Bull. Korean Math. Soc. **51** (2014), No. 3, pp. 739–763
http://dx.doi.org/10.4134/BKMS.2014.51.3.739

ON THE NONLINEAR MATRIX EQUATION $X + \sum_{i=1}^{m} A_i^* X^{-q} A_i = Q (0 < q \leq 1)$

Xiaoyan Yin, Ruiping Wen, and Liang Fang

Abstract. In this paper, the nonlinear matrix equation

$$X + \sum_{i=1}^{m} A_i^* X^{-q} A_i = Q \ (0 < q \leq 1)$$

is investigated. Some necessary conditions and sufficient conditions for the existence of positive definite solutions for the matrix equation are derived. Two iterative methods for the maximal positive definite solution are proposed. A perturbation estimate and an explicit expression for the condition number of the maximal positive definite solution are obtained. The theoretical results are illustrated by numerical examples.

1. Introduction

In this paper, we consider the following nonlinear matrix equation

$$X + \sum_{i=1}^{m} A_i^* X^{-q} A_i = Q \tag{1.1}$$

where $0 < q \leq 1$, $A_1, A_2, \ldots, A_m, Q$ are $n \times n$ nonsingular complex matrices with Q Hermitian positive definite, and A^* is the conjugate transpose of a matrix A. This type of nonlinear matrix equations with $m = 1$ have many applications in control theory, dynamic programming, statistics, stochastic filtering, nano research and etc., see for instance [6, 8, 13, 28] and the references therein. When $m > 1$, Eq.(1.1) arises in solving a large-scale system of linear equations in many physical calculations. Following [2], consider a linear system $Mx = f$ where the positive definite matrix M arises from a finite difference

Received March 27, 2013; Revised August 19, 2013.
2010 *Mathematics Subject Classification.* 15A24, 15A45, 65H05.
Key words and phrases. nonlinear matrix equation, positive definite solution, perturbation estimate, condition number.

approximation to an elliptic partial differential equation. As an example, let

$$M = \begin{pmatrix} Q & 0 & \cdots & 0 & A_1 \\ 0 & Q & \cdots & 0 & A_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & Q & A_m \\ A_1^* & A_2^* & \cdots & A_m^* & Q \end{pmatrix}.$$

We can rewrite $M = \tilde{M} + D$ for

$$\tilde{M} = \begin{pmatrix} X & 0 & \cdots & 0 & A_1 \\ 0 & X & \cdots & 0 & A_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & X & A_m \\ A_1^* & A_2^* & \cdots & A_m^* & Q \end{pmatrix}, \; D = \begin{pmatrix} Q-X & 0 & \cdots & 0 & 0 \\ 0 & Q-X & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & Q-X & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Moreover, we can decompose $\tilde{M}$ to the LU decomposition

$$\tilde{M} = \begin{pmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \\ A_1^*X^{-q} & A_2^*X^{-q} & \cdots & A_m^*X^{-q} & I \end{pmatrix} \begin{pmatrix} X & 0 & \cdots & 0 & A_1 \\ 0 & X & \cdots & 0 & A_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & X & A_m \\ 0 & 0 & \cdots & 0 & X \end{pmatrix}.$$

Such a decomposition of $\tilde{M}$ exists if and only if X is a positive definite solution of the matrix equations $X + \sum_{i=1}^m A_i^* X^{-q} A_i = Q$. Solving the linear system $\tilde{M}y = f$ is equivalent to solving two linear systems with a lower and upper block triangular system matrix. To compute the solution of $Mx = f$ from y, the Woodbury formula can be applied.

In the last few years there has been a constantly increasing interest in developing the theory, applications and numerical solutions for the definite solutions to the nonlinear matrix equations of the form (1.1). When $m = 1$ and q is a positive integer, Eq.(1.1) has been extensively investigated by many authors, for example [8, 12, 16, 18, 25]. In case $m = 1$ and $0 < q \leq 1$, Hasanov and other authors [9, 10, 24] derived necessary conditions and sufficient conditions for the existence of positive definite solutions for the matrix equation $X \pm A^* X^{-q} A = Q$ and provided iterative methods for obtaining positive definite solutions of these equations. Inversion free iteration methods for the maximal positive definite solution for the matrix equation $X + A^* X^{-\alpha} A = Q$ with the case $0 < \alpha \leq 1$ and the minimal positive definite solution for $X + A^* X^{-\alpha} A = Q$ with the case $\alpha \geq 1$ can be found in [19, 20]. When $m \geq 1$, $q = 1$ and $Q = I$, He and Long [11] gave some necessary conditions and sufficient conditions for the existence of a positive definite solution of Eq.(1.1). Then based on the matrix differentiation, Duan et al. [4] derived a perturbation bound for the maximal positive definite solution of $X + \sum_{i=1}^m A_i^* X^{-1} A_i = I$. In addition, Duan [3, 5] and Y. Lim [15] proved that the nonlinear matrix equation $X - \sum_{i=1}^m A_i^* X^{-q} A_i = Q$ always has a unique positive definite solution. Similar nonlinear matrix equations such as $X^s \pm A^*X^{-t}A = Q$ [17, 27], $X + A^*F(X)A = Q$ [21], $X^r + \Sigma_{i=1}^m A_i^* X^{\delta_i} A_i = I$ [23] have been investigated by many authors.

Based on these, we continue to study the matrix equation

$$X + \sum_{i=1}^{m} A_i^* X^{-q} A_i = Q$$

with $0 < q \leq 1$ and Q Hermite positive definite. In Section 2, we derive some sufficient conditions and necessary conditions for the matrix equation to have positive definite solutions. Two iterative methods for obtaining the maximal positive definite solution are also proposed. Perturbation of the positive definite solutions is considered in Section 3. We obtain a perturbation estimate and an explicit expression of the condition number for the maximal positive definite solution of the matrix equation. Section 4 offers several numerical examples to illustrate the effectiveness of the theoretical results.

Throughout this paper, we denote by $C^{n\times n}, H^{n\times n}$ the set of all $n\times n$ complex matrices, all $n\times n$ Hermitian matrices, respectively. The notation $A \geq 0 (A > 0)$ means that A is Hermitian positive semidefinite (positive definite). We denote by $\sigma_1(A)$ and $\sigma_n(A)$ the maximal and minimal singular values of A, respectively. Similarly, $\lambda_1(A)$ and $\lambda_n(A)$ stand for the maximal and the minimal eigenvalues of A, respectively. For $A, B \in H^{n\times n}$, we write $A \geq B (A > B)$ if $A - B \geq 0 (> 0)$ and let

$$(A, B) = \{X | A < X < B\}, \quad (A, B] = \{X | A < X \leq B\}.$$

For $n \times n$ complex matrix $A = (a_1, a_2, \ldots, a_n) = (a_{ij})$ and a matrix B, $A \otimes B = (a_{ij}B)$ is a Kronecker product; $\mathrm{vec}(A)$ is a vector defined by $\mathrm{vec}(A) = (a_1^T, a_2^T, \ldots, a_n^T)^T$. Unless otherwise noted, the symbol $\|\cdot\|_F$ stands for the Frobenius norm, and $\|\cdot\|$ the spectral norm (i.e., $\|A\| = \sqrt{\rho(AA^*)} = \sigma_1(A)$) and the Euclidean vector norm.

2. Positive definite solutions

In this section, we provide several necessary conditions and sufficient conditions for Eq.(1.1) to have positive definite solutions and also we propose two iterative methods for obtaining the maximal positive definite solution of Eq.(1.1).

We start with several lemmas which we need to prove our main results:

Lemma 2.1 ([26]). *If $A > B > 0$ (or $A \geq B > 0$), then $A^r > B^r$ (or $A^r \geq B^r$) for all $r \in (0, 1]$, and $A^r < B^r$ (or $0 < A^r \leq B^r$) for all $r \in [-1, 0)$.*

Lemma 2.2 ([1]). *Let A, B be positive definite. Then for any unitary invariant norm $|||\cdot|||$, we have*

$$|||B^tA^tB^t||| \leq |||(BAB)^t|||, \quad \text{if} \quad 0 \leq t \leq 1;$$
$$|||(BAB)^t||| \leq |||B^tA^tB^t|||, \quad \text{if} \quad t \geq 1.$$

Lemma 2.3 ([26])**.** *If* $0 < q \leq 1$*, and* X *and* Y *are positive definite matrices of the same order with* $X, Y \geq bI > 0$*, then* $\|X^q - Y^q\| \leq qb^{q-1}\|X - Y\|$ *and* $\|X^{-q} - Y^{-q}\| \leq qb^{-(q+1)}\|X - Y\|$.

Lemma 2.4 ([28])**.** *If* C *and* P *are Hermitian matrices of the same order with* $P > 0$*, then* $CPC + P^{-1} \geq 2C$.

Lemma 2.5 ([7])**.** *Let* A *and* B *be positive operators on a Hilbert space* H *such that* $M_1 I \geq A \geq m_1 I > 0$, $M_2 I \geq A \geq m_2 I > 0$ *and* $0 < A \leq B$. *Then*

$$A^t \leq (\frac{M_1}{m_1})^{t-1} B^t \text{ and } A^t \leq (\frac{M_2}{m_2})^{t-1} B^t$$

hold for any $t \geq 1$.

Lemma 2.6. *For any* $n \times n$ *matrix* B *and positive definite matrix* P*, we have*

$$\lambda_1(B^*PB) \leq \lambda_1(P)\lambda_1(B^*B),$$
$$\lambda_n(B^*PB) \geq \lambda_n(P)\lambda_n(B^*B).$$

Proof. Since $P > 0$, by spectral decomposition theorem, there exists a unitary matrix U such that $P = U\mathrm{diag}(\lambda_1(P), \ldots, \lambda_n(P))U^*$. Then $\lambda_n(P)I \leq P \leq \lambda_1(P)I$. It follows that $\lambda_n(P)B^*B \leq B^*PB \leq \lambda_1(P)B^*B$, which gives

$$\lambda_1(B^*PB) \leq \lambda_1(P)\lambda_1(B^*B) \quad \text{and} \quad \lambda_n(B^*PB) \geq \lambda_n(P)\lambda_n(B^*B). \qquad \square$$

Theorem 2.1. *If Eq.*(1.1) *has a positive definite solution* X*, then for each* $i = 1, 2, \ldots, m$*, we have*

$$X^q \in (A_i Q^{-1} A_i^*, (Q - \sum_{i=1}^{m} A_i^* Q^{-q} A_i)^q).$$

The proof is similar to that of Theorem 2.2 in [9] and is omitted here.

Theorem 2.2. *If Eq.*(1.1) *has a positive definite solution* X*, then*

$$\sum_{i=1}^{m} \sigma_n^2(Q^{-q/2} A_i Q^{-1/2}) \leq \frac{q^q}{(q+1)^{q+1}} \quad \text{and} \quad X \leq \mu Q,$$

where μ *is a solution of the equation* $x^q(1 - x) = \sum_{i=1}^{m} \sigma_n^2(Q^{-q/2} A_i Q^{-1/2})$ *in* $[\frac{q}{q+1}, 1]$.

Proof. Consider the following sequence

$$\mu_0 = 1, \quad \mu_{k+1} = 1 - \frac{\sum_{i=1}^{m} \sigma_n^2(Q^{-q/2} A_i Q^{-1/2})}{\mu_k^q}, \quad k = 0, 1, 2, \ldots.$$

Obviously, $\mu_0 > 0$. Let X be a positive definite solution of Eq.(1.1). Then $X = Q - \sum_{i=1}^{m} A_i^* X^{-q} A_i < Q = \mu_0 Q$. Assuming that $\mu_k > 0$, and $X < \mu_k Q$, we have from Lemma 2.1 that

$$X = Q - \sum_{i=1}^{m} A_i^* X^{-q} A_i < Q - \sum_{i=1}^{m} A_i^* (\mu_k Q)^{-q} A_i$$

$$= Q^{1/2}[I - \frac{\sum_{i=1}^{m} Q^{-1/2}A_i^* Q^{-q} A_i Q^{-1/2}}{\mu_k^q}]Q^{1/2}$$
$$\leq Q^{1/2}[1 - \frac{\sum_{i=1}^{m} \sigma_n^2(Q^{-q/2}A_i Q^{-1/2})}{\mu_k^q}]Q^{1/2} = \mu_{k+1}Q.$$

which gives $\mu_{k+1} > 0$ and $X < \mu_{k+1}Q$. Thus $\mu_k > 0$ and $X < \mu_k Q$ for $k = 0, 1, 2, \ldots$, by induction.

It is easy to see that $\mu_1 < \mu_0$. Suppose $\mu_k < \mu_{k-1}$. Then $\mu_k^q < \mu_{k-1}^q$ and

$$\mu_{k+1} = 1 - \frac{\sum_{i=1}^{m} \sigma_n^2(Q^{-q/2}A_i Q^{-1/2})}{\mu_k^q} < 1 - \frac{\sum_{i=1}^{m} \sigma_n^2(Q^{-q/2}A_i Q^{-1/2})}{\mu_{k-1}^q} = \mu_k$$

which means that sequence $\{\mu_k\}$ is monotonically decreasing. Notice that $X < \mu_k Q$ implies $\mu_k > \lambda_n(Q^{-1/2}XQ^{-1/2})$ for each $k = 0, 1, 2, \ldots$. Thus $\{\mu_k\}$ is convergent. Denote $\lim_{k\to\infty} \mu_k = \mu$. Then

$$\mu > 0, \quad X \leq \mu Q \quad \text{and} \quad \mu = 1 - \frac{\sum_{i=1}^{m} \sigma_n^2(Q^{-q/2}A_i Q^{-1/2})}{\mu^q},$$

i.e., μ is a solution of the equation $x^q(1-x) = \sum_{i=1}^{m} \sigma_n^2(Q^{-q/2}A_i Q^{-1/2})$. It follows that

$$\sum_{i=1}^{m} \sigma_n^2(Q^{-q/2}A_i Q^{-1/2}) \leq \max_{x\in[0,1]} f(x) = f(\frac{q}{q+1}) = \frac{q^q}{(q+1)^{q+1}},$$

where $f(x) = x^q(1-x)$.

Next we show that $\mu \in [\frac{q}{q+1}, 1]$. Obviously, $\mu_0 = 1 > \frac{q}{q+1}$. Assuming that $\mu_k > \frac{q}{q+1}$, we have

$$\mu_{k+1} = 1 - \frac{\sum_{i=1}^{m} \sigma_n^2(Q^{-q/2}A_i Q^{-1/2})}{\mu_k^q} \geq 1 - \frac{1}{\mu_k^q}\frac{q^q}{(q+1)^{q+1}} > 1 - \frac{1}{q+1} = \frac{q}{q+1}.$$

Hence $\mu_k > \frac{q}{q+1}$ for each $k = 0, 1, 2, \ldots$ which implies that $\mu \geq \frac{q}{q+1}$. □

Consider the following scalar equations:

$$x^q(1-x) = \sum_{i=1}^{m} \sigma_n^2(Q^{-q/2}A_i Q^{-1/2}), \tag{2.1}$$

$$x^q(1-x) = \sum_{i=1}^{m} \sigma_1^2(Q^{-q/2}A_i Q^{-1/2}). \tag{2.2}$$

Let

$$f(x) = x^q(1-x), \ x \in [0,1].$$

It is not difficult to know that $f(x)$ is monotonically increasing on $[0, \frac{q}{q+1}]$, monotonically decreasing on $[\frac{q}{q+1}, 1]$, and

$$\max_{x\in[0,1]} f(x) = f(\frac{q}{q+1}) = \frac{q^q}{(q+1)^{q+1}}.$$

Thus, if

$$\sum_{i=1}^{m} \sigma_1^2(Q^{-q/2}A_iQ^{-1/2}) < \frac{q^q}{(q+1)^{q+1}}, \tag{2.3}$$

then scalar equations (2.1) and (2.2) have two positive solutions $\alpha_1, \alpha_2(\alpha_1 < \frac{q}{q+1} < \alpha_2)$, and $\beta_1, \beta_2(\beta_1 < \frac{q}{q+1} < \beta_2)$, respectively. It is not difficult to verify that

$$0 < \alpha_1 \leq \beta_1 < \frac{q}{q+1} < \beta_2 \leq \alpha_2 < 1. \tag{2.4}$$

Note that if (2.3) holds, then $\alpha_2 = \mu$ where μ is as defined in Theorem 2.2.

Denote the following matrix sets:

$$\varphi_1 = \{X > 0 \mid \beta_1 Q \leq X \leq \beta_2 Q\},$$
$$\varphi_2 = \{X > 0 \mid \beta_2 Q \leq X \leq \alpha_2 Q\},$$
$$\varphi_3 = \{X > 0 \mid \alpha_2 Q < X < Q\}.$$

We have the following theorem:

Theorem 2.3. *Suppose that* $\sum_{i=1}^{m} \sigma_1^2(Q^{-q/2}A_iQ^{-1/2}) < \frac{q^q}{(q+1)^{q+1}}$. *Then Eq.*(1.1)

(i) *has no positive definite solution in* φ_1, φ_3;

(ii) *has positive definite solutions in* φ_2; *Moreover, if* $\sum_{i=1}^{m} \|A_i\|^2\|Q^{-1}\|^{q+1} < \frac{q^q}{(q+1)^{q+1}}$, *then the positive definite solution in* φ_2 *is unique, which is the maximal positive definite solution.*

Proof. (i) Let X be any positive definite solution of Eq.(1.1). Applying Lemma 2.2, we have

$$\begin{aligned}\frac{1}{\lambda_n(Q^{-q/2}X^qQ^{-q/2})} &= \|Q^{q/2}X^{-q}Q^{q/2}\| \leq \|Q^{1/2}X^{-1}Q^{1/2}\|^q \\ &= \frac{1}{\lambda_n^q(Q^{-1/2}XQ^{-1/2})}.\end{aligned}$$

Combining this with Lemma 2.6, we have

$$\begin{aligned}\lambda_n(Q^{-1/2}XQ^{-1/2}) &= \lambda_n(I - \sum_{i=1}^{m} Q^{-1/2}A_i^*X^{-q}A_iQ^{-1/2}) \\ &= 1 - \lambda_1(\sum_{i=1}^{m} Q^{-1/2}A_i^*X^{-q}A_iQ^{-1/2}) \\ &\geq 1 - \sum_{i=1}^{m} \lambda_1(Q^{-1/2}A_i^*X^{-q}A_iQ^{-1/2}) \\ &= 1 - \sum_{i=1}^{m} \lambda_1(Q^{-1/2}A_i^*Q^{-q/2}Q^{q/2}X^{-q}Q^{q/2}Q^{-q/2}A_iQ^{-1/2})\end{aligned}$$

$$\begin{aligned}
&\geq 1 - \lambda_1(Q^{q/2}X^{-q}Q^{q/2})\sum_{i=1}^{m}\lambda_1(Q^{-1/2}A_i^*Q^{-q}A_iQ^{-1/2})\\
&= 1 - \frac{\sum_{i=1}^{m}\sigma_1^2(Q^{-q/2}A_iQ^{-1/2})}{\lambda_n(Q^{-q/2}X^qQ^{-q/2})}\\
&\geq 1 - \frac{\sum_{i=1}^{m}\sigma_1^2(Q^{-q/2}A_iQ^{-1/2})}{\lambda_n^q(Q^{-1/2}XQ^{-1/2})}.
\end{aligned}$$

Thus

$$\sum_{i=1}^{m}\sigma_1^2(Q^{-q/2}A_iQ^{-1/2}) \geq [1-\lambda_n(Q^{-1/2}XQ^{-1/2})]\lambda_n^q(Q^{-1/2}XQ^{-1/2}),$$

namely, $\lambda_n(Q^{-1/2}XQ^{-1/2}) \leq \beta_1$ or $\lambda_n(Q^{-1/2}XQ^{-1/2}) \geq \beta_2$. Thus Eq.(1.1) has no positive definite solution in φ_1.

Similarly,

$$\begin{aligned}
\lambda_1(Q^{-q/2}X^qQ^{-q/2}) &= \|Q^{-q/2}X^qQ^{-q/2}\| \leq \|Q^{-1/2}XQ^{-1/2}\|^q\\
&= \lambda_1^q(Q^{-1/2}XQ^{-1/2}),
\end{aligned}$$

and from Lemma 2.6,

$$\begin{aligned}
\lambda_1(Q^{-1/2}XQ^{-1/2}) &= \lambda_1(I - \sum_{i=1}^{m}Q^{-1/2}A_i^*X^{-q}A_iQ^{-1/2})\\
&= 1 - \lambda_n(\sum_{i=1}^{m}Q^{-1/2}A_i^*X^{-q}A_iQ^{-1/2})\\
&\leq 1 - \sum_{i=1}^{m}\lambda_n(Q^{-1/2}A_i^*Q^{-q/2}Q^{q/2}X^{-q}Q^{q/2}Q^{-q/2}A_iQ^{-1/2})\\
&\leq 1 - \lambda_n(Q^{q/2}X^{-q}Q^{q/2})\sum_{i=1}^{m}\lambda_n(Q^{-1/2}A_i^*Q^{-q}A_iQ^{-1/2})\\
&= 1 - \frac{\sum_{i=1}^{m}\sigma_n^2(Q^{-q/2}A_iQ^{-1/2})}{\lambda_1(Q^{-q/2}X^qQ^{-q/2})}\\
&\leq 1 - \frac{\sum_{i=1}^{m}\sigma_n^2(Q^{-q/2}A_iQ^{-1/2})}{\lambda_1^q(Q^{-1/2}XQ^{-1/2})}.
\end{aligned}$$

Thus,

$$\sum_{i=1}^{m}\sigma_n^2(Q^{-q/2}A_iQ^{-1/2}) \leq [1-\lambda_1(Q^{-1/2}XQ^{-1/2})]\lambda_1^q(Q^{-1/2}XQ^{-1/2}).$$

Consequently, $\alpha_1 \leq \lambda_1(Q^{-1/2}XQ^{-1/2}) \leq \alpha_2$, which implies that Eq.(1.1) has no positive definite solution in φ_3.

(ii) Consider the following mapping G :

$$G(X) = Q - \sum_{i=1}^{m} A_i^* X^{-q} A_i.$$

G is continuous on φ_2. If $X \in \varphi_2$, then

$$\begin{aligned}
\lambda_n(Q^{-1/2}G(X)Q^{-1/2}) &= \lambda_n(I - \sum_{i=1}^{m} Q^{-1/2}A_i^* X^{-q} A_i Q^{-1/2}) \\
&\geq \lambda_n[I - \frac{1}{\beta_2^q}\sum_{i=1}^{m} Q^{-1/2}A_i^* Q^{-q} A_i Q^{-1/2}] \\
&\geq \lambda_n[I - \frac{1}{\beta_2^q}\sum_{i=1}^{m} \sigma_1^2(Q^{-q/2}A_i Q^{-1/2})I] \\
&= 1 - \frac{1}{\beta_2^q}\sum_{i=1}^{m} \sigma_1^2(Q^{-q/2}A_i Q^{-1/2}) = \beta_2, \\
\lambda_1(Q^{-1/2}G(X)Q^{-1/2}) &= \lambda_1(I - \sum_{i=1}^{m} Q^{-1/2}A_i^* X^{-q} A_i Q^{-1/2}) \\
&\leq \lambda_1(I - \frac{1}{\alpha_2^q}\sum_{i=1}^{m} Q^{-1/2}A_i^* Q^{-q} A_i Q^{-1/2}) \\
&\leq \lambda_1[I - \frac{1}{\alpha_2^q}\sum_{i=1}^{m} \sigma_n^2(Q^{-q/2}A_i Q^{-1/2})I] \\
&= 1 - \frac{1}{\alpha_2^q}\sum_{i=1}^{m} \sigma_n^2(Q^{-q/2}A_i Q^{-1/2}) = \alpha_2.
\end{aligned}$$

Therefore, $\beta_2 I \leq Q^{-1/2}G(X)Q^{-1/2} \leq \alpha_2 I$ and consequently, $\beta_2 Q \leq G(X) \leq \alpha_2 Q$. By Schauder fixed point theorem, we know that $G(X)$ has a fixed point in φ_2. That is, Eq.(1.1) has a positive definite solution X in φ_2.

Now suppose $\sum_{i=1}^{m} \|A_i\|^2 \|Q^{-1}\|^{q+1} < \frac{q^q}{(q+1)^{q+1}}$. Let

$$\rho = \frac{(q+1)^{q+1}}{q^q}\|Q^{-1}\|^{q+1}\sum_{i=1}^{m}\|A_i\|^2.$$

Then $\rho < 1$. Denote

$$\Omega = \{X > 0 : \frac{q}{q+1}Q \leq X \leq Q].$$

Obviously, $G(X) = Q - \sum_{i=1}^{m} A_i^* X^{-q} A_i \leq Q$ for any $X \in \Omega$. It follows from Lemma 2.1 that

$$G(X) = Q - \sum_{i=1}^{m} A_i^* X^{-q} A_i$$

$$\geq Q^{1/2}[I - \frac{(1+q)^q}{q^q} \sum_{i=1}^{m} Q^{-1/2} A_i^* Q^{-q} A_i Q^{-1/2}] Q^{1/2}$$
$$\geq Q^{1/2}[1 - \frac{(1+q)^q}{q^q} \sum_{i=1}^{m} \|A_i\|^2 \|Q^{-1}\|^{q+1}] Q^{1/2} > \frac{q}{q+1} Q,$$

which gives $G(\Omega) \subseteq \Omega$.

Notice that for any $X, Y \in \Omega$,

$$X, Y \geq \frac{q}{q+1} \lambda_n(Q) I = \frac{q}{(q+1)\|Q^{-1}\|} I.$$

Consequently, we have by Lemma 2.3 that

$$\begin{aligned} \|G(X) - G(Y)\| &= \|\sum_{i=1}^{m} A_i^*(X^{-q} - Y^{-q}) A_i\| \\ &\leq \sum_{i=1}^{m} \|A_i\|^2 \|X^{-q} - Y^{-q}\| \\ &\leq [\frac{(q+1)^{q+1}}{q^q} \|Q^{-1}\|^{q+1} \sum_{i=1}^{m} \|A_i\|^2] \cdot \|X - Y\| \\ &= \rho \|X - Y\|, \end{aligned}$$

which means that $G(X)$ is a contraction on Ω. By Banach's fixed-point theorem, $G(X)$ has a unique fixed point on Ω, denoted by X_L. That is, Eq.(1.1) has a unique positive definite solution X_L in Ω. Combining the fact that $\varphi_2 \subset \Omega$, we obtain that $X_L \in \varphi_2$.

Next, we prove that X_L is the maximal positive definite solution of Eq.(1.1). Let X be an arbitrary positive definite solution of Eq.(1.1). Then $X \leq \alpha_2 Q$ according to Theorem 2.2. Since $G(X)$ is monotonically increasing, then

$$X = G(X) \leq G(\alpha_2 Q),\ X = G^k(X) \leq G^k(\alpha_2 Q) \to X_L,\ k \to \infty.$$

Thus $X \leq X_L$ which means that X_L is the maximal positive definite solution of Eq.(1.1). □

Remark 2.1. From the proof of Theorem 2.3(ii), we know that if

$$\sum_{i=1}^{m} \|A_i\|^2 \|Q^{-1}\|^{q+1} < \frac{q^q}{(q+1)^{q+1}},$$

then the maximal positive definite solution X_L is the unique positive definite solution of Eq.(1.1) satisfying $X > \frac{q}{q+1} Q$.

Corollary 2.1. *If* $\sum_{i=1}^{m} \|A_i\|^2 \|Q^{-1}\|^{q+1} < \frac{q^q}{(q+1)^{q+1}}$, *then the maximal positive definite solution* X_L *satisfies*

$$\|X_L^{-1}\| < (1 + \frac{1}{q}) \|Q^{-1}\|.$$

Moreover, for any other positive definite solution X of Eq.(1.1)*, we have*

$$\|X^{-1}\| > (1+\frac{1}{q})\|Q\|^{-1}.$$

Proof. Since $\sum_{i=1}^{m}\|A_i\|^2\|Q^{-1}\|^{q+1} < \frac{q^q}{(q+1)^{q+1}}$, then the maximal positive definite solution X_L exists and satisfies $X_L \geq \beta_2 Q > \frac{q}{q+1}Q$ according to Theorem 2.3. Thus $X_L^{-1} < (1+\frac{1}{q})Q^{-1}$ which gives $\|X_L^{-1}\| < (1+\frac{1}{q})\|Q^{-1}\|$.

Let $X \neq X_L$ be any positive definite solution to Eq.(1.1). Then

$$\lambda_n(Q^{-1/2}XQ^{-1/2}) \leq \beta_1 < \frac{q}{q+1}$$

from the proof of Theorem 2.3(i) and consequently $\|X^{-1}\| = \lambda_n^{-1}(X) > (1+\frac{1}{q})\lambda_n(Q^{-1}) = (1+\frac{1}{q})\|Q\|^{-1}$. □

Next we give two iterative methods for the maximal positive definite solution X_L .

As first method we consider the following fixed point iteration:

$$\begin{cases} X_0 = \gamma Q, \ \gamma \in [\mu, 1], \\ X_{k+1} = Q - \sum_{i=1}^{m} A_i^* X_k^{-q} A_i, \end{cases} \tag{2.5}$$

where μ is as defined in Theorem 2.2.

Theorem 2.4. *If Eq.*(1.1) *has a positive definite solution, then the sequence* $\{X_k\}$ *in iteration* (2.5) *is monotonically decreasing and converges to the maximal positive definite solution* X_L*.*

Proof. Let X be a positive definite solution of Eq.(1.1). Then $X_0 = \gamma Q \geq \mu Q \geq X$ according to Theorem 2.2. Assuming that $X_{k-1} \geq X$, we have

$$X_k = Q - \sum_{i=1}^{m} A_i^* X_{k-1}^{-q} A_i \geq Q - \sum_{i=1}^{m} A_i^* X^{-q} A_i = X.$$

Hence $X_k \geq X$ for each $k = 0, 1, 2, \ldots$.

Similarly, we prove that the sequence $\{X_k\}$ is monotonically decreasing by induction.

$$\begin{aligned} X_1 &= Q - \sum_{i=1}^{\mathrm{m}} A_i^* X_0^{-q} A_i \\ &= Q - \sum_{i=1}^{m} A_i^* (\gamma Q)^{-q} A_i \\ &= Q^{1/2}[I - \frac{\sum_{i=1}^{m} Q^{-1/2} A_i^* Q^{-q} A_i Q^{-1/2}}{\gamma^q}]Q^{1/2} \\ &\leq Q^{1/2}(1 - \frac{\sum_{i=1}^{m} \sigma_n^2(Q^{-q/2} A_i Q^{-1/2})}{\gamma^q})Q^{1/2} \leq \gamma Q = X_0, \end{aligned}$$

where the last inequality holds since

$$\gamma^q(1-\gamma) \le \mu^q(1-\mu) = \sum_{i=1}^{m} \sigma_n^2(Q^{-q/2}A_iQ^{-1/2}).$$

In fact, this can be obtained from the decreasing property of $f(x) = x^q(1-x)$ in $[\frac{q}{q+1}, 1]$ and the fact that $\frac{q}{q+1} \le \mu \le \gamma \le 1$. Suppose $X_k \le X_{k-1}$. Then

$$X_{k+1} = Q - \sum_{i=1}^{m} A_i^* X_k^{-q} A_i \le Q - \sum_{i=1}^{m} A_i^* X_{k-1}^{-q} A_i = X_k,$$

which implies that the sequence $\{X_k\}$ is monotonically decreasing. Hence $\{X_k\}$ is convergent. Denote $\lim_{k\to\infty} X_k = X_l$. Then $X_l \ge X$ for any positive definite solution X which gives $X_l = X_L$. □

Our next algorithm is an inversion-free iteration:

$$\begin{cases} Y_0 = (\gamma Q)^{-1}, \quad \gamma \in [\mu, 1], \\ X_k = Q - \sum_{i=1}^{m} A_i^* Y_k^q A_i, \\ Y_{k+1} = Y_k(2I - X_kY_k), \end{cases} \tag{2.6}$$

where μ is as defined in Theorem 2.2.

Theorem 2.5. *If Eq.*(1.1) *has a positive definite solution, then the sequences* $\{X_k\}$ *and* $\{Y_k\}$ *from* (2.6) *satisfy*

$$X_0 \ge X_1 \ge X_2 \ge \cdots, \quad \lim_{k\to\infty} X_k = X_L; \quad Y_0 \le Y_1 \le Y_2 \le \cdots, \quad \lim_{k\to\infty} Y_k = X_L^{-1},$$

where X_L *is the maximal positive definite solution of Eq.*(1.1).

Proof. Let X be a positive definite solution of Eq.(1.1). Then $X \le \mu Q$ according to Theorem 2.2. Combining this with Lemma 2.1, we obtain that

$$Y_0 = (\gamma Q)^{-1} \le (\mu Q)^{-1} \le X^{-1},$$

$$X_0 = Q - \sum_{i=1}^{m} A_i^* Y_0^q A_i \ge Q - \sum_{i=1}^{m} A_i^* X^{-q} A_i = X.$$

By Lemma 2.4, we have

$$Y_1 = 2Y_0 - Y_0X_0Y_0 \le X_0^{-1} \le X^{-1}.$$

Since

$$\begin{aligned} X_0 &= Q - \sum_{i=1}^{m} \frac{A_i^* Q^{-q} A_i}{\gamma^q} \\ &\le Q^{1/2}[1 - \sum_{i=1}^{m} \frac{\sigma_n^2(Q^{-q/2}A_iQ^{-1/2})}{\gamma^q}]Q^{1/2} \le \gamma Q = Y_0^{-1} \end{aligned}$$

from the definition of γ, then

$$Y_1 - Y_0 = Y_0 - Y_0X_0Y_0 = Y_0[Y_0^{-1} - X_0]Y_0 \ge 0.$$

It follows that

$$X_1 = Q - \sum_{i=1}^{m} A_i^* Y_1^q A_i \geq Q - \sum_{i=1}^{m} A_i^* X^{-q} A_i = X$$

and

$$X_1 - X_0 = \sum_{i=1}^{m} A_i^* Y_0^q A_i - \sum_{i=1}^{m} A_i^* Y_1^q A_i = \sum_{i=1}^{m} A_i^* (Y_0^q - Y_1^q) A_i < 0.$$

Hence $X \leq X_1 \leq X_0$ and $Y_0 \leq Y_1 \leq X^{-1}$. Assuming that

$$X \leq X_k \leq X_{k-1}, \quad Y_{k-1} \leq Y_k \leq X^{-1}, \ k = 2, 3, \ldots,$$

we have

$$\begin{aligned} Y_k &= 2Y_{k-1} - Y_{k-1} X_{k-1} Y_{k-1} \leq X_{k-1}^{-1} \leq X_k^{-1}, \\ Y_{k+1} &= 2Y_k - Y_k X_k Y_k \leq X_k^{-1} \leq X^{-1}, \\ X_{k+1} &= Q - \sum_{i=1}^{m} A_i^* Y_{k+1}^q A_i > Q - \sum_{i=1}^{m} A_i^* X^{-q} A_i = X, \end{aligned}$$

and consequently,

$$Y_{k+1} - Y_k = Y_k (Y_k^{-1} - X_k) Y_k \geq 0, \ X_{k+1} - X_k = \sum_{i=1}^{m} A_i^* (Y_k^q - Y_{k+1}^q) A_i \leq 0.$$

Hence we have by induction that $X_0 \geq X_1 \geq X_2 \geq \cdots \geq X_k \geq X$ and $Y_0 \leq Y_1 \leq Y_2 \leq \cdots \leq Y_k \leq X^{-1}$ hold for each $k = 0, 1, 2, \ldots$ and so the sequences $\{X_k\}$ and $\{Y_k\}$ are convergent. Denote $\lim_{k\to\infty} X_k = X_l$ and $\lim_{k\to\infty} Y_k = Y$. Taking limit in the iteration (2.6) leads to $Y = X_l^{-1}$ and $X_l = Q - \sum_{i=1}^{m} A_i^* X_l^{-q} A_i$. Moreover, since $X_k \geq X, k = 1, 2, 3, \ldots$ for any positive definite solution X, we have $X_l = X_L$. □

Theorem 2.6. *If Eq.*(1.1) *has a positive definite solution and after k iterative steps of iteration* (2.6)*, we have* $\|I - X_k Y_k\| < \epsilon$*, then*

$$\|X_k + \sum_{i=1}^{m} A_i^* X_k^{-q} A_i - Q\| \leq \epsilon q \alpha \lambda_1^{1-q}(Q) \sum_{i=1}^{m} \|A_i\|^2,$$

where $\alpha = \min\{\lambda_n^{2/q-1}(A_i Q^{-1} A_i^*) \lambda_1^{1-1/q}(A_i Q^{-1} A_i^*) : i = 1, 2, \ldots, m\}$.

Proof. According to Theorem 2.1, we have $X_L^q > A_i Q^{-1} A_i^* > 0$ for each $i = 1, 2, \ldots, m$. Denote $m_i = \lambda_n(A_i Q^{-1} A_i^*)$ and $M_i = \lambda_1(A_i Q^{-1} A_i^*)$. From the proof of Theorem 2.5, we have

$$\lambda_n(Q^{-1}) I \leq Q^{-1} \leq \frac{1}{\gamma} Q^{-1} = Y_0 \leq Y_k \leq X_k^{-1} \leq X_L^{-1}.$$

Then we obtain that

$$\|X_k^{-q} - Y_k^q\| \leq q \cdot \lambda_n^{q-1}(Q^{-1}) \|X_k^{-1} - Y_k\| = q \cdot \lambda_1^{1-q}(Q) \|X_k^{-1} - Y_k\|$$

from Lemma 2.3, and

$$\frac{1}{\gamma}Q^{-1} \leq X_k^{-1} \leq X_L^{-1} \leq (\frac{M_i}{m_i})^{1/q-1}(A_iQ^{-1}A_i^*)^{-1/q}$$
$$\leq M_i^{1/q-1}m_i^{1-2/q}I,\ i=1,2,\dots$$

from Lemma 2.5. Since

$$\begin{aligned} X_k + \sum_{i=1}^m A_i^* X_k^{-q} A_i - Q &= X_k - X_{k+1} + \sum_{i=1}^m A_i^*(X_k^{-q} - Y_{k+1}^q)A_i \\ &= \sum_{i=1}^m A_i^*(Y_{k+1}^q - Y_k^q)A_i + \sum_{i=1}^m A_i^*(X_k^{-q} - Y_{k+1}^q)A_i \\ &= \sum_{i=1}^m A_i^*(X_k^{-q} - Y_k^q)A_i. \end{aligned}$$

Then we have

$$\begin{aligned} \|X_k + \sum_{i=1}^m A_i^* X_k^{-q} A_i - Q\| &= \|\sum_{i=1}^m A_i^*(X_k^{-q} - Y_k^q)A_i\| \\ &\leq \sum_{i=1}^m \|A_i\|^2 \|X_k^{-q} - Y_k^q\| \\ &\leq q\lambda_1^{1-q}(Q)\sum_{i=1}^m \|A_i\|^2\|X_k^{-1} - Y_k\| \\ &\leq q\lambda_1^{1-q}(Q)\sum_{i=1}^m \|A_i\|^2\|X_k^{-1}\|\|I - X_kY_k\| \\ &\leq \epsilon q\lambda_1^{1-q}(Q)\sum_{i=1}^m \|A_i\|^2\|X_k^{-1}\| \\ &\leq \epsilon q\lambda_1^{1-q}(Q)M_i^{1/q-1}m_i^{1-2/q}\sum_{i=1}^m \|A_i\|^2 \\ &\leq \epsilon q\alpha\lambda_1^{1-q}(Q)\sum_{i=1}^m \|A_i\|^2, \end{aligned}$$

where $\alpha = \min\{\lambda_n^{1-2/q}(A_iQ^{-1}A_i^*)\lambda_1^{1/q-1}(A_iQ^{-1}A_i^*) : i=1,2,\dots,m\}$. □

3. Perturbation estimates for X_L

Consider the perturbed matrix equation

$$\tilde{X} + \sum_{i=1}^m \tilde{A}_i{}^* \tilde{X}^{-q} \tilde{A}_i = \tilde{Q}, \tag{3.1}$$

where $\tilde{A}_i$ and $\tilde{Q}$ are the slightly perturbed matrices of the matrices A_i and Q, respectively. In this section, we show that if $\|\tilde{A}_i - A\|$ and $\|\tilde{Q} - Q\|$ are sufficiently small, then the maximal solution $\tilde{X}_L$ to the perturbed matrix equation (3.1) exists. We derive a perturbation estimate for the maximal positive definite solution X_L and give an explicit expression of the Rice condition number of X_L.

Denote $\Delta Q = \tilde{Q} - Q,\ \Delta X = \tilde{X}_L - X_L,\ \Delta A_i = \tilde{A}_i - A_i, i = 1, 2, \ldots, m.$

Theorem 3.1. *Let*

$$\text{(i)}\quad \theta := \frac{q^q}{(q+1)^{q+1}} - \sum_{i=1}^{m} \|A_i\|^2 \|Q^{-1}\|^{q+1} > 0, \tag{3.2}$$

$$\text{(ii)}\quad \|\Delta Q\| \leq \frac{1}{\|Q^{-1}\|} \cdot (1 - \sqrt[q+1]{1-\theta}), \tag{3.3}$$

$$\text{(iii)}\quad \sum_{i=1}^{m} (\|\tilde{A}_i\|^2 - \|A_i\|^2) < \frac{(q+1)^{q+1} - q^q}{(q+1)^{q+1}\|Q^{-1}\|^{q+1}}\theta. \tag{3.4}$$

Then nonlinear matrix equations

$$X + \sum_{i=1}^{m} A_i^* X^{-q} A_i = Q \text{ and } \tilde{X} + \sum_{i=1}^{m} \tilde{A}_i{}^* \tilde{X}^{-q} \tilde{A}_i = \tilde{Q}$$

have maximal positive definite solutions X_L and $\tilde{X}_L$, respectively. Moreover,

$$\|\Delta X\| \leq \frac{1}{\xi} \cdot (\|\Delta Q\| + 2\sum_{i=1}^{m} \|X_L^{-q} A_i\| \cdot \|\Delta A_i\| + \sum_{i=1}^{m} \|X_L^{-q}\| \cdot \|\Delta A_i\|^2), \tag{3.5}$$

where

$$\xi = 1 - qb^{-(q+1)} \sum_{i=1}^{m} \|\tilde{A}_i\|^2, \quad b = \frac{q}{q+1} \min\{\lambda_n(Q), \lambda_n(\tilde{Q})\}.$$

Proof. Since $\theta > 0$, we know from Theorem 2.3 that Eq.(1.1) has the maximal positive definite solution $X_L \in [\beta_2 Q, \alpha_2 Q]$. Notice that $\theta < 1$ and

$$\|\tilde{Q}^{-1}\| \leq \|Q^{-1}\| + \|Q^{-1}\| \cdot \|\Delta Q\| \cdot \|\tilde{Q}^{-1}\| \leq \|Q^{-1}\| + (1 - \sqrt[q+1]{1-\theta})\|\tilde{Q}^{-1}\|,$$

which gives

$$\|\tilde{Q}^{-1}\|^{q+1} \leq \frac{\|Q^{-1}\|^{q+1}}{1-\theta}.$$

Consequently, we have

$$\begin{aligned} &\sum_{i=1}^{m} \|\tilde{A}_i\|^2 \|\tilde{Q}^{-1}\|^{q+1} \\ &< \frac{\|Q^{-1}\|^{q+1}}{1-\theta} \left[\sum_{i=1}^{m} \|A_i\|^2 + \frac{(q+1)^{q+1} - q^q}{(q+1)^{q+1}\|Q^{-1}\|^{q+1}}\theta \right] \end{aligned}$$

$$= \frac{\sum_{i=1}^m (q+1)^{q+1}\|A_i\|^2\|Q^{-1}\|^{q+1} + [(q+1)^{q+1} - q^q]\theta}{(1-\theta)(q+1)^{q+1}}$$
$$= \frac{q^q}{(q+1)^{q+1}} \cdot \frac{\sum_{i=1}^m \|A_i\|^2 \frac{(q+1)^{q+1}}{q^q}\|Q^{-1}\|^{q+1} + [\frac{(q+1)^{q+1}}{q^q} - 1]\theta}{1-\theta}$$
$$= \frac{q^q}{(q+1)^{q+1}}. \tag{3.6}$$

Applying Theorem 2.3, we obtain that the perturbed matrix equation (3.1) has the maximal positive definite solution $\tilde{X}_L \in [\tilde{\beta}_2 Q, \tilde{\alpha}_2 Q]$, where $\tilde{\beta}_2$ and $\tilde{\alpha}_2$ are the biggest positive solutions of the polynomial equations $x^q(1-x) = \sum_{i=1}^m \sigma_1^2(\tilde{Q}^{-q/2}\tilde{A}_i\tilde{Q}^{-1/2})$ and $x^q(1-x) = \sum_{i=1}^m \sigma_n^2(\tilde{Q}^{-q/2}\tilde{A}_i\tilde{Q}^{-1/2})$, respectively.

In the following, we show the estimate (3.5):

Since $X_L \geq \beta_2 Q > \frac{q}{q+1}\lambda_n(Q)I$ and $\tilde{X}_L \geq \tilde{\beta}_2\tilde{Q} > \frac{q}{q+1}\lambda_n(\tilde{Q})I$. Let $b = \frac{q}{q+1}\min\{\lambda_n(Q), \lambda_n(\tilde{Q})\}$. Then $X_L, \tilde{X}_L > bI$ and consequently,

$$\|X_L^{-q} - \tilde{X}_L^{-q}\| \leq qb^{-(q+1)}\|\Delta X\|$$

from Lemma 2.3.

Since $X_L + \sum_{i=1}^m A_i^* X_L^{-q} A_i = Q$ and $\tilde{X}_L + \sum_{i=1}^m \tilde{A}_i^* \tilde{X}_L^{-q}\tilde{A}_i = \tilde{Q}$, then

$$\tilde{X}_L - X_L + \sum_{i=1}^m \tilde{A}_i^*\tilde{X}_L^{-q}\tilde{A}_i - \sum_{i=1}^m A_i^* X_L^{-q} A_i = \tilde{Q} - Q,$$

i.e.,

$$\begin{aligned}\Delta X &= \Delta Q + \sum_{i=1}^m \tilde{A}_i{}^*(X_L^{-q} - \tilde{X}_L^{-q})\tilde{A}_i + \sum_{i=1}^m A_i^* X_L^{-q} A_i - \sum_{i=1}^m \tilde{A}_i^* X_L^{-q}\tilde{A}_i \\ &= \Delta Q + \sum_{i=1}^m \tilde{A}_i{}^*(X_L^{-q} - \tilde{X}_L^{-q})\tilde{A}_i - \sum_{i=1}^m \Delta A_i^* X_L^{-q} A_i \\ &\quad - \sum_{i=1}^m \Delta A_i^* X_L^{-q}\Delta A_i - \sum_{i=1}^m A_i^* X_L^{-q}\Delta A_i.\end{aligned}$$

Hence

$$\begin{aligned}\|\Delta X\| &\leq \|\Delta Q\| + \sum_{i=1}^m \|\tilde{A}_i{}^*(X_L^{-q} - \tilde{X}_L^{-q})\tilde{A}_i\| + \sum_{i=1}^m \|\Delta A_i^* X_L^{-q} A_i\| \\ &\quad + \sum_{i=1}^m \|\Delta A_i^* X_L^{-q}\Delta A_i\| + \sum_{i=1}^m \|A_i^* X_L^{-q}\Delta A_i\| \\ &\leq \|\Delta Q\| + \|X_L^{-q} - \tilde{X}_L^{-q}\|\sum_{i=1}^m \|\tilde{A}_i\|^2 + \sum_{i=1}^m \|\Delta A_i^* X_L^{-q}\Delta A_i\|\end{aligned}$$

$$+ 2\sum_{i=1}^{m} \|\Delta A_i^* X_L^{-q} A_i\|$$
$$\leq \|\Delta Q\| + qb^{-(q+1)}\|\Delta X\| \sum_{i=1}^{m} \|\tilde{A}_i\|^2 + 2\sum_{i=1}^{m} \|X_L^{-q} A_i\| \cdot \|\Delta A_i\|$$
$$+ \sum_{i=1}^{m} \|X_L^{-q}\| \cdot \|\Delta A_i\|^2.$$

Denote $\xi = 1 - qb^{-(q+1)} \sum_{i=1}^{m} \|\tilde{A}_i\|^2$. Notice from the proof of (3.6) that

$$\sum_{i=1}^{m} \|Q^{-1}\|^{q+1} \cdot \|\tilde{A}_i\|^2 < \sum_{i=1}^{m} \frac{\|Q^{-1}\|^{q+1}}{1-\theta} \cdot \|\tilde{A}_i\|^2$$
$$< \frac{\|Q^{-1}\|^{q+1}}{1-\theta} \left[\sum_{i=1}^{m} \|A_i\|^2 + \frac{(q+1)^{q+1} - q^q}{(q+1)^{q+1}\|Q^{-1}\|^{q+1}} \theta \right]$$
$$< \frac{q^q}{(q+1)^{q+1}}.$$

Then

$$qb^{-(q+1)} \sum_{i=1}^{m} \|\tilde{A}_i\|^2$$
$$= \begin{cases} q \cdot \frac{(q+1)^{q+1}}{q^{q+1}} \|Q^{-1}\|^{q+1} \sum_{i=1}^{m} \|\tilde{A}_i\|^2 < 1 & \text{if } b = \frac{q}{q+1}\lambda_n(Q), \\ q \cdot \frac{(q+1)^{q+1}}{q^{q+1}} \|\tilde{Q}^{-1}\|^{q+1} \sum_{i=1}^{m} \|\tilde{A}_i\|^2 < 1 & \text{if } b = \frac{q}{q+1}\lambda_n(\tilde{Q}). \end{cases}$$

Therefore, $\xi > 0$ and we have

$$\|\Delta X\| \leq \frac{1}{\xi} \cdot (\|\Delta Q\| + 2\sum_{i=1}^{m} \|X_L^{-q} A_i\| \cdot \|\Delta A_i\| + \sum_{i=1}^{m} \|X_L^{-q}\| \cdot \|\Delta A_i\|^2). \qquad \square$$

By the theory of condition number developed by Rice [22], we give in this following an explicit expression of the condition number of the maximal positive definite solution X_L.

The complex case.

Lemma 3.1 ([14]). *Let X be any $n \times n$ positive definite matrix, $0 < q \leq 1$. Then*
(i) $X^{-q} = \frac{\sin q\pi}{\pi} \int_0^\infty (\lambda I + X)^{-1} \lambda^{-q} d\lambda$,
(ii) $X^{-q} = \frac{\sin q\pi}{q\pi} \int_0^\infty (\lambda I + X)^{-1} X (\lambda I + X)^{-1} \lambda^{-q} d\lambda$.

From Theorem 3.1, we see that if $\|(\Delta A_1, \ldots, \Delta A_m, \Delta Q)\|_F$ is sufficiently small, then the maximal positive solution $\tilde{X}_L$ to the perturbed matrix equation (3.1) exists. Subtracting (1.1) from (3.1) gives rise to

$$\Delta X + \sum_{i=1}^{m} [\tilde{A}_i^* \tilde{X}_L^{-q} \tilde{A}_i - A_i^* X_L^{-q} A_i] = \Delta Q,$$

i.e.,

$$\Delta X + \sum_{i=1}^{m}[A_i^*(\tilde{X}_L^{-q} - X_L^{-q})A_i + \tilde{A_i}^*(\tilde{X}_L^{-q} - X_L^{-q})\Delta A_i$$
$$+ \Delta A_i^*(\tilde{X}_L^{-q} - X_L^{-q})A_i]$$
$$= \Delta Q - \sum_{i=1}^{m}[(\Delta A_i^* X_L^{-q} A_i + \Delta A_i^* X_L^{-q}\Delta A_i + A_i^* X_L^{-q}\Delta A_i)]. \tag{3.7}$$

Applying Lemma 3.1, we have

(3.8)

$$\begin{aligned}
&\tilde{X}_L^{-q} - X_L^{-q}\\
&= \frac{\sin q\pi}{\pi}\int_0^\infty [(\lambda I + X_L + \Delta X)^{-1} - (\lambda I + X_L)^{-1}]\lambda^{-q} d\lambda\\
&= \frac{\sin q\pi}{\pi}\int_0^\infty -(\lambda I + X_L)^{-1}\Delta X(\lambda I + X_L + \Delta X)^{-1}\lambda^{-q} d\lambda\\
&= \frac{\sin q\pi}{\pi}\int_0^\infty -(\lambda I + X_L)^{-1}\Delta X(\lambda I + X_L)^{-1}\lambda^{-q} d\lambda\\
&\quad + \frac{\sin q\pi}{\pi}\int_0^\infty (\lambda I + X_L)^{-1}\Delta X(\lambda I + X_L + \Delta X)^{-1}\Delta X(\lambda I + X_L)^{-1}\lambda^{-q} d\lambda.
\end{aligned}$$

Combining (3.8) with (3.7), we obtain that

(3.9)

$$\Delta X - \frac{\sin q\pi}{\pi}\sum_{i=1}^{m}\int_0^\infty A_i^*(\lambda I + X_L)^{-1}\Delta X(\lambda I + X_L)^{-1}A_i\lambda^{-q} d\lambda = E + h(\Delta X),$$

where $E = \Delta Q - \sum_{i=1}^{m}[(\Delta A_i^* X_L^{-q} A_i + \Delta A_i^* X_L^{-q}\Delta A_i + A_i^* X_L^{-q}\Delta A_i)]$,

$$\begin{aligned}
&h(\Delta X)\\
&= -\frac{\sin q\pi}{\pi}\sum_{i=1}^{m}[A_i^*\int_0^\infty (\lambda I + X_L)^{-1}\Delta X(\lambda I + X_L + \Delta X)^{-1}\Delta X(\lambda I + X_L)^{-1}\lambda^{-q} d\lambda A_i\\
&\quad + \frac{\sin q\pi}{\pi}\sum_{i=1}^{m}[\tilde{A_i}^*\int_0^\infty (\lambda I + X_L)^{-1}\Delta X(\lambda I + X_L + \Delta X)^{-1}\lambda^{-q} d\lambda\Delta A_i\\
&\quad + \Delta A_i^*\int_0^\infty (\lambda I + X_L)^{-1}\Delta X(\lambda I + X_L + \Delta X)^{-1}\lambda^{-q} d\lambda A_i].
\end{aligned}$$

Lemma 3.2. *Let* $\sum_{i=1}^{m}\|A_i\|^2\|Q^{-1}\|^{q+1} < \frac{q^q}{(q+1)^{q+1}}$. *Then the linear operator* $\boldsymbol{L} : H^{n\times n} \to H^{n\times n}$ *defined by*

$$\boldsymbol{L}W = W - \frac{\sin q\pi}{\pi}\sum_{i=1}^{m}\int_0^\infty A_i^*(\lambda I + X_L)^{-1}W(\lambda I + X_L)^{-1}A_i\lambda^{-q} d\lambda \tag{3.10}$$

is invertible.

Proof. It suffices to show that for any matrix $V \in H^{n\times n}$, the following equation

$$\mathbf{L}W = V \tag{3.11}$$

has a unique solution. Define the operator $\mathbf{M} : H^{n\times n} \to H^{n\times n}$ by

$$\mathbf{M}Z = \frac{\sin q\pi}{\pi}\sum_{i=1}^{m}\int_0^\infty X_L^{-1/2}A_i^*(\lambda I + X_L)^{-1}X_L^{1/2}ZX_L^{1/2}(\lambda I + X_L)^{-1}A_iX_L^{-1/2}\lambda^{-q}d\lambda,$$
$$Z \in H^{n\times n}.$$

Let $Y = X_L^{-1/2}WX_L^{-1/2}$. Thus (3.11) is equivalent to

$$Y - \mathbf{M}Y = X_L^{-1/2}VX_L^{-1/2}. \tag{3.12}$$

Notice that $\|X_L^{-1}\| < \frac{q+1}{q}\|Q^{-1}\|$. According to Lemma 3.1(ii), we have

$$\begin{aligned}
&\|\mathbf{M}Y\| \\
&= \|\tfrac{\sin q\pi}{\pi}\sum_{i=1}^{m}\int_0^\infty X_L^{-1/2}A_i^*(\lambda I+X_L)^{-1}X_L^{1/2}YX_L^{1/2}(\lambda I+X_L)^{-1}A_iX_L^{-1/2}\lambda^{-q}d\lambda\| \\
&\le \|Y\|\cdot\|\sum_{i=1}^{m}\tfrac{\sin q\pi}{\pi}\int_0^\infty X_L^{-1/2}A_i^*(\lambda I + X_L)^{-1}X_L(\lambda I + X_L)^{-1}A_iX_L^{-1/2}\lambda^{-q}d\lambda\| \\
&= \|Y\|\cdot\|\sum_{i=1}^{m}q\cdot X_L^{-1/2}A_i^*\cdot\tfrac{\sin q\pi}{q\pi}\int_0^\infty(\lambda I+X_L)^{-1}X_L(\lambda I+X_L)^{-1}\lambda^{-q}d\lambda\cdot A_iX_L^{-1/2}\| \\
&= q\|Y\|\cdot\|\sum_{i=1}^{m}X_L^{-1/2}A_i^*X_L^{-q}A_iX_L^{-1/2}\| \\
&\le q\|Y\|\cdot\sum_{i=1}^{m}\|X_L^{-q/2}A_iX_L^{-1/2}\|^2 \\
&\le q\|Y\|\cdot\sum_{i=1}^{m}\|A_i\|^2\|X_L^{-1}\|^{q+1} \\
&\le q\|Y\|\cdot\sum_{i=1}^{m}\|A_i\|^2(\frac{q+1}{q})^{q+1}\|Q^{-1}\|^{q+1} < \|Y\|.
\end{aligned}$$

Then $\|\mathbf{M}\| < 1$ which implies that $\mathbf{I}-\mathbf{M}$ is invertible. Therefore, for any matrix $V \in H^{n\times n}$, equation (3.12) has a unique solution Y. Thus equation (3.11) has a unique solution W for any $V \in H^{n\times n}$ which implies that the operator $\mathbf{L}$ is invertible. The proof is then completed. □

Let $B_i = X_L^{-q}A_i, i = 1, 2, \ldots, m$. We can rewrite (3.9) as

$$\begin{aligned}
\Delta X &= \tilde{X}_L - X_L \\
&= \mathbf{L}^{-1}(\Delta Q - \sum_{i=1}^{m}B_i^*\Delta A_i - \sum_{i=1}^{m}\Delta A_i^*B_i) \\
&\quad - \mathbf{L}^{-1}(\sum_{i=1}^{m}\Delta A_i^*X_L^{-q}\Delta A_i) + \mathbf{L}^{-1}(h(\Delta X)).
\end{aligned}$$

Then we have
(3.13)

$$\begin{aligned}\Delta X &= \tilde{X}_L - X_L\\ &= \mathbf{L}^{-1}(\Delta Q - \sum_{i=1}^{m} B_i^*\Delta A_i - \sum_{i=1}^{m}\Delta A_i^* B_i) + O(\|(\Delta A_1,\ldots,\Delta A_m,\Delta Q)\|_F^2)\end{aligned}$$

$(\Delta A_1,\ldots,\Delta A_m,\Delta Q)\to 0$. By Rice's condition number theory [22], we define the condition number of the maximal positive definite solution X_L of Eq.(1.1) as follows:

$$C(X_L) = \lim_{\delta\to 0} \sup_{\substack{\|(\frac{\Delta A_1}{\mu_1},\ldots,\frac{\Delta A_m}{\mu_m},\frac{\Delta Q}{\rho})\|_F\le\delta\\ \Delta A_i\in C^{n\times n},\Delta Q\in H^{n\times n}}} \frac{\|\Delta X\|_F}{\xi\delta}, \tag{3.14}$$

where $\xi,\rho,\mu_1,\ldots,\mu_m$ are positive parameters. Taking $\xi=\rho=\mu_1=\cdots=\mu_m=1$ in (3.14) gives the absolute condition number $C_{\rm abs}(X_L)$ and taking $\xi=\|X_L\|_F$, $\rho=\|Q\|_F$, $\mu_i=\|A_i\|_F$, $i=1,2,\ldots,m$ gives the relative condition number $C_{\rm rel}(X_L)$.

Substituting (3.13) into (3.14), we get

$$\begin{aligned}C(X_L) &= \frac{1}{\xi}\max_{\substack{(\frac{\Delta A_1}{\mu_1},\ldots,\frac{\Delta A_m}{\mu_m},\frac{\Delta Q}{\rho})\neq 0\\ \Delta A_i\in C^{n\times n},\Delta Q\in H^{n\times n}}} \frac{\|\mathbf{L}^{-1}[\Delta Q - \sum_{i=1}^m (B_i^*\Delta A_i + \Delta A_i^* B_i)]\|_F}{\|(\frac{\Delta A_1}{\mu_1},\ldots,\frac{\Delta A_m}{\mu_m},\frac{\Delta Q}{\rho})\|_F}\\ &= \frac{1}{\xi}\max_{\substack{(E_1,\ldots,E_m,H)\neq 0\\ E_i\in C^{n\times n},H\in H^{n\times n}}} \frac{\|\mathbf{L}^{-1}[\rho H - \sum_{i=1}^m \mu_i(B_i^*E_i + E_i^*B_i)]\|_F}{\|(E_1,\ldots,E_m,H)\|_F}\\ &= \frac{1}{\xi}\max_{\substack{(E_1,\ldots,E_m,H)\neq 0\\ E_i\in C^{n\times n},H\in H^{n\times n}}} \frac{\|\mathbf{L}^{-1}[\rho H + \sum_{i=1}^m \mu_i B_i^*(-E_i) + \sum_{i=1}^m (-E_i)^*B_i]\|_F}{\|(-E_1,\ldots,-E_m,H)\|_F}\\ &= \frac{1}{\xi}\max_{\substack{(K_1,\ldots,K_m,H)\neq 0\\ K_i\in C^{n\times n},H\in H^{n\times n}}} \frac{\|\mathbf{L}^{-1}[\rho H + \sum_{i=1}^m \mu_i(B_i^*K_i + K_i^*B_i)]\|_F}{\|(K_1,\ldots,K_m,H)\|_F}.\end{aligned}$$

Let L be the matrix of the operator $\mathbf{L}$. Then it is not difficult to see that

$$L = I\otimes I - \frac{\sin q\pi}{\pi}\sum_{i=1}^{m}\int_0^\infty [(\lambda I + X_L)^{-1}A_i]^T \otimes [(\lambda I + X_L)^{-1}A_i]^*\lambda^{-q}d\lambda.$$

Denote by $\eta = \mathrm{vec}(H) = a + jb$, $w_i = \mathrm{vec}(K_i) = u^{(i)} + jv^{(i)}$, where $a,b,u^{(i)},v^{(i)}\in R^{n^2}$, and j is the imaginary unit. Let

$$g_1 = \begin{pmatrix} a\\ b\end{pmatrix},\ g_2^{(i)} = \begin{pmatrix} u^{(i)}\\ v^{(i)}\end{pmatrix},\ i=1,2,\ldots,m,\ g = \begin{pmatrix} g_1\\ g_2^{(1)}\\ \vdots\\ g_2^{(m)}\end{pmatrix},$$

$$L^{-1}(I\otimes B_i^*) = L^{-1}(I\otimes (X_L^{-q}A_i)^*) = U_1^{(i)} + j\Omega_1^{(i)},\ i=1,2,\ldots,m,$$

$$L^{-1}(B_i^T \otimes I)\Pi = L^{-1}((X_L^{-q}A_i)^T \otimes I)\Pi = U_2^{(i)} + j\Omega_2^{(i)},\ i = 1, 2, \ldots, m,$$

where $U_1^{(i)}$, $U_2^{(i)}$, $\Omega_1^{(i)}$, $\Omega_2^{(i)} \in R^{n^2 \times n^2}$, and Π is the vec-permutation matrix, such that $\mathrm{vec}(K^T) = \Pi \mathrm{vec} K$. Denote

$$L^{-1} = S + j\Sigma,\ S, \Sigma \in R^{n^2 \times n^2},$$
$$S_c = \begin{bmatrix} S & -\Sigma \\ \Sigma & S \end{bmatrix}, U_c^{(i)} = \begin{bmatrix} U_1^{(i)} + U_2^{(i)} & \Omega_2^{(i)} - \Omega_1^{(i)} \\ \Omega_1^{(i)} + \Omega_2^{(i)} & U_1^{(i)} - U_2^{(i)} \end{bmatrix}.$$

Then we obtain that

$$\begin{aligned}
&C(X_L)\\
&= \frac{1}{\xi} \max_{\substack{(K_1,\ldots,K_m,H)\neq 0 \\ K_i \in C^{n\times n}, H \in H^{n\times n}}} \frac{\|\mathbf{L}^{-1}[\rho H + \sum_{i=1}^m \mu_i (B_i^* K_i + K_i^* B_i)]\|_F}{\|(K_1,\ldots,K_m,H)\|_F}\\
&= \frac{1}{\xi} \max_{\substack{(K_1,\ldots,K_m,H)\neq 0 \\ K_i \in C^{n\times n}, H \in H^{n\times n}}} \frac{\|\rho L^{-1}\mathrm{vec}(H) + \sum_{i=1}^m \mu_i L^{-1}\mathrm{vec}(B_i^* K_i + K_i^* B_i)\|}{\|\mathrm{vec}(K_1,\ldots,K_m,H)\|}\\
&= \frac{1}{\xi} \max_{\substack{(K_1,\ldots,K_m,H)\neq 0 \\ K_i \in C^{n\times n}, H \in H^{n\times n}}} \frac{\|\rho L^{-1}\mathrm{vec}(H) + \sum_{i=1}^m \mu_i [L^{-1}(I\otimes B_i^*)\mathrm{vec}(K_i) + L^{-1}(B_i^T \otimes I)\mathrm{vec}(K_i^*)]\|}{\|\mathrm{vec}(K_1,\ldots,K_m,H)\|}\\
&= \frac{1}{\xi} \max_{\substack{(K_1,\ldots,K_m,H)\neq 0 \\ K_i \in C^{n\times n}, H \in H^{n\times n}}} \frac{\|\rho S_c \binom{a}{b} + \sum_{i=1}^m \mu_i U_c^{(i)} \binom{u^{(i)}}{v^{(i)}}\|}{\|\mathrm{vec}(K_1,\ldots,K_m,H)\|}\\
&= \frac{1}{\xi} \max_{(g_1, g_2^{(1)},\ldots,g_2^{(m)})\neq 0} \frac{\|\rho S_c g_1 + \sum_{i=1}^m \mu_i U_c^{(i)} g_2^{(i)}\|}{\|g\|}\\
&= \frac{1}{\xi} \max_{g\neq 0} \frac{\|(\rho S_c, \mu_1 U_c^{(1)}, \ldots, \mu_m U_c^{(m)})g\|}{\|g\|} = \frac{1}{\xi}\|(\rho S_c, \mu_1 U_c^{(1)}, \ldots, \mu_m U_c^{(m)})\|.
\end{aligned}$$

Theorem 3.2. *Let* $\sum_{i=1}^m \|A_i\|^2 \|Q^{-1}\|^{q+1} < \frac{q^q}{(q+1)^{q+1}}$. *Then the condition number* $C(X_L)$ *defined by* (3.14) *has the following explicit expression*

$$C(X_L) = \frac{1}{\xi}\|(\rho S_c, \mu_1 U_c^{(1)}, \ldots, \mu_m U_c^{(m)})\|, \tag{3.15}$$

where $S_c, U_c^{(i)}, i = 1, 2, \ldots, m$ *are defined as above.*

Remark 3.1. From (3.15), we have the relative condition number

$$C_{\mathrm{rel}}(X_L) = \frac{\|(\|Q\|_F S_c, \|A_1\|_F U_c^{(1)}, \ldots, \|A_m\|_F U_c^{(m)})\|}{\|X_L\|_F}. \tag{3.16}$$

The real case

Next we consider the real case, i.e., all the coefficient matrices $A_1, \ldots, A_m, Q$ of Eq.(1.1) are real. In such a case the corresponding maximal solution X_L is also real. Similar to Theorem 3.2, we obtain the following theorem.

Theorem 3.3. *Let $A_1, \dots, A_m, Q$ be real. Suppose that*

$$\sum_{i=1}^{m} \|A_i\|^2 \|Q^{-1}\|^{q+1} < \frac{q^q}{(q+1)^{q+1}}.$$

Then the condition number $C(X_L)$ defined by (3.14) has the explicit expression

$$C(X_L) = \frac{1}{\xi} \|(\rho S_r, \mu_1 U_r^{(1)}, \dots, \mu_m U_r^{(m)})\|, \tag{3.17}$$

where

$$S_r = [I \otimes I - \frac{\sin q\pi}{\pi} \sum_{i=1}^{m} \int_0^{\infty} [(\lambda I + X_L)^{-1} A_i]^T \otimes [(\lambda I + X_L)^{-1} A_i]^T \lambda^{-q} d\lambda]^{-1},$$

$$U_r^{(i)} = S_r [I \otimes (A_i^T X_L^{-q}) + ((A_i^T X_L^{-q}) \otimes I)\Pi], \quad i = 1, 2, \dots, m.$$

Remark 3.2. In the real case the relative condition number is given by

$$C_{\mathrm{rel}}(X_L) = \frac{\|(\|Q\|_F S_r, \|A_1\|_F U_r^{(1)}, \dots, \|A_m\|_F U_r^{(m)})\|}{\|X_L\|_F}. \tag{3.18}$$

4. Numerical experiments

In this section, some simple examples are given to illustrate the results of the previous sections. All the tests are carried out using MATLAB 7.1 with machine precision around 10^{-16}. The practical stopping criterion used is the residual $\|X + \sum_{i=1}^{m} A_i^* X^{-q} A_i - Q\| < 10^{-10}$.

Example 4.1. Consider Eq.(1.1) with the case $m = 2$, $q = 0.3$, and the matrices A_1, A_2 and Q as follows:

$$A_1 = \begin{pmatrix} -0.45 & 0.45 & 0.85 & -1.2 & 0.75 \\ 0.55 & 1.05 & 0.4 & 0.75 & 0.9 \\ -0.9 & 0.95 & -0.7 & 0.85 & -0.9 \\ 0.7 & -0.85 & 0.4 & 0.7 & 0.75 \\ 0.25 & 0.65 & 0.75 & -0.6 & 0.65 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -0.54 & 0.57 & 1.02 & -1.35 & 0.93 \\ 0.69 & 1.26 & 0.51 & 0.63 & 1.11 \\ -1.08 & 1.14 & 0.87 & 1.02 & -1.11 \\ 0.87 & -1.02 & 0.51 & 0.84 & 0.93 \\ 0.33 & 0.81 & 0.93 & -0.72 & 0.78 \end{pmatrix},$$

$$Q = \begin{pmatrix} 68.6 & 28.8 & 21.2 & 25.2 & 21.6 \\ 28.8 & 52.4 & 9.6 & 10.8 & 20.4 \\ 21.2 & 9.6 & 38.0 & 12.0 & 13.2 \\ 25.2 & 10.8 & 12.0 & 48.9 & 9.6 \\ 21.6 & 20.4 & 13.2 & 9.6 & 40.4 \end{pmatrix}.$$

By computation, $(\|A_1\|^2 + \|A_2\|^2)\|Q^{-1}\|^{q+1} = 0.3019 < \frac{q^q}{(q+1)^{q+1}} = 0.4955$, $\frac{q}{q+1} = 0.2308$, $\beta_2 = 0.8164$ and $\alpha_2 = 0.9992$. According to Theorem 2.5, take

$\gamma = 1$, using iteration (2.6) and iterating 8 steps, then we get the maximal positive definite solution to Eq.(1.1):

$$X_L \approx X_8 = \begin{pmatrix} 66.8612 & 29.6685 & 20.3249 & 24.7669 & 20.0718 \\ 29.6685 & 49.7674 & 9.6956 & 10.7331 & 20.6371 \\ 20.3249 & 9.6956 & 36.3003 & 13.3999 & 11.2874 \\ 24.7669 & 10.7331 & 13.3999 & 45.7122 & 10.4319 \\ 20.0718 & 20.6371 & 11.2874 & 10.4319 & 37.8838 \end{pmatrix}$$

with the residual $\|X_8 + \sum_{i=1}^{m} A^* X_8^{-q} A - Q\| = 8.4947e-012$. Moreover, from $\lambda_n(X_8 - \beta_2 Q) = 0.2338$ and $\lambda_n(\alpha_2 Q - X_8) = 0.0012$, we know that $X_L \in [\beta_2 Q, \alpha_2 Q]$.

Example 4.2. Let

$$A = \frac{\sqrt{3}}{45}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}, \ B = \frac{A + A^*}{2},$$

$q = 0.5,\ X = \mathrm{diag}(0.725, 2, 3, 2, 1),\ Q = X + A^* X^{-q} A + B^* X^{-q} B.$

Consider the perturbed matrix equation

$$\tilde{X} + \tilde{A}_j^* \tilde{X}^{-q} \tilde{A}_j + \tilde{B}_j^* \tilde{X}^{-q} \tilde{B}_j = \tilde{Q}_j,$$

where $\epsilon_j = 0.1^{2j}$, $\tilde{A}_j = A + \epsilon_j(I + E)$, $\tilde{B}_j = B + \epsilon_j(I + 2E)$ $\tilde{X}_j = X + \epsilon_j(I - E)$, $\tilde{Q}_j = \tilde{X}_j + \tilde{A}_j^* \tilde{X}_j^{-q} \tilde{A}_j + \tilde{B}_j^* \tilde{X}_j^{-q} \tilde{B}_j$, with

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Now we compute the perturbation bounds for Eq.(1.1).

By computation, $(\|A\|^2 + \|B\|^2)\|Q^{-1}\|^{q+1} = 0.0286 < \frac{q^q}{(q+1)^{q+1}} = 0.3849$ and $\lambda_n(X - \frac{q}{q+1}Q) = 0.4804 > 0$ which implies that $X = X_L$ from Remark 2.1. Obviously, $\tilde{X}_j$ are positive definite solutions of the perturbed matrix equations $\tilde{X} + \tilde{A}_j^* \tilde{X}^{-q} \tilde{A}_j + \tilde{B}_j^* \tilde{X}^{-q} \tilde{B}_j = \tilde{Q}_j$. Moreover, it is not difficult to verify that the corresponding equations $\tilde{X} + \tilde{A}_j^* \tilde{X}^{-q} \tilde{A}_j + \tilde{B}_j^* \tilde{X}^{-q} \tilde{B}_j = \tilde{Q}_j$ and $\tilde{X}_j$ satisfy the assumption $(\|\tilde{A}_j\|^2 + \|\tilde{B}_j\|^2)\|\tilde{Q}_j^{-1}\|^{q+1} < \frac{q^q}{(q+1)^{q+1}}$ and the conditions $\lambda_n(\tilde{X}_j - \frac{q}{q+1}\tilde{Q}_j) > 0$ for each $j = 1, 2, 3, 4, 5$. Thus by Remark 2.1, $\tilde{X}_j$ $(j = 1, 2, \ldots, 5)$ are the maximal positive definite solutions of the corresponding perturbed matrix equations, respectively. We denote $\tilde{X}^j = \tilde{X}_L^j$ and let $\Delta X^{(j)} = \tilde{X}_L^j - X_L$. All the conditions of Theorem 3.1 are satisfied for $j = 1, 2, 3, 4, 5$. The results are given in the following table.

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
true error $\frac{\|\Delta X^{(j)}\|}{\|X_L\|}$	0.0133	$1.3333e{-}004$	$1.3333e{-}006$	$1.3333e{-}008$	$1.3333e{-}010$
our result (3.5)	0.0277	$2.4599e{-}004$	$2.4581e{-}006$	$2.4581e{-}008$	$2.4581e{-}010$

Example 4.3. Consider Eq.(1.1) with $q = 0.5$ and

$$A_1 = \begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1.1 & 0 \\ 0 & 1.2 \end{pmatrix},$$

where $a_1 = 0.25+10^{-k}$ and $a_2 = 0.35+10^{-k}$. Denote $\theta = \sum_{i=1}^{m} \|A_i\|^2 \|Q^{-1}\|^{q+1} - \frac{q^q}{(q+1)^{q+1}}$. Results for $C_{\mathrm{rel}}(X_L)$ by (3.18) with different vales of k are listed below where $C_{\mathrm{rel}}(X_L)$ is the relative condition number of the maximal positive definite solution.

k	1	2	3	4	5	6
θ	-0.1032	-0.2140	-0.2235	-0.2244	-0.2245	-0.2245
$C_{\mathrm{rel}}(X_L)$	1.2588	1.1452	1.1362	1.1353	1.1352	1.1352

From the numerical results in the second line, we see that the condition of Theorem 3.3 is always satisfied for each $k = 1, 2, \ldots, 6$. The numerical results listed in the third line show that the maximal positive definite solution X_L is well-conditioned in such cases.

Acknowledgement. The research was supported by National Science Foundation of China (11101322, 61373174), Natural Science Foundation of Shanxi Province(2010011006), and the Fundamental Research Funds for the Central Universities(K5051370007).

References

[1] R. Bhatia, *Matrix Analysis*, Graduate Texts in Mathematics, Springer, Verlag, 1997.

[2] B. L. Buzbee, G. H. Golub, and C. W. Nilson, *On direct methods for solving Poisson's equations*, SIAM J. Numer. Anal. **7** (1970), no. 4, 627–656.

[3] X. F. Duan and A. P. Liao, *On Hermitian positive definite solution of the matrix equation $X - \sum_{i=1}^{m} A_i^* X^r A_i = Q$*, J. Comput. Appl. Math. **229** (2009), no. 1, 27–36.

[4] X. F. Duan, C. M. Li, and A. P. Liao, *Solutions and perturbation analysis for the nonlinear matrix equation $X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = I$*, Appl. Math. Comput. **218** (2011), no. 8, 4458–4466.

[5] X. F. Duan, A. P. Liao, and B. Tang, *On the nonlinear matrix equation $X - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q$*, Linear Algebra Appl. **429** (2008), no. 1, 110–121.

[6] J. C. Engwerda, *On the existence of a positive definite solution of the matrix equation $X + A^T X^{-1} A = I$*, Linear Algebra Appl. **194** (1993), 91–108.

[7] T. Furuta, *Operator inequalities associated with Hölder-Mccarthy and Kantorovich inequalities*, J. Inequal. Appl. **6** (1998), no. 2, 137–148.

[8] C. H. Guo, Y. C. Kuo, and W. W. Lin, *Numerical solution of nonlinear matrix equations arising from Green's function calculations in nano research*, J. Comput. Appl. Math. **236** (2012), no. 17, 4166–4180.

[9] V. I. Hasanov, *Positive definite solutions of the matrix equations $X \pm A^* X^{-q} A = Q$*, Linear Algebra Appl. **404** (2005), 166–182.

[10] V. I. Hasanov and S. M. El-Sayed, *On the positive definite solutions of nonlinear matrix equation $X + A^* X^{-\delta} A = Q$*, Linear Algebra Appl. **412** (2006), no. 2, 154–160.

[11] Y. He and J. Long, *On the Hermitian positive definite solution of the nonlinear matrix equation* $X+\sum_{i=1}^{m} A_i^* X^{-1} A_i = I$, Appl. Math. Comput. **216** (2010), no. 12, 3480–3485.

[12] I. G. Ivanov, *On positive definite solutions of the family of matrix equations* $X + A^* X^{-n} A = Q$, J. Comput. Appl. Math. **193** (2006), no. 1, 277–301.

[13] P. Lancaster and L. Rodman, *Algebraic Riccati Equations*, Oxford Science Publishers, Oxford, 1995.

[14] J. Li and Y. H. Zhang, *Perturbation analysis of the matrix equation* $X - A^* X^{-p} A = Q$, Linear Algebra Appl. **431** (2009), no. 9, 936–945.

[15] Y. Lim, *Solving the nonlinear matrix equation* $X = Q+\Sigma_{i=1}^{m} M_i X^{\delta_i} M_i^*$ *via a contraction principal*, Linear Algebra Appl. **430** (2009), no. 4, 1380–1383.

[16] W. W. Lin and S. F. Xu, *Convergence analysis of structure-preserving doubling algorithms for Riccati-type matrix equations*, SIAM J. Matrix Anal. Appl. **28** (2006), no. 1, 26–39.

[17] X. G. Liu and H. Gao, *On the positive definite solutions of the matrix equation* $X^s \pm A^* X^{-t} A = I_n$, Linear Algebra Appl. **368** (2003), 83–79.

[18] B. Meini, *Efficient computation of the extreme solutions of* $X + A^* X^{-1} A = Q$ *and* $X - A^* X^{-1} A = Q$, Math. Comput. **71** (2002), no. 239, 1189–1204.

[19] Z. Y. Peng and S. M. El-Sayed, *On positive definite solution of a nonlinear matrix equation*, Numer. Linear Algebra Appl. **14** (2007), no. 2, 99–113.

[20] Z. Y. Peng, S. M. El-Sayed, and X. L. Zhang, *Iterative methods for the extremal positive definite solution of the matrix equation* $X + A^* X^{-\alpha} A = Q$, J. Comput. Appl. Math. **2000** (2007), no. 2, 520–527.

[21] C. M. Ran and C. B. R. Martine, *On the nonlinear matrix equation* $X+A^*F(X)A = Q$: *solutions and perturbation theory*, Linear Algebra Appl. **346** (2002), 15–26.

[22] J. R. Rice, *A theory of condition*, SIAM J. Numer. Anal. **3** (1966), no. 2, 287–310.

[23] A. M. Sarhan, N. M. El-Shazy, and E. M. Shehata, *On the existence of extremal positive definite solutions of the nonlinear matrix equation* $X^r + \Sigma_{i=1}^{m} A_i^* X^{\delta_i} A_i = I$, Math. Comput. Model. **51** (2010), no. 9-10, 1107–1117.

[24] S. M. El-Sayed and M. G. Petkov, *Iterative methods for nonlinear matrix equation* $X + A^* X^{-\alpha} A = I$, Linear Algebra Appl. **403** (2005), no. 1, 45–52.

[25] J. G. Sun and S. F. Xu, *Perturbation analysis of the maximal solution of the matrix equation* $X + A^* X^{-1} A = P$.Π, Linear Algebra Appl. **362** (2003), 211–228.

[26] J. Wang, Y. H. Zhang, and B. R. Zhu, *On Hermitian positive definite solutions of matrix equation* $X + A^* X^{-q} A = I (q > 0)$, Math. Num. Sin. **26** (2004), no. 1, 61–72.

[27] X. Y. Yin, S. Y. Liu, and L.Fang, *Solutions and perturbation estimates for the matrix equation* $X^s + A^* X^{-t} A = Q$, Linear Algebra Appl. **431** (2009), no. 9, 1409–1421.

[28] X. Zhan, *Computing the extremal positive definite solutions of a matrix equation*, SIAM J. Scient. Comput. **17** (1996), no. 5, 1167–1174.

Xiaoyan Yin
Department of Mathematics
Xidian University
Xi'an 710071, P. R. China
E-mail address: yinxiaoyan0105@126.com

Ruiping Wen
Department of Mathematics
Taiyuan Normal University
Taiyuan 030012, P. R. China
E-mail address: wenrp@163.com

Liang Fang
Department of Mathematics
Xidian University
Xi'an 710071, P. R. China
E-mail address: fl@xidian.edu.cn