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GROUP INVERSE AND GENERALIZED DRAZIN INVERSE OF BLOCK MATRICES IN A BANACH ALGEBRA

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Abstract. Necessary and sufficient conditions for the existence of the group inverse of an anti-triangular block matrix in Banach algebras are presented. Using these results, we give the formulae for the generalized Drazin inverse of a block matrix.

1. Introduction

Let $\mathcal{A}$ be a complex unital Banach algebra with unit 1. The sets of all invertible and quasinilpotent elements ($\sigma(a) = \{0\}$) of $\mathcal{A}$ will be denoted by $\mathcal{A}^{-1}$ and $\mathcal{A}^{qnil}$, respectively.

The group inverse of $a \in \mathcal{A}$ is the unique element $a^\# \in \mathcal{A}$ which satisfies

$$a^\# a a^\# = a^\#, \qquad a a^\# a = a, \qquad a a^\# = a^\# a.$$

If the group inverse of a exists, a is group invertible. Denote by $\mathcal{A}^\#$ the set of all group invertible elements of $\mathcal{A}$.

The generalized Drazin inverse of $a \in \mathcal{A}$ (or Koliha–Drazin inverse of a) is the unique element $a^d \in \mathcal{A}$ which satisfies

$$a^d a a^d = a^d, \qquad a a^d = a^d a, \qquad a - a^2 a^d \in \mathcal{A}^{qnil}.$$

Recall that $a^\pi = 1 - aa^d$ is the spectral idempotent of a corresponding to the set $\{0\}$ [9]. We use $\mathcal{A}^d$ to denote the set of all generalized Drazin invertible elements of $\mathcal{A}$.

We state the following result which is proved for matrices [8, Theorem 2.1], for bounded linear operators [7, Theorem 2.3] and for elements of Banach algebras [4].

Lemma 1.1 ([4, Example 4.5]). *Let $a, b \in \mathcal{A}^d$ and let $ab = 0$. Then*

$$(a+b)^d = \sum_{n=0}^{\infty} (b^d)^{n+1} a^n a^\pi + \sum_{n=0}^{\infty} b^\pi b^n (a^d)^{n+1}.$$

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The group inverse and Drazin inverse of an operator or a matrix has various applications in singular differential equations and singular difference equations, Markov chains, numerical analysis, probability statistical and so on [2]. Several authors have investigated representations for the group inverse and Drazin inverse of an anti-triangular block operator or matrix, under some conditions on the individual blocks [1, 3, 5, 6, 10].

Liu and Yang [10] studied necessary and sufficient conditions for the existence of the group inverse of an anti-triangular block matrix, using the rank of block matrices.

If $p = p^2 \in \mathcal{A}$ is an idempotent, we can represent element $a \in \mathcal{A}$ as

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{11} = pap$, $a_{12} = pa(1-p)$, $a_{21} = (1-p)ap$, $a_{22} = (1-p)a(1-p)$.

Let

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A} \tag{1}$$

relative to the idempotent $p \in \mathcal{A}$ and $bc \in (p\mathcal{A}p)^{\#}$. For $d = 0$ in (1), x is an anti-triangular block matrix. We are going to study the equivalent conditions for the existence and the representations for the group inverse of the anti-triangular block matrix. Then, applying these results, we show some expressions for the generalized Drazin inverse of a block matrix in (1), where $d \in ((1-p)\mathcal{A}(1-p))^d$, under certain conditions involving the group inverse of the product bc.

Lemma 1.2. *Let $p \in \mathcal{A}$ be an idempotent, $b \in p\mathcal{A}(1-p)$, $c \in (1-p)\mathcal{A}p$ and $bc \in (p\mathcal{A}p)^{\#}$. If $(bc)^{\pi}b = 0$ or $c(bc)^{\pi} = 0$, then $cb \in ((1-p)\mathcal{A}(1-p))^{\#}$ and $(cb)^{\#} = c[(bc)^{\#}]^2 b$.*

Proof. Denote by $z = c[(bc)^{\#}]^2 b$. It follows that $zcb = c(bc)^{\#}b = cbz$ and $zcbz = z$. If $(bc)^{\pi}b = 0$ or $c(bc)^{\pi} = 0$, then $cbzcb = cbc(bc)^{\#}b = cb$. So, $cb \in ((1-p)\mathcal{A}(1-p))^{\#}$ and $(cb)^{\#} = z$. □

2. Results

First, we present the necessary and sufficient conditions for the existence of the group inverse of the anti-triangular block matrix.

Theorem 2.1. *Let $x = \left[\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right] \in \mathcal{A}$ relative to the idempotent $p \in \mathcal{A}$ and $bc \in (p\mathcal{A}p)^{\#}$. Then $a(bc)^{\pi} = 0$, $c(bc)^{\pi} = 0$ and $(bc)^{\pi}b = 0$ if and only if $x \in \mathcal{A}^{\#}$ and $x^{\#} = t$, where*

$$t = \begin{bmatrix} (bc)^{\pi}a(bc)^{\#} & (bc)^{\#}b - (bc)^{\pi}a(bc)^{\#}a(bc)^{\#}b \\ c(bc)^{\#} & -c(bc)^{\#}a(bc)^{\#}b \end{bmatrix}. \tag{2}$$

Proof. If $a(bc)^{\pi} = 0$, $c(bc)^{\pi} = 0$ and $(bc)^{\pi}b = 0$, we can easy verify that $xt = tx$, $txt = t$ and $xtx = x$. Thus, $x \in \mathcal{A}^{\#}$ and $x^{\#} = t$.

Suppose that $x \in \mathcal{A}^\#$ and the group inverse $x^\#$ is represented by t in (2). Then, $[xx^\#]_{21} = [x^\# x]_{21}$ gives

$$c(bc)^\# a(bc)^\pi = c(bc)^\pi a(bc)^\#. \tag{3}$$

From $[xx^\# x]_{11} = [x]_{11}$, $[xx^\# x]_{21} = [x]_{21}$ and $[xx^\# x]_{12} = [x]_{12}$, we have

$$a(bc)^\pi = a(bc)^\pi a(bc)^\# a(bc)^\pi + bc(bc)^\# a(bc)^\pi, \tag{4}$$

$$c(bc)^\pi = c(bc)^\pi a(bc)^\# a(bc)^\pi, \tag{5}$$

$$(bc)^\pi b = a(bc)^\pi a(bc)^\# b. \tag{6}$$

The equality $[x^\# x x^\#]_{11} = [x^\#]_{11}$ implies

$$(bc)^\pi a(bc)^\# a(bc)^\pi a(bc)^\# = 0. \tag{7}$$

Applying the equality (5) twice and (7), we get

$$c(bc)^\pi = c(bc)^\pi a(bc)^\# a(bc)^\pi = c(bc)^\pi a(bc)^\# a(bc)^\pi a(bc)^\# a(bc)^\pi = 0.$$

By (4), (3) and the previous equality, observe that

$$a(bc)^\pi = a(bc)^\pi a(bc)^\# a(bc)^\pi + bc(bc)^\pi a(bc)^\# = a(bc)^\pi a(bc)^\# a(bc)^\pi$$

which yields

$$a(bc)^\pi = a(bc)^\pi a(bc)^\# a(bc)^\pi a(bc)^\# a(bc)^\pi.$$

Now, (7) gives $a(bc)^\pi = 0$. Using this equality and (6), we deduce that $(bc)^\pi b = 0$. □

If we state the condition $(bc)^\pi a = 0$ instead of the assumption $a(bc)^\pi = 0$ of Theorem 2.1, we obtain the following result in an analogous manner as in the proof of Theorem 2.1.

Theorem 2.2. *Let $x = \left[\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right] \in \mathcal{A}$ relative to the idempotent $p \in \mathcal{A}$ and $bc \in (p\mathcal{A}p)^\#$. Then $(bc)^\pi a = 0$, $c(bc)^\pi = 0$ and $(bc)^\pi b = 0$ if and only if $x \in \mathcal{A}^\#$ and $x^\# = u$, where*

$$u = \begin{bmatrix} (bc)^\# a(bc)^\pi & (bc)^\# b \\ c(bc)^\# - c(bc)^\# a(bc)^\# a(bc)^\pi & -c(bc)^\# a(bc)^\# b \end{bmatrix}. \tag{8}$$

Observe that in Theorem 2.1 and Theorem 2.2 we obtain the same expressions for the group inverse as in [10, Theorem 2.4 and Theorem 2.5] but under different conditions.

Since the hypothesis $bc \in (p\mathcal{A}p)^{-1}$ implies $(bc)^\pi = 0$, the next corollary follows by Theorem 2.1 and Theorem 2.2.

Corollary 2.1. *Let $x = \left[\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right] \in \mathcal{A}$ relative to the idempotent $p \in \mathcal{A}$ and $bc \in (p\mathcal{A}p)^{-1}$. Then $x \in \mathcal{A}^\#$ and*

$$x^\# = \begin{bmatrix} 0 & (bc)^\# b \\ c(bc)^\# & -c(bc)^\# a(bc)^\# b \end{bmatrix}.$$

Now, we use the previous results to determine the new expressions of the generalized Drazin inverse of a block matrix x in (1), where $bc \in (p\mathcal{A}p)^{\#}$ and $d \in ((1-p)\mathcal{A}(1-p))^d$.

Theorem 2.3. *Let x be defined as in* (1)*, $d \in ((1-p)\mathcal{A}(1-p))^d$ and $bc \in (p\mathcal{A}p)^{\#}$. If*

$$bd = 0, \quad a(bc)^{\pi} = 0, \quad c(bc)^{\pi} = 0, \quad \text{and} \quad (bc)^{\pi}b = 0,$$

then $x \in \mathcal{A}^d$ and

$$x^d = \begin{bmatrix} 0 & 0 \\ 0 & d^d(cb)^{\pi} \end{bmatrix} + \begin{bmatrix} p & 0 \\ 0 & d^{\pi} \end{bmatrix} t + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & d^{\pi}d^n \end{bmatrix} t^{n+1}, \tag{9}$$

where t is defined as in (2)*.*

Proof. We can write

$$x = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} := y + z. \tag{10}$$

The hypothesis $bd = 0$ gives $yz = 0$. Applying Theorem 2.1, note that $y \in \mathcal{A}^{\#}$ and $y^{\#} = t$. By Lemma 1.2, we deduce that $cb \in ((1-p)\mathcal{A}(1-p))^{\#}$ and $(cb)^{\pi} = (1-p) - cbc[(bc)^{\#}]^2 b = (1-p) - c(bc)^{\#}b$. Now, we have

$$y^{\pi} = 1 - yy^{\#} = \begin{bmatrix} (bc)^{\pi} & -(bc)^{\pi}a(bc)^{\#}b \\ 0 & (cb)^{\pi} \end{bmatrix}.$$

Obviously, $z \in \mathcal{A}^d$,

$$z^d = \begin{bmatrix} 0 & 0 \\ 0 & d^d \end{bmatrix} \quad \text{and} \quad z^{\pi} = \begin{bmatrix} p & 0 \\ 0 & d^{\pi} \end{bmatrix}.$$

Using Lemma 1.1 and $yy^{\pi} = 0$, we obtain $x \in \mathcal{A}^d$ and

$$x^d = z^d y^{\pi} + z^{\pi} y^{\#} + \sum_{n=1}^{\infty} z^{\pi} z^n (y^{\#})^{n+1}$$

implying (9). □

Applying Theorem 2.3, we show the next result.

Corollary 2.2. *Let x be defined as in* (1)*, t be defined as in* (2) *and $bc \in (p\mathcal{A}p)^{\#}$. Suppose that $bd = 0$, $a(bc)^{\pi} = 0$, $c(bc)^{\pi} = 0$ and $(bc)^{\pi}b = 0$.*

(i) *If $d \in ((1-p)\mathcal{A}(1-p))^{\#}$, then $x \in \mathcal{A}^{\#}$ and*

$$x^{\#} = \begin{bmatrix} 0 & 0 \\ 0 & d^{\#}(cb)^{\pi} \end{bmatrix} + \begin{bmatrix} p & 0 \\ 0 & d^{\pi} \end{bmatrix} t. \tag{11}$$

(ii) *If $d \in ((1-p)\mathcal{A}(1-p))^{qnil}$, then $x \in \mathcal{A}^d$ and*

$$x^d = t + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & d^n \end{bmatrix} t^{n+1}.$$

Proof. (i) Denote by y the right hand side of (11). By Theorem 2.3 and $dd^\pi = 0$, we get $x \in \mathcal{A}^d$ and $x^d = y$. We can check that $x = x^2y$ and so $x^\# = y$.

(ii) This part follows from $d^d = 0$ and Theorem 2.3. □

If we replace the assumption $bd = 0$ of Theorem 2.3 with $dc = 0$, we prove the following theorem.

Theorem 2.4. *Let x be defined as in* (1)*, $d \in ((1-p)\mathcal{A}(1-p))^d$ and $bc \in (p\mathcal{A}p)^\#$. If*

$$dc = 0, \quad a(bc)^\pi = 0, \quad c(bc)^\pi = 0 \quad \text{and} \quad (bc)^\pi b = 0,$$

then $x \in \mathcal{A}^d$ and

$$x^d = t\begin{bmatrix} p & 0 \\ 0 & d^\pi \end{bmatrix} + \begin{bmatrix} 0 & -(bc)^\pi a(bc)^\# bd^d \\ 0 & (cb)^\pi d^d \end{bmatrix} + \sum_{n=1}^{\infty} t^{n+1}\begin{bmatrix} 0 & 0 \\ 0 & d^\pi d^n \end{bmatrix}, \tag{12}$$

where t is defined as in (2)*.*

Proof. If x is represented as in (10), then $zy = 0$. Similarly as in the proof of Theorem 2.3, by Lemma 1.1, we verify the formula (12). □

As a consequence of Theorem 2.4, we present the next formula for the group inverse $x^\#$ involving the group inverse of d.

Corollary 2.3. *Let x be defined as in* (1)*, t be defined as in* (2)*, $d \in ((1-p)\mathcal{A}(1-p))^\#$ and $bc \in (p\mathcal{A}p)^\#$. If $dc = 0$, $a(bc)^\pi = 0$, $c(bc)^\pi = 0$ and $(bc)^\pi b = 0$, then $x \in \mathcal{A}^\#$ and*

$$x^\# = t\begin{bmatrix} p & 0 \\ 0 & d^\pi \end{bmatrix} + \begin{bmatrix} 0 & -(bc)^\pi a(bc)^\# bd^\# \\ 0 & (cb)^\pi d^\# \end{bmatrix}.$$

By Theorem 2.2, we get the following representations for the generalized Drazin inverse of x.

Theorem 2.5. *Let x be defined as in* (1)*, $d \in ((1-p)\mathcal{A}(1-p))^d$ and $bc \in (p\mathcal{A}p)^\#$. If*

$$bd = 0, \quad (bc)^\pi a = 0, \quad c(bc)^\pi = 0 \quad \text{and} \quad (bc)^\pi b = 0,$$

then $x \in \mathcal{A}^d$ and

$$\begin{aligned} x^d = {} & \begin{bmatrix} 0 & 0 \\ -d^d c(bc)^\# a(bc)^\pi & d^d(cb)^\pi \end{bmatrix} + \begin{bmatrix} p & 0 \\ 0 & d^\pi \end{bmatrix} u \\ & + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & d^\pi d^n \end{bmatrix} u^{n+1}, \end{aligned} \tag{13}$$

where u is defined as in (8)*.*

Proof. Assume that x is presented as in (10), then $z \in \mathcal{A}^d$, $y \in \mathcal{A}^\#$, $y^\# = u$ and

$$y^\pi = \begin{bmatrix} (bc)^\pi & 0 \\ -c(bc)^\# a(bc)^\pi & (cb)^\pi \end{bmatrix}.$$

Since $yz = 0$, applying Lemma 1.1, we get the representation (13) of x^d. □

The next corollary follows directly from Theorem 2.5.

Corollary 2.4. *Let x be defined as in* (1)*, u be defined as in* (8) *and $bc \in (p\mathcal{A}p)^{\#}$. Suppose that $bd = 0$, $(bc)^{\pi}a = 0$, $c(bc)^{\pi} = 0$ and $(bc)^{\pi}b = 0$.*

(i) *If $d \in ((1-p)\mathcal{A}(1-p))^{\#}$, then $x \in \mathcal{A}^{\#}$ and*

$$x^{\#} = \begin{bmatrix} 0 & 0 \\ -d^{\#}c(bc)^{\#}a(bc)^{\pi} & d^{\#}(cb)^{\pi} \end{bmatrix} + \begin{bmatrix} p & 0 \\ 0 & d^{\pi} \end{bmatrix} u.$$

(ii) *If $d \in ((1-p)\mathcal{A}(1-p))^{qnil}$, then $x \in \mathcal{A}^d$ and*

$$x^d = u + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & d^n \end{bmatrix} u^{n+1}.$$

Using the condition $dc = 0$ instead of $bd = 0$ in Theorem 2.5, we can obtain the another formula for x^d.

Theorem 2.6. *Let x be defined as in* (1)*, $d \in ((1-p)\mathcal{A}(1-p))^d$ and $bc \in (p\mathcal{A}p)^{\#}$. If*

$$dc = 0, \quad (bc)^{\pi}a = 0, \quad c(bc)^{\pi} = 0 \quad \text{and} \quad (bc)^{\pi}b = 0,$$

then $x \in \mathcal{A}^d$ and

$$x^d = u \begin{bmatrix} p & 0 \\ 0 & d^{\pi} \end{bmatrix} + \sum_{n=1}^{\infty} u^{n+1} \begin{bmatrix} 0 & 0 \\ 0 & d^{\pi}d^n \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & (cb)^{\pi}d^d \end{bmatrix},$$

where u is defined as in (8)*.*

We verify the next result by Theorem 2.6.

Corollary 2.5. *Let x be defined as in* (1)*, u be defined as in* (8)*, $d \in ((1-p)\mathcal{A}(1-p))^{\#}$ and $bc \in (p\mathcal{A}p)^{\#}$. If $dc = 0$, $(bc)^{\pi}a = 0$, $c(bc)^{\pi} = 0$ and $(bc)^{\pi}b = 0$, then $x \in \mathcal{A}^{\#}$ and*

$$x^{\#} = u \begin{bmatrix} p & 0 \\ 0 & d^{\pi} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & (cb)^{\pi}d^d \end{bmatrix}.$$

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