \bar{x} - \gamma A^*(A\bar{x} - b) - \bar{x}, x - \bar{x}\rangle \leq 0,\ x \in C,$$

where $\gamma > 0$ is any positive scalar. In the terminology of projection, we see that (1.16) is equivalent to the fixed point equation

(1.17) $$\bar{x} = P_C(\bar{x} - \gamma A^*(A\bar{x} - b)).$$

It is not hard to see that for $0 < \gamma < \frac{2}{||A||^2}$, the mapping $x \to P_C(x - \gamma A^*(Ax - b))$ is nonexpansive. Therefore, finding the least-squares solution of the constrained linear inverse problem (1.13) is equivalent to finding the minimum-norm fixed point of the nonexpansive mapping $x \to P_C(x-\gamma A^*(Ax-b))$.

Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let $T_i : C \to C$, for $i = 1, 2, \ldots, N$, be relatively nonexpansive mapping and $A : C \to E^*$ be a continuous monotone mapping with $F := \cap_{i=1}^N F(T) \cap VI(C, A) \neq \emptyset$.

It is our purpose in this paper to introduce an iterative scheme (see (3.1)) which converges strongly, to a minimum-norm (with respect to the generalized projection) point of F, that is, to a point $x^* \in F$ such that $x^* = \Pi_F(0)$. Our theorems improve most of the results that have been proved for this important class of nonlinear mappings.

2. Preliminaries

Let E be a Banach space and let $S(E) = \{x \in E : ||x|| = 1\}$. Then a Banach space E is said to be *smooth* provided that the limit

(2.1) $$\lim_{t\to 0} \frac{||x + ty|| - ||x||}{t},$$

exists for each $x, y \in S(E)$. The norm of E is said to be *uniformly smooth* if the limit (2.1) is attained uniformly for (x, y) in $S(E) \times S(E)$ (see [17]).

The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : \|x\| = \|y\| = 1;\ \epsilon = \|x - y\| \right\}.$$

E is called *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$.

In the sequel, we shall make use of the following lemmas.

Lemma 2.1 ([22]). *Let C be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space E. If $A : C \to E^*$ is continuous monotone mapping, then $VI(C, A)$ is closed and convex.*

Lemma 2.2 ([13])**.** *Let E be a strictly convex and smooth Banach space, let C be a nonempty, closed and convex subset of E, and let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Lemma 2.3 ([1])**.** *Let K be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then $\forall y \in K$,*

$$\phi(y, \Pi_K x) + \phi(\Pi_K x, x) \leq \phi(y, x).$$

Lemma 2.4 ([8])**.** *Let E be a real smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \to 0$ as $n \to \infty$.*

We make use of the function $V : E \times E^* \to \mathbb{R}$ defined by

$$V(x, x^*) = ||x||^2 - 2\langle x, x^*\rangle + ||x||^2 \text{ for all } x \in E \text{ and } x^* \in E,$$

studied by Alber [1]. That is, $V(x, y) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$. We know the following lemma.

Lemma 2.5 ([1])**.** *Let E be a reflexive strictly convex and smooth Banach space with E^* as its dual. Then*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.6 ([1])**.** *Let C be a convex subset of a real smooth Banach space E. Let $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle z - x_0, Jx - Jx_0\rangle \leq 0, \ \forall z \in C.$$

Lemma 2.7 ([20])**.** *Let E be a uniformly convex Banach space and $B_R(0)$ be a closed ball of E. Then, there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$||\alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_N x_N||^2 \leq \sum_{i=0}^{N} \alpha_i ||x_i||^2 - \alpha_i\alpha_j g(||x_i - x_j||)$$

for $\alpha_i \in (0, 1)$ such that $\sum_{i=0}^{N} \alpha_i = 1$ and $x_i \in B_R(0) := \{x \in E : ||x|| \leq R\}$, for some $R > 0$.

Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E. Let $A \subseteq E \times E^*$ be a monotone mapping satisfying

$$D(A) \subset C \subset \cap_{r>0} J^{-1}R(J + rA). \tag{2.2}$$

Then we have the following lemmas.

Lemma 2.8 ([2]). *Let E be a smooth and strictly convex Banach space, C be a nonempty closed convex subset of E, and $A \subset E \times E^*$ a monotone operator satisfying* (2.2). *Let Q_r be the resolvent of A defined by $Q_r = (J + rA)^{-1}J$, for $r > 0$ and $\{r_n\}$ a sequence of $(0, \infty)$ such that $\lim_{n\to\infty} r_n = \infty$. If $\{x_n\}$ is a bounded sequence of C such that $Q_{r_n}x_n \rightharpoonup z$, then $z \in A^{-1}(0)$.*

Lemma 2.9 ([7]). *Let E be a smooth and strictly convex Banach space, C be a nonempty, closed and convex subset of E, and $A \subset E \times E^*$ a monotone operator satisfying* (2.2) *and $A^{-1}(0)$ is nonempty. Let Q_r be the resolvent of A defined by $Q_r = (J + rA)^{-1}J$ for $r > 0$. Then for each $r > 0$,*

$$\phi(p, Q_r x) + \phi(Q_r x, x) \le \phi(p, x)$$

for all $p \in A^{-1}(0)$ and $x \in C$.

Lemma 2.10 ([19]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \le (1 - \beta_n)a_n + \beta_n\delta_n, \ n \ge n_0,$$

where $\{\beta_n\} \subset (0, 1)$ and $\{\delta_n\} \subset R$ satisfying the following conditions:

$$\lim_{n\to\infty} \beta_n = 0, \ \sum_{n=1}^{\infty} \beta_n = \infty, \ \textit{and} \ \limsup_{n\to\infty} \delta_n \le 0.$$

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.11 ([12]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all* (*sufficiently large*) *numbers $k \in \mathbb{N}$*:

$$a_{m_k} \le a_{m_k+1} \ \textit{and} \ a_k \le a_{m_k+1}.$$

In fact, $m_k = \max\{j \le k : a_j < a_{j+1}\}$.

3. Main result

We now prove the following theorem.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let $A : C \to E^*$ be a continuous monotone mapping satisfying* (2.2) *and $||Ax|| \le ||Ax - Ap||$, $\forall x \in C$ and $p \in VI(C, A)$. Let $T_i : C \to C$, $i = 1, 2, \dots, N$, be a finite family of relatively nonexpansive mappings. Assume that $F := \cap_{i=1}^N F(T_i) \cap VI(C, A)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$(3.1) \qquad \begin{cases} x_0 \in C, \ \textit{chosen arbitrarily}, \\ y_n = \Pi_C\big[(1 - \alpha_n)(J + r_n A)^{-1}Jx_n\big], \\ x_{n+1} = \Pi_C J^{-1}(\beta_0 J y_n + \sum_{i=1}^N \beta_i J T_i y_n), \ \forall n \ge 0, \end{cases}$$

where $\alpha_n \in (0,1)$*,* $\{\beta_i\}_{i=0}^N \subset [c,d] \subset (0,1)$ *and* $\{r_n\}$ *a sequence of* $(0,\infty)$ *satisfying*: $\lim_{n\to\infty} \alpha_n = 0$*,* $\sum_{n=1}^{\infty} \alpha_n = \infty$*,* $\sum_{i=0}^{N} \beta_i = 1$ *and* $\lim_{n\to\infty} r_n = \infty$*. Then* $\{x_n\}$ *converges strongly to the minimum-norm element of* F*.*

Proof. Let $p \in \Pi_F(0)$ and $w_n := (J + r_n A)^{-1} J x_n := Q_{r_n} x_n$. Then, since by Remark 1.3 we have that $p \in A^{-1}(0)$, from Lemma 2.9 we get that

$$\phi(p, w_n) = \phi(p, Q_{r_n} x_n) \le \phi(p, x_n).$$

Now from (3.1), Lemma 2.3 and property of ϕ and (3) we get that

$$\begin{aligned}
\phi(p, y_n) &= \phi(p, \Pi_C(1-\alpha_n)w_n) \le \phi(p, (1-\alpha_n)w_n) \\
&= \phi(p, J^{-1}(\alpha_n J0 + (1-\alpha_n)Jw_n) \\
&= ||p||^2 - 2\langle p, \alpha_n J0 + (1-\alpha_n)Jw_n\rangle + ||\alpha_n J0 + (1-\alpha_n)Jw_n||^2 \\
&\le ||p||^2 - 2\alpha_n\langle p, J0\rangle - 2(1-\alpha_n)\langle p, Jw_n\rangle \\
&\quad + \alpha_n||J0||^2 + (1-\alpha_n)||Jw_n||^2 \\
&= \alpha_n\phi(p,0) + (1-\alpha_n)\phi(p, w_n) \\
&\le \alpha_n\phi(p,0) + (1-\alpha_n)\phi(p, x_n). \qquad (3.2)
\end{aligned}$$

Moreover, from (3.1), Lemma 2.3, Lemma 2.7, relatively nonexpansiveness of T and (3.2) we get that

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, \Pi_C J^{-1}(\beta_0 Jy_n + \sum_{i=1}^{N} \beta_i JT_i y_n) \\
&\le \phi(p, J^{-1}(\beta_0 Jy_n + \sum_{i=1}^{N} \beta_i JT_i y_n) \\
&= ||p||^2 - 2\langle p, \beta_0 Jy_n + \sum_{i=1}^{N} \beta_i JT_i y_n\rangle + ||\beta_0 Jy_n + \sum_{i=1}^{N} \beta_i JT_i y_n||^2 \\
&\le ||p||^2 - 2\beta_0\langle p,\ Jy_n\rangle - 2\sum_{i=1}^{N} \beta_i\langle p, JT_i y_n\rangle \\
&\quad + \beta_0||Jy_n||^2 + \sum_{i=1}^{N} \beta_i||JT_i y_n||^2 - \beta_0\beta_i g(||Jy_n - T_i y_n) \\
&= \beta_0\phi(p, y_n) + \sum_{i=1}^{N} \beta_i\phi(p, T_i y_n) - \beta_0\beta_i g(||Jy_n - T_i y_n) \\
&\le \beta_0\phi(p, y_n) + (1-\beta_0)\phi(p, y_n) - \beta_0\beta_i g(||Jy_n - T_i y_n) \\
&\le \phi(p, y_n) - \beta_0\beta_i g(||Jy_n - T_i y_n) \qquad (3.3) \\
&\le \alpha_n\phi(p,0) + (1-\alpha_n)\phi(p, x_n) \qquad (3.4)
\end{aligned}$$

for each $i \in \{1, 2, \dots, N\}$. Thus, by induction,

$$\phi(p, x_{n+1}) \leq \max\{\phi(p, 0), \phi(p, x_0)\}, \ \forall n \geq 0,$$

which implies that $\{x_n\}$ is bounded and hence $\{y_n\}$ and $\{w_n\}$ are bounded. Now let $z_n = (1 - \alpha_n)w_n$. Then we note that $y_n = \Pi_C z_n$. Furthermore, since $\alpha_n \to 0$ we get that

$$z_n - w_n = \alpha_n(-w_n) \to 0 \text{ as } n \to \infty. \tag{3.5}$$

Thus, using Lemma 2.3, Lemma 2.5 and property of ϕ we obtain that

$$\begin{aligned}
\phi(p, y_n) &\leq \phi(p, z_n) = V(p, Jz_n) \\
&\leq V(p, Jz_n - \alpha_n(J0 - Jp)) - 2\langle z_n - p, -\alpha_n(J0 - Jp)\rangle \\
&= \phi(p, J^{-1}(\alpha_n Jp + (1 - \alpha_n)Jw_n) + 2\alpha_n\langle z_n - p, J0 - Jp\rangle \\
&\leq \alpha_n\phi(p, p) + (1 - \alpha_n)\phi(p, w_n) + 2\alpha_n\langle z_n - p, J0 - Jp\rangle \\
&= (1 - \alpha_n)\phi(p, w_n) + 2\alpha_n\langle z_n - p, J0 - Jp\rangle \\
&\leq (1 - \alpha_n)\phi(p, x_n) + 2\alpha_n\langle z_n - p, J0 - Jp\rangle. \qquad (3.6)
\end{aligned}$$

Furthermore, from (3.3) and (3.6) we have that

$$\begin{aligned}
\phi(p, x_{n+1}) &\leq (1 - \alpha_n)\phi(p, x_n) + 2\alpha_n\langle z_n - p, J0 - Jp\rangle \\
&\quad - \beta_0\beta_i g(||Jy_n - JT_i y_n||) \qquad (3.7) \\
&\leq (1 - \alpha_n)\phi(p, x_n) + 2\alpha_n\langle z_n - p, J0 - Jp\rangle. \qquad (3.8)
\end{aligned}$$

Now, we consider two cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\phi(p, x_n)\}$ is non-increasing. In this situation, $\{\phi(p, x_n)\}$ is convergent. Then from (3.7) we have that

$$\beta_0\beta_i g(||Jy_n - JT_i y_n||) \to 0, \tag{3.9}$$

which implies, by the property of g that

$$Jy_n - JT_i y_n \to 0 \text{ as } n \to \infty, \tag{3.10}$$

and hence, since J^{-1} is uniformly continuous on bounded sets we obtain that

$$y_n - T_i y_n \to 0 \text{ as } n \to \infty \tag{3.11}$$

for each $i \in \{1, 2, \dots, N\}$.

Furthermore, Lemma 2.3, property of ϕ and the fact that $\alpha_n \to 0$ as $n \to \infty$, imply that

$$\begin{aligned}
\phi(y_n, \Pi_C z_n) &\leq \phi(y_n, z_n) \\
&= \phi(y_n, J^{-1}(\alpha_n J0 + (1 - \alpha_n)Jw_n) \\
&\leq \alpha_n\phi(y_n, 0) + (1 - \alpha_n)\phi(w_n, w_n) \\
&\leq \alpha_n\phi(y_n, 0) + (1 - \alpha_n)\phi(w_n, w_n) \to 0 \text{ as } n \to \infty, \qquad (3.12)
\end{aligned}$$

and hence

$$y_n - z_n \to 0 \text{ as } n \to \infty. \tag{3.13}$$

Since $\{z_n\}$ is bounded and E is reflexive, we choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z$ and $\limsup_{n\to\infty}\langle z_n - p, Jw - Jp\rangle = \lim_{i\to\infty}\langle z_{n_i} - p, J0 - Jp\rangle$. Then, from (3.5) and (3.13) we get that

$$y_{n_i} \rightharpoonup z,\ w_{n_i} \rightharpoonup z \text{ as } i \to \infty. \tag{3.14}$$

Thus, since T satisfies condition (R3) we obtain from (3.11) that $z \in F(T_i)$ for each $i \in \{1, 2, \ldots, N\}$ and hence $z \in \cap_{i=1}^N F(T_i)$. Furthermore, since $w_{n_i} \rightharpoonup z$, Lemma 2.8 implies that $z \in A^{-1}(0)$ and hence by Remark 1.3 we have that $z \in VI(C, A)$.

Thus, from the above discussions we obtain that $z \in F = \cap_{i=1}^N F(T_i) \cap VI(C, A)$. Therefore, by Lemma 2.6, we immediately obtain that

$$\limsup_{n\to\infty}\langle z_n - p, J0 - Jp\rangle = \lim_{i\to\infty}\langle z_{n_i} - p, J0 - Jp\rangle = \langle z - p, J0 - Jp\rangle \le 0.$$

It follows from Lemma 2.10 and (3.8) that $\phi(p, x_n) \to 0$ as $n \to \infty$. Consequently, by Lemma 2.4 we get that $x_n \to p$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\phi(p, x_{n_i}) < \phi(p, x_{n_i+1})$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.11, there exist a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, $\phi(p, x_{m_k}) \le \phi(p, x_{m_k+1})$ and $\phi(p, x_k) \le \phi(p, x_{m_k+1})$ for all $k \in \mathbb{N}$. Then from (3.7) and the fact that $\alpha_n \to 0$ we obtain that

$$g(||Jy_{m_k} - JT_i y_{m_k}||) \to 0, \text{ as } k \to \infty$$

for each $i \in \{1, 2, \ldots, N\}$. Thus, using the same proof as in Case 1, we obtain that $y_{m_k} - T_i y_{m_k} \to 0$, $y_{m_k} - z_{m_k} \to 0$, as $k \to \infty$, and hence we obtain that

$$\limsup_{k\to\infty}\langle z_{m_k} - p, J0 - Jp\rangle \le 0. \tag{3.15}$$

Then from (3.8) we have that

$$\phi(p, x_{m_k+1}) \le (1 - \alpha_{m_k})\phi(p, x_{m_k}) + 2\alpha_{m_k}\langle z_{m_k} - p, J0 - Jp\rangle. \tag{3.16}$$

Since $\phi(p, x_{m_k}) \le \phi(p, x_{m_k+1})$, (3.16) implies that

$$\begin{aligned}\alpha_{m_k}\phi(p, x_{m_k}) &\le \phi(p, x_{m_k}) - \phi(p, x_{m_k+1}) + 2\alpha_{m_k}\langle z_{m_k} - p, J0 - Jp\rangle \\ &\le 2\alpha_{m_k}\langle z_{m_k} - p, J0 - Jp\rangle.\end{aligned}$$

In particular, since $\alpha_{m_k} > 0$, we get

$$\phi(p, x_{m_k}) \le 2\langle z_{m_k} - p, J0 - Jp\rangle.$$

Then, from (3.15) we obtain that $\phi(p, x_{m_k}) \to 0$ as $k \to \infty$. This together with (3.16) gives $\phi(p, x_{m_k+1}) \to 0$ as $k \to \infty$. But $\phi(p, x_k) \le \phi(p, x_{m_k+1})$ for all $k \in \mathbb{N}$, thus we obtain that $x_k \to p$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to p which is the minimum-norm element of F and the proof is complete. □

If in Theorem 3.1, we assume that $N = 1$, then we get the following corollary.

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let $A : C \to E^*$ be a continuous monotone mapping satisfying* (2.2) *and $||Ax|| \leq ||Ax - Ap||$, $\forall x \in C$ and $p \in VI(C, A)$. Let $T : C \to C$ be a relatively nonexpansive mapping. Assume that $F := F(T) \cap VI(C, A)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily}, \\ y_n = \Pi_C\big[(1-\alpha_n)(J + r_nA)^{-1}Jx_n\big], \\ x_{n+1} = \Pi_C J^{-1}(\beta Jy_n + (1-\beta)JTy_n), \forall n \geq 0, \end{cases} \tag{3.17}$$

where $\alpha_n \in (0, 1)$, $\beta \subset [c, d] \subset (0, 1)$ and $\{r_n\}$ a sequence of $(0, \infty)$ satisfying: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\lim_{n\to\infty} r_n = \infty$. *Then $\{x_n\}$ converges strongly to the minimum-norm element of F.*

If in Theorem 3.1, we assume that $T_i \equiv I$, for $i = 1, 2, \ldots, N$, identity map on C, then we get the following corollary.

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let $A : C \to E^*$ be a continuous monotone mapping satisfying* (2.2) *and $||Ax|| \leq ||Ax - Ap||$, $\forall x \in C$ and $p \in VI(C, A)$. Assume that $VI(C, A)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily}, \\ x_{n+1} = \Pi_C\big[(1-\alpha_n)(J + r_nA)^{-1}Jx_n\big], \ \forall n \geq 0, \end{cases} \tag{3.18}$$

where $\alpha_n \in (0, 1)$, and $\{r_n\}$ a sequence of $(0, \infty)$ satisfying: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\lim_{n\to\infty} r_n = \infty$. *Then $\{x_n\}$ converges strongly to the minimum-norm element of $VI(C, A)$.*

We make use of Remark 1.3 to restate the above theorem.

Theorem 3.4. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let $A : C \to E^*$ be a continuous monotone mapping satisfying* (2.2) *and $||Ax|| \leq ||Ax - Ap||$, $\forall x \in C$ and $p \in VI(C, A)$. Let $T_i : C \to C$, $i = 1, 2, \ldots, N$, be a finite family of relatively nonexpansive mappings. Assume that $F := \cap_{i=1}^N F(T_i) \cap A^{-1}(0)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily}, \\ y_n = \Pi_C\big[(1-\alpha_n)(J + r_nA)^{-1}Jx_n\big], \\ x_{n+1} = \Pi_C J^{-1}(\beta_0 Jy_n + \sum_{i=1}^N \beta_i JT_iy_n), \ \forall n \geq 0, \end{cases} \tag{3.19}$$

where $\alpha_n \in (0,1)$, $\{\beta_i\}_{i=0}^N \subset [c,d] \subset (0,1)$ *and* $\{r_n\}$ *a sequence of* $(0,\infty)$ *satisfying*: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{i=0}^{N} \beta_i = 1$, *and* $\lim_{n\to\infty} r_n = \infty$. *Then* $\{x_n\}$ *converges strongly to the minimum-norm element of* F.

A monotone mapping $A \subset E \times E^*$ is said to be *maximal monotone* if its graph is not properly contained in the graph of any monotone mapping. We know that if A is maximal monotone mapping, then $A^{-1}(0)$ ia closed and convex (see [17] for more details). The following lemma is well-known.

Lemma 3.5 ([15]). *Let* E *be a smooth, strictly convex and reflexive Banach space, let* C *be a nonempty, closed and convex subset of* E *and let* $A \subset E \times E^*$ *be a monotone mapping. Then* A *is maximal if and only if* $R(J + rA) = E^*$ *for all* $r > 0$.

We note from the above lemma that if A is maximal, then it satisfies condition (2.2) and hence we have the following corollary.

Corollary 3.6. *Let* C *be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space* E. *Let* $A : C \to E^*$ *be a maximal monotone mapping. Let* $T_i : C \to C$, $i = 1, 2, \ldots, N$, *be a finite family of relatively nonexpansive mappings. Assume that* $F := \cap_{i=1}^N F(T_i) \cap A^{-1}(0)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated by*

$$\text{(3.20)} \qquad \begin{cases} x_0 \in C, \ \text{chosen arbitrarily}, \\ y_n = \Pi_C\big[(1-\alpha_n)(J + r_nA)^{-1}Jx_n\big], \\ x_{n+1} = \Pi_C J^{-1}(\beta_0 Jy_n + \sum_{i=1}^{N} \beta_i JT_i y_n), \ \forall n \geq 0, \end{cases}$$

where $\alpha_n \in (0,1)$, $\{\beta_i\}_{i=0}^N \subset [c,d] \subset (0,1)$ *and* $\{r_n\}$ *a sequence of* $(0,\infty)$ *satisfying*: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{i=0}^{N} \beta_i = 1$, *and* $\lim_{n\to\infty} r_n = \infty$. *Then* $\{x_n\}$ *converges strongly to the minimum-norm element of* F.

If in Corollary 3.6, we assume that $T_i \equiv I$, for $i = 1, 2, \ldots, N$, identity map on C, then we get the following corollary.

Corollary 3.7. *Let* C *be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space* E. *Let* $A : C \to E^*$ *be a maximal monotone mapping. Assume that* $A^{-1}(0)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated by*

$$\text{(3.21)} \qquad \begin{cases} x_0 \in C, \ \text{chosen arbitrarily}, \\ x_{n+1} = \Pi_C\big[(1-\alpha_n)(J + r_nA)^{-1}Jx_n\big], \ \forall n \geq 0, \end{cases}$$

where $\alpha_n \in (0,1)$ *and* $\{r_n\}$ *a sequence of* $(0,\infty)$ *satisfying*: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} r_n = \infty$. *Then* $\{x_n\}$ *converges strongly to the minimum-norm element of* $A^{-1}(0)$.

If in Theorem 3.1, we assume that $A \equiv 0$, then the assumption that E be 2-uniformly convex may be relaxed. In fact, we have the following corollary.

Corollary 3.8. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let $T_i : C \to C$, $i = 1, 2, \ldots, N$, be a finite family of relatively nonexpansive mappings. Assume that $F := \cap_{i=1}^{N} F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$(3.22) \qquad \begin{cases} x_0 \in C, \ \text{chosen arbitrarily}, \\ y_n = \Pi_C\big[(1-\alpha_n)x_n\big], \\ x_{n+1} = \Pi_C J^{-1}(\beta_0 J y_n + \sum_{i=1}^{N} \beta_i J T_i y_n), \ \forall n \geq 0, \end{cases}$$

where $\alpha_n \in (0,1)$ and $\{\beta_i\}_{i=0}^{N} \subset [c,d] \subset (0,1)$ satisfying: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{i=0}^{N} \beta_i = 1$. Then $\{x_n\}$ converges strongly to the minimum-norm element of F.

If in Corollary 3.8, we assume that $N = 1$, then we get the following corollary.

Corollary 3.9. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let $T : C \to C$, be a relatively nonexpansive mappings. Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$(3.23) \qquad \begin{cases} x_0 \in C, \ \text{chosen arbitrarily}, \\ y_n = \Pi_C\big[(1-\alpha_n)x_n\big], \\ x_{n+1} = \Pi_C J^{-1}(\beta J y_n + (1-\beta) J T y_n), \forall n \geq 0, \end{cases}$$

where $\beta \in (0,1)$ and $\alpha_n \in (0,1)$ satisfying: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to the minimum-norm element of $F(T)$.

If $E = H$, a real Hilbert space, then E is uniformly convex and uniformly smooth real Banach space. In this case, $J = I$, identity map on H and $\Pi_C = P_C$, projection mapping from H onto C. Thus, the following corollaries hold.

Corollary 3.10. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a continuous monotone mapping satisfying (2.2) and $||Ax|| \leq ||Ax - Ap||$, $\forall x \in C$ and $p \in VI(C, A)$. Let $T_i : C \to C$, $i = 1, 2, \ldots, N$, be a finite family of nonexpansive mappings. Assume that $F := \cap_{i=1}^{N} F(T_i) \cap VI(C, A)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$(3.24) \qquad \begin{cases} x_0 \in C, \ \text{chosen arbitrarily}, \\ y_n = P_C\big[(1-\alpha_n)(I + r_n A)^{-1} x_n\big], \\ x_{n+1} = P_C(\beta_0 y_n + \sum_{i=1}^{N} \beta_i T_i y_n), \ \forall n \geq 0, \end{cases}$$

where $\alpha_n \in (0,1)$, $\{\beta_i\}_{i=0}^{N} \subset [c,d] \subset (0,1)$ and $\{r_n\}$ a sequence of $(0,\infty)$ satisfying: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{i=0}^{N} \beta_i = 1$, and $\lim_{n\to\infty} r_n = \infty$. Then $\{x_n\}$ converges strongly to the minimum-norm element of F.

Corollary 3.11. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a maximal monotone mapping. Let $T_i : C \to C$, $i = 1, 2, \ldots, N$, be a finite family of nonexpansive mappings.*

Assume that $F := \cap_{i=1}^{N} F(T_i) \cap A^{-1}(0)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated by*

$$\text{(3.25)} \qquad \begin{cases} x_0 \in C, \ \text{chosen arbitrarily}, \\ y_n = P_C\left[(1-\alpha_n)(I + r_n A)^{-1} x_n\right], \\ x_{n+1} = P_C(\beta_0 y_n + \sum_{i=1}^{N} \beta_i T_i y_n), \ \forall n \geq 0, \end{cases}$$

where $\alpha_n \in (0,1)$, $\{\beta_i\}_{i=0}^{N} \subset [c,d] \subset (0,1)$ *and* $\{r_n\}$ *a sequence of* $(0,\infty)$ *satisfying*: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{i=0}^{N} \beta_i = 1$, *and* $\lim_{n\to\infty} r_n = \infty$. *Then* $\{x_n\}$ *converges strongly to the minimum-norm element of* F.

4. Application

In this section, we study the problem of finding a minimizer of a continuously Fréchet differentiable convex functional which has minimum-norm in Banach spaces. The following is deduced from Corollary 3.7.

Theorem 4.1. *Let* E *be a uniformly convex and uniformly smooth real Banach space. Let* f *be a continuously Fréchet differentiable convex functional on* E *and* $\bigtriangledown f$ *is maximal monotone with* $F := (\bigtriangledown f)^{-1}(0) = \{z \in E : f(z) = \min_{y\in E} f(y)\} \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\text{(4.1)} \qquad \begin{cases} x_0 \in C \ \text{chosen arbitrarily}, \\ x_{n+1} = \Pi_C((1-\alpha_n)(J + r_n \bigtriangledown f)^{-1} J x_n), \end{cases}$$

where $\alpha_n \in (0,1)$ *and* $\{r_n\}$ *a sequence of* $(0,\infty)$ *satisfying*: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} r_n = \infty$. *Then* $\{x_n\}$ *converges strongly to the minimum-norm element of* F.

Remark 4.2. Our theorems improve most of the results that have been proved for these important class of non-linear mappings. In particular, Corollary 3.3 improves Theorem 3.1 of [5] and hence results of [6] in the sense that our convergence is strong in a more general class of continuous monotone mappings in a more general Banach spaces provided that A satisfies (2.2).

Moreover, Corollary 3.7 improves Theorem 3.3 of [4] in the sense that our convergence is valid in a more general Banach spaces that does not require computations of C_n, Q_n and $C_n \cap Q_n$ for each $n \geq 0$ provided that A is maximal monotone mapping.

In addition, Corollary 3.2 improves Theorem 3.1 of [3] in the sense that our convergence is for a more general class of relatively nonexpansive and continuous monotone mappings in a more general Banach spaces provided that A satisfies (2.2).

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