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NORMAL FAMILY OF MEROMORPHIC FUNCTIONS

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Abstract. We study normality for families of meromorphic functions which is related to an extended version of a Hayman's conjecture on value distribution, and prove several normality criteria for meromorphic functions and certain non-homogeneous differential polynomials.

1. Introduction and results

We shall use the usual notations and classical results of Nevanlinna's theory (see [16]). Let f, g be non-constant meromorphic functions and c be a finite complex number. We say that f and g share the value c if $f - c$ and $g - c$ have the same zeros (see [16]). Throughout the paper, we denote by $\mathbb{C}$ the complex plane and by D a domain in $\mathbb{C}$.

A family $\mathcal{F}$ of functions meromorphic in D is said to be normal if each sequence in $\mathcal{F}$ has a subsequence which converges spherically uniformly on compact subsets of D [12].

Hayman [7] proposed a well-known conjecture on value distribution: If f is a transcendental meromorphic functions and n is a positive integer, then $f^n f'$ assumes every finite non-zero value infinitely often. This conjecture, following partial results by Clunie [5], Mues [10] and Hayman [7], was finally confirmed by Bergweiler and Eremenko [2], Chen and Fang [4], independently.

In 1993, C. C. Yang, L. Yang and Y. F. Wang [15] considered an extended version of the above Hayman's conjecture and proved that if f is a transcendental entire function, and $k, n(\geq 2)$ are positive integers, then $f(f^{(k)})^n$ assumes every finite non-zero value infinitely often. They also pointed out, but without proof, that the same conclusion holds for $n = 1$. Although the current partial answers to this Yang's problem are affirmative (see [1, 3, 8, 9, 11, 13, 14, 18]), the verification of its validity may need more time.

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According to Bloch's principle (see [17, p. 222]), it is natural to consider the normality for family of meromorphic functions corresponding to the above Yang's problem. In this direction, Pang and Zalcman proved the following results.

Theorem A ([11]). *Let k be a positive integer and let $\mathcal{F}$ be a family of functions holomorphic in a unit disc Δ such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least k. Suppose that there exist positive integer n and finite non-zero complex number c such that $f^n f^{(k)} \neq c$ for each $f \in \mathcal{F}$ and all $z \in \Delta$. Then $\mathcal{F}$ is normal in Δ.*

In this paper, we study the normality for families of meromorphic functions corresponding to the above Yang's problem and prove the following results.

Theorem 1. *Let $k \geq 3$ be a positive integer and let c be a finite non-zero complex number. Let $\mathcal{F}$ be a family of functions meromorphic in D such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least k and poles of multiplicity at least 2. If, for each pair of functions f and g in $\mathcal{F}$, $ff^{(k)}$ and $gg^{(k)}$ share the value c, then $\mathcal{F}$ is normal in D.*

Example. Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_j\}$, where $f_j(z) = jz^{k-1}$, $z \in D$, $j = 1, 2, \ldots$. We see that each $f_j \in \mathcal{F}$ has only zero of multiplicity $k-1$ and that $f_j f_j^{(k)}$ and $f_l f_l^{(k)}$ share c for each pair of functions f_j and f_l in $\mathcal{F}$. But $\mathcal{F}$ is not normal at $z = 0$.

The above example shows that the hypothesis that each $f \in \mathcal{F}$ has only zeros of multiplicity at least k is best possible for Theorem 1.

For normality criteria concerning differential polynomials, we have:

Theorem 2. *Let $k \geq 3$ be a positive integer and let $a_1(z), a_2(z), \ldots, a_{k+2}(z)$ be functions holomorphic in D with $a_{k+2}(z) \neq 0$. Let $\mathcal{F}$ be a family of functions meromorphic in D such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least k and poles of multiplicity at least 2. Set*

$$E(f) = \left\{ z \in D : \sum_{m=0}^{k} a_m(z) f(z) f^{(k-m)}(z) + a_{k+1}(z) f(z) + a_{k+2}(z) = 0 \right\},$$

where $a_0(z) \equiv 1$ and $f^{(0)} \equiv f$. If there exists a constant $M > 0$ such that $|f^{(k)}(z)| \leq M$ for each $f \in \mathcal{F}$ and all $z \in E(f)$, then $\mathcal{F}$ is normal in D.

Suppose that $a_m(z) \equiv 0$ for $m = 1, 2, \ldots, k+1$. Then the following corollaries are immediate results of Theorem 2.

Corollary 1. *Let $k \geq 3$ be a positive integer and let $c(z)$ be a non-vanishing holomorphic function in D. Let $\mathcal{F}$ be a family of functions meromorphic in D such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least k and poles of multiplicity at least 2. If $f(z)f^{(k)}(z) \neq c(z)$ for each $f \in \mathcal{F}$ and all $z \in D$, then $\mathcal{F}$ is normal in D.*

Corollary 2. *Let $k \geq 3$ be a positive integer and let $c(z)$ be a non-vanishing holomorphic function in D. Let $\mathcal{F}$ be a family of functions holomorphic in D such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least k. If $f(z)f^{(k)}(z) \neq c(z)$ for each $f \in \mathcal{F}$ and all $z \in D$, then $\mathcal{F}$ is normal in D.*

Obviously Corollary 2 generalizes Theorem A from a non-zero constant c to a function $c(z)$ which is holomorphic and non-vanishing in D for the cases $k \geq 3$ and $n = 1$.

2. Lemmas

Lemma 1 ([17, p. 216])**.** *Let $\mathcal{F}$ be a family of functions meromorphic in the unit disc $\triangle$, let k, l be positive integers and α be a real number with $-l < \alpha < k$. Suppose that all zeros of functions in $\mathcal{F}$ have multiplicity at least k and all poles of functions in $\mathcal{F}$ have multiplicity at least l and that $\mathcal{F}$ is not normal at $z_0 \in \triangle$. Then there exist functions $f_j \in \mathcal{F}$, points $z_j \in \triangle$, positive numbers ρ_j and a non-constant function g which is meromorphic in $\mathbb{C}$ such that $z_j \to z_0$, $\rho_j \to 0$ and $\rho_j^{-\alpha} f(z_j + \rho_j \zeta) \to g(\zeta)$ spherically uniformly on compact sets of $\mathbb{C}$.*

Lemma 2 ([6, Lemma 2])**.** *Let f be a non-constant meromorphic function and let $Q_1[f]$, $Q_2[f]$ be differential polynomials in f. Let n be a positive integer and $f^n Q_1[f] = Q_2[f]$. If $\gamma_{Q_2} \leq n$, then $m(r, Q_1[f]) = S(r, f)$, where γ_{Q_2} is the degree of $Q_2[f]$.*

To state our lemmas, we need the following notations.

Let f be a non-constant meromorphic function in $\mathbb{C}$ and let k be a positive integer. We denote by $N_{k)}(r, 1/f)$ the counting function for zeros of f with multiplicity at most k, $N_{(k}(r, 1/f)$ the counting function for zeros of f with multiplicity at least k, and $N_k(r, 1/f)$ the counting function for zeros of f with multiplicity k. As usual, we use $\overline{N}_{k)}(r, 1/f), \overline{N}_{(k}(r, 1/f)$ and $\overline{N}_k(r, 1/f)$ to denote the corresponding reduced ones, without regard to multiplicity.

Lemma 3. *Let $k \geq 3$ be a positive integer and let f be a transcendental meromorphic function in $\mathbb{C}$. If f has only zeros of multiplicity at least k and poles of multiplicity at least 2, then $ff^{(k)}$ assumes every finite non-zero complex number infinitely often.*

Proof. Let c be a finite non-zero complex number. Set

$$F = ff^{(k)} - c. \tag{2.1}$$

By (2.1) we have

$$T(r, F) = O(T(r, f)). \tag{2.2}$$

Rewriting (2.1) as $F - ff^{(k)} = -c$, which leads to

$$(-c)\frac{F'}{F} = (F - ff^{(k)})\frac{F'}{F} = F' - ff^{(k)}\frac{F'}{F} = f'f^{(k)} + ff^{(k+1)} - ff^{(k)}\frac{F'}{F},$$

so that

$$f\phi = (-c)\frac{F'}{F}, \tag{2.3}$$

where

$$\phi = \frac{f'}{f}f^{(k)} + f^{(k+1)} - f^{(k)}\frac{F'}{F}. \tag{2.4}$$

We shall show that F is not a constant. Otherwise we have $ff^{(k)} \equiv b$ for some constant b. Since f has only zeros of multiplicity at least k, we find $b \neq 0$ and thus $f \neq 0$, which means that $b/f^2(\equiv f^{(k)}/f)$ must be an entire function. This together with Nevanlinna's first fundamental theorem yields

$$2T(r, f) + O(1) = T(r, b/f^2) = m(r, b/f^2) = m(r, f^{(k)}/f) = S(r, f),$$

so that f is a constant. It is impossible. Hence F is not a constant.

Now we can derive from (2.3) that $\phi \not\equiv 0$. By applying Lemma 2 to (2.3) and noting (2.2) we obtain

$$m(r, \phi) = S(r, f). \tag{2.5}$$

From (2.1) we see that any pole of f must be a simple pole of F'/F, which and (2.3) means that any pole of f with multiplicity m (≥ 2) must be a zero of ϕ with multiplicity $m-1$. Thus we have

$$N_{(2}(r, f) \leq N(r, 1/\phi) + \overline{N}(r, 1/\phi). \tag{2.6}$$

If z_0 is a zero of f with multiplicity n $(\geq k+1)$, then we see from (2.1) that F' has zeros at z_0 with multiplicity at least n and, so is F'/F simply noting $F(z_0) = -c$. From this and (2.3) it follows that z_0 will never be a pole of ϕ. Therefore, we deduce from (2.4) that ϕ can only have poles at the zeros of F and the zeros of f with multiplicity at most k. This and (2.4) gives

$$N(r, \phi) \leq \overline{N}_{k)}(r, 1/f) + \overline{N}(r, 1/F). \tag{2.7}$$

From (2.2),(2.3) and Nevanlinna's first fundamental theorem, we have

$$m(r, f) \leq m(r, 1/\phi) + S(r, f) = T(r, \phi) - N(r, 1/\phi) + S(r, f). \tag{2.8}$$

Since f has no simple poles, from (2.5)-(2.8) we obtain

$$\begin{aligned} T(r, f) &= m(r, f) + N_{(2}(r, f) \leq T(r, \phi) + \overline{N}(r, 1/\phi) + S(r, f) \\ &\leq 2T(r, \phi) + S(r, f) \leq 2\overline{N}_{k)}(r, 1/f) + 2\overline{N}(r, 1/F) + S(r, f). \end{aligned} \tag{2.9}$$

Noting that f has only zeros with multiplicity at least k, thus we have

$$\overline{N}_{k)}(r, 1/f) = \overline{N}_k(r, 1/f) = \frac{1}{k}N_k(r, 1/f) \leq \frac{1}{k}T(r, f) + O(1). \tag{2.10}$$

From (2.9) and (2.10) we obtain

$$\left(1 - \frac{2}{k}\right)T(r, f) \leq 2\overline{N}(r, 1/F) + S(r, f),$$

so that F has infinitely many zeros since $k \geq 3$. Lemma 3 is proved. □

Lemma 4. *Let $k \geq 2$ be a positive integer and let c be a non-zero constant. Suppose that f is a rational function but not a polynomial and that f has only zeros of multiplicity at least k and poles of multiplicity at least 2, then $ff^{(k)} - c$ has at least two distinct zeros.*

Proof. If, to the contrary, $ff^{(k)} - c$ has at most one zero. We set

$$f = \frac{\alpha(z-a_1)^{m_1}(z-a_2)^{m_2}\cdots(z-a_s)^{m_s}}{(z-b_1)^{n_1}(z-b_2)^{n_2}\cdots(z-b_t)^{n_t}} = \frac{Q(z)}{P(z)}, \text{ say,} \tag{2.11}$$

where a_μ ($\mu = 1, 2, \ldots, s$), b_ν ($\nu = 1, 2, \ldots, t$) and $\alpha (\neq 0)$ are constants, $P(z)$ and $Q(z)$ are relatively prime polynomials. By the assumptions we have $m_\mu \geq k$ ($\mu = 1, 2, \ldots, s$) and $n_\nu \geq 2$ ($\nu = 1, 2, \ldots, t$). For simplicity we write

$$q = m_1 + m_2 + \cdots + m_s \geq ks, \tag{2.12}$$

$$p = n_1 + n_2 + \cdots + n_t \geq 2t. \tag{2.13}$$

By differentiating (2.11) k times we have

$$f^{(k)} = \frac{\alpha(z-a_1)^{m_1-k}(z-a_2)^{m_2-k}\cdots(z-a_s)^{m_s-k}G(z)}{(z-b_1)^{n_1+k}(z-b_2)^{n_2+k}\cdots(z-b_t)^{n_t+k}}, \tag{2.14}$$

where $G(z)$ is a polynomial of degree at most $k(s+t-1)$, with constants as coefficients. In fact we have

$$G(z) = (q-p)(q-p-1)\cdots[q-p-(k-1)]z^{k(s+t-1)} + d_1 z^{k(s+t-1)-1} + \cdots + d_{k(s+t-1)}.$$

From (2.11) and (2.14) we get

$$ff^{(k)} = \frac{\alpha^2(z-a_1)^{2m_1-k}(z-a_2)^{2m_2-k}\cdots(z-a_s)^{2m_s-k}G(z)}{(z-b_1)^{2n_1+k}(z-b_2)^{2n_2+k}\cdots(z-b_t)^{2n_t+k}} = \frac{Q_1(z)}{P_1(z)}, \text{ say,} \tag{2.15}$$

where $P_1(z)$ and $Q_1(z)$ are also relatively prime polynomials.

If we write $G(z) = \beta(z-c_1)^{l_1}(z-c_2)^{l_2}\cdots(z-c_q)^{l_q}$, where $l_1, l_2, \ldots, l_q$ are non-negative integers; $c_1, c_2, \ldots, c_q$ and β are constants with $\beta \neq 0$, and then substitute it into (2.15), then by differentiating (2.15) with the same method as (2.14) follows from (2.11), we can deduce that

$$(ff^{(k)})' = \frac{\alpha^2(z-a_1)^{2m_1-k-1}(z-a_2)^{2m_2-k-1}\cdots(z-a_s)^{2m_s-k-1}H(z)}{(z-b_1)^{2n_1+k+1}(z-b_2)^{2n_2+k+1}\cdots(z-b_t)^{2n_t+k+1}}, \tag{2.16}$$

where $H(z)$ is a polynomial with degree at most $\deg(G) + s + t - 1$.

We now divide our argument into two cases.

Case 1. If $ff^{(k)} - c$ has exactly one zero, then by (2.15) we may write

$$ff^{(k)} = c + \frac{\gamma(z-z_0)^l}{(z-b_1)^{2n_1+k}(z-b_2)^{2n_2+k}\cdots(z-b_t)^{2n_t+k}} = \frac{Q_1(z)}{P_1(z)}, \tag{2.17}$$

where l is a positive integer and γ is a non-zero constant. Since $c \neq 0$ we deduce from (2.15) and (2.17) that $a_\mu \neq z_0$ for $\mu = 1, 2, \ldots, s$.

By differentiating (2.17) and noting (2.13) we have

$$(2.18)\qquad (ff^{(k)})' = \frac{\gamma(z-z_0)^{l-1}U(z)}{(z-b_1)^{2n_1+k+1}(z-b_2)^{2n_2+k+1}\cdots(z-b_t)^{2n_t+k+1}},$$

where $U(z) = [l-2p-kt]z^t + e_1z^{t-1} + \cdots + e_t$ and $e_1, \ldots, e_t$ are constants.

Next, we shall distinguish two subcases.

Subcase 1.1. If $l \neq 2p+kt$, then by (2.17) we have $\deg(P_1) \leq \deg(Q_1)$. This together with (2.15) implies that $2p+kt = \deg(P_1) \leq \deg(Q_1) \leq 2q - ks + k(s+t-1)$, which leads to $p < q$.

Noting that $a_\mu \neq z_0$ for $\mu = 1, 2, \ldots, s$, thus we have from (2.16) and (2.18)

$$2q-(k+1)s = \sum_{\mu=1}^{s}(2m_\mu - k - 1) \leq \deg(U) = t.$$

By this together with (2.12), (2.13), we have $2q \leq (k+1)s + t \leq \frac{3}{2}q + \frac{1}{2}p$ since $k \geq 2$. This contradicts $p < q$.

Subcase 1.2. If $l = 2p+kt$, then we see from (2.17) that $\deg(Q_1) \leq \deg(P_1)$, and thus by (2.15) we get

$$(2.19)\qquad 2q - ks + \deg(G) = \deg(Q_1) \leq \deg(P_1) = 2p + kt.$$

Since $a_\mu \neq z_0$ for $\mu = 1, 2, \ldots, s$, again from (2.16), (2.18) we have $l - 1 \leq \deg(H)$ and thus

$$(2.20)\qquad 2p + kt = l \leq \deg(H) + 1 \leq \deg(G) + s + t.$$

Using (2.19), (2.20) and noting (2.12), (2.13), we obtain $2q \leq ks+s+t \leq \frac{3}{2}q+\frac{1}{2}p$ since $k \geq 2$, which leads to $q \leq p$.

However, if we substitute the fact $\deg(G) \leq k(s+t-1)$ into (2.20), then we have $2p+kt \leq k(s+t-1)+s+t$, which together with (2.12), (2.13) implies that $2p \leq ks+s+t-k < \frac{3}{2}q + \frac{1}{2}p$, contradicting $q \leq p$.

Case 2. If $ff^{(k)} - c$ has no zeros, then we see that $l = 0$ in equality (2.17) and thus $l \neq 2p+kt$. By the same proceeding as in the subcase 1.1, we can also get a contradiction.

Hence, $ff^{(k)} - c$ has at least two distinct zeros. Lemma 4 is proved. □

Lemma 5. *Let $k \geq 3$ be a positive integer and let c be a non-zero constant. If f is a non-constant meromorphic function such that f has only zeros of multiplicity at least k and poles of multiplicity at least 2, then $ff^{(k)} - c$ has at least two distinct zeros.*

Proof. Suppose first that f is a polynomial. Then $ff^{(k)}$ is also a polynomial with degree at least k since f has only zeros of multiplicity at least k, and thus $ff^{(k)} - c$ has at least one zero. If $ff^{(k)} - c$ has exactly one zero, say z_0, then there exist a non-zero constant λ and a positive integer $m(\geq k)$ such that $ff^{(k)} = c + \lambda(z-z_0)^m$, which, however, must only have simple zero since $c \neq 0$. This is impossible because f is a polynomial and all its zeros have multiplicity at least $k \geq 3$.

If f is rational but not a polynomial, then Lemma 5 follows from Lemma 4 immediately. Finally, if f is transcendental, then by Lemma 3 we know that $ff^{(k)} - c$ can assume zero infinitely often. The proof of Lemma 5 is complete. □

3. Proof of Theorem 1

Suppose, to the contrary, that $\mathcal{F}$ is not normal at $z_0 \in D$. Then by Lemma 1 with $\alpha = k/2$, there exist points $z_j \in D$, functions $f_j \in \mathcal{F}$ and positive numbers ρ_j such that $z_j \to z_0,\ \rho_j \to 0$ and

$$g_j(\zeta) = \rho_j^{-\frac{k}{2}} f_j(z_j + \rho_j\zeta) \to g(\zeta)$$

locally uniformly in $\mathbb{C}$ with respect to the spherical metric, where g is a non-constant meromorphic function. By Hurwitz's theorem, we see that all zeros of g have multiplicity at least k and all poles of g have multiplicity at least 2.

On every compact subset of $\mathbb{C}$, we have

$$f_j(z_j + \rho_j\zeta)f_j^{(k)}(z_j + \rho_j\zeta) - c = g_j(\zeta)g_j^{(k)}(\zeta) - c \longrightarrow g(\zeta)g^{(k)}(\zeta) - c, \tag{3.1}$$

spherically uniformly.

If $gg^{(k)} \equiv c$, then $g \neq 0, \infty$ since $c \neq 0$ and thus $c/g^2 (\equiv g^{(k)}/g)$ is an entire function. This together with Nevanlinna's first fundamental theorem provides

$$2T(r, g) + O(1) = T(r, c/g^2) = m(r, c/g^2) = m(r, g^{(k)}/g) = S(r, g),$$

which implies that g is a constant. It is a contradiction. Hence $gg^{(k)} - c \not\equiv 0$.

By Lemma 5 we know that $gg^{(k)} - c$ has at least two distinct zeros, say ζ_0 and ζ_0^*. Thus there exists a positive number δ and disjoint plane domains D_1 and D_2 such that $gg^{(k)} - c$ has no other zeros in $D_1 \bigcup D_2$ apart from ζ_0 and ζ_0^*, where $D_1 = \{\zeta \in \mathbb{C} : |\zeta - \zeta_0| < \delta\}$ and $D_2 = \{\zeta \in \mathbb{C} : |\zeta - \zeta_0^*| < \delta\}$.

In view of $gg^{(k)} - c \not\equiv 0$, by Hurwitz's theorem and (3.1), we see that there exist points $\zeta_j \in D_1$ and $\zeta_j^* \in D_2$ such that $\zeta_j \to \zeta_0,\ \zeta_j^* \to \zeta_0^*$ and

$$f_j(z_j+\rho_j\zeta_j)f_j^{(k)}(z_j+\rho_j\zeta_j)-c = 0, \quad f_j(z_j+\rho_j\zeta_j^*)f_j^{(k)}(z_j+\rho_j\zeta_j^*)-c = 0 \tag{3.2}$$

for sufficiently large j. By the hypotheses of Theorem 1, $f_1f_1^{(k)}$ and $f_jf_j^{(k)}$ share the value c for all integers $j \geq 2$. It follows from (3.2) that for j large enough

$$f_1(z_j+\rho_j\zeta_j)f_1^{(k)}(z_j+\rho_j\zeta_j)-c = 0, \qquad f_1(z_j+\rho_j\zeta_j^*)f_1^{(k)}(z_j+\rho_j\zeta_j^*)-c = 0. \tag{3.3}$$

We now claim that $f_1f_1^{(k)} - c \not\equiv 0$. Since otherwise we can deduce that f_1 must be a constant by the same way as we have used above in this section, which contradicts (3.3) and thus our claim is proved. Therefore, the set of all zeros of $f_1f_1^{(k)} - c$ has no accumulation points. By considering $z_j + \rho_j\zeta_j \to z_0$ and $z_j + \rho_j\zeta_j^* \to z_0$, it follows that, for sufficiently large j, $z_j + \rho_j\zeta_j = z_0$ and $z_j + \rho_j\zeta_j^* = z_0$, which leads to $\zeta_j = \zeta_j^*$. This contradicts the fact that $\zeta_j \in D_1,\ \zeta_j^* \in D_2$ and $D_1 \bigcap D_2 = \emptyset$. Theorem 1 is proved.

4. Proof of Theorem 2

If, to the contrary, $\mathcal{F}$ is not normal at $z_0 \in D$, then by Lemma 1 with $\alpha = k/2$, there exist points $z_j \in D$, functions $f_j \in \mathcal{F}$ and positive numbers ρ_j such that $z_j \to z_0$, $\rho_j \to 0$ and $g_j(\zeta) = \rho_j^{-\frac{k}{2}} f_j(z_j + \rho_j\zeta)$ converges locally uniformly to a non-constant meromorphic function $g(\zeta)$ in $\mathbb{C}$. By Hurwitz's theorem, all zeros of g have multiplicity at least k and all poles of g have multiplicity at least 2. Clearly $a_{k+2}(z_0) \neq 0, \infty$ since $a_{k+2}(z)$ is holomorphic and non-vanishing in D. Hence by Lemma 5 we see that equation $g(\zeta)g^{(k)}(\zeta) + a_{k+2}(z_0) = 0$ must have a solution in $\mathbb{C}$.

We may now assume that there exists $\xi_0 \in \mathbb{C}$ such that $g(\xi_0)g^{(k)}(\xi_0) + a_{k+2}(z_0) = 0$. Then $g(\xi_0) \neq \infty$ since $a_{k+2}(z)$ is holomorphic in D. So there exists $\delta > 0$ such that $g(\zeta)$ is analytic in $D_{2\delta} = \{\zeta \in \mathbb{C} : |\zeta - \xi_0| < 2\delta\}$. For $r = 1, 2, \ldots, k$, since $g_j^{(r)}(\zeta) \to g^{(r)}(\zeta)$ uniformly in $D_\delta = \{\zeta \in \mathbb{C} : |\zeta - \xi_0| < \delta\}$, thus all $g_j^{(r)}(\zeta)$ are also analytic in D_δ for sufficiently large j. By an elementary computation we have

$$\begin{aligned}
& g_j(\zeta)g_j^{(k)}(\zeta) \\
&= f_j(z_j+\rho_j\zeta)\left\{\sum_{m=0}^{k} a_m(z_j+\rho_j\zeta)f_j^{(k-m)}(z_j+\rho_j\zeta) + a_{k+1}(z_j+\rho_j\zeta)\right\} \\
&\quad - f_j(z_j+\rho_j\zeta)\left\{\sum_{m=1}^{k} a_m(z_j+\rho_j\zeta)f_j^{(k-m)}(z_j+\rho_j\zeta) + a_{k+1}(z_j+\rho_j\zeta)\right\} \\
&= f_j(z_j+\rho_j\zeta)\left\{\sum_{m=0}^{k} a_m(z_j+\rho_j\zeta)f_j^{(k-m)}(z_j+\rho_j\zeta) + a_{k+1}(z_j+\rho_j\zeta)\right\} \\
&\quad - \sum_{m=1}^{k} a_m(z_j+\rho_j\zeta)\rho_j^m g_j(\zeta)g_j^{(k-m)}(\zeta) - \rho_j^{\frac{k}{2}} g_j(\zeta)a_{k+1}(z_j+\rho_j\zeta),
\end{aligned} \tag{4.1}$$

where $g_j^{(0)} \equiv g_j$. Noting that $\rho_j \to 0$, $z_j \to z_0 \in D$ and that for $m = 1, 2, \ldots, k+1$ $a_m(z)$ are analytic in D, thus there exists a constant $L > 0$, depending only on z_0, such that $|a_m(z_j + \rho_j\zeta)| \leq L$ for sufficiently large j and $\zeta \in D_\delta$. Therefore, we have uniformly $\sum_{m=1}^{k} a_m(z_j+\rho_j\zeta)\cdot\rho_j^m\cdot g_j(\zeta)g_j^{(k-m)}(\zeta) \to 0$ in $D_{\delta/2} = \{\zeta \in \mathbb{C} : |\zeta - \xi_0| < \delta/2\}$. By this and (4.1) we have uniformly in $D_{\delta/2}$

$$\begin{aligned}
& f_j(z_j+\rho_j\zeta)\left\{\sum_{m=0}^{k} a_m(z_j+\rho_j\zeta)f_j^{(k-m)}(z_j+\rho_j\zeta) + a_{k+1}(z_j+\rho_j\zeta)\right\} \\
&\quad + a_{k+2}(z_j+\rho_j\zeta) \\
&= g_j(\zeta)g_j^{(k)}(\zeta) + \sum_{m=1}^{k} a_m(z_j+\rho_j\zeta)\rho_j^m g_j(\zeta)g_j^{(k-m)}(\zeta)
\end{aligned}$$

$$+\rho_j^{\frac{k}{2}} g_j(\zeta) a_{k+1}(z_j+\rho_j\zeta) + a_{k+2}(z_j+\rho_j\zeta)$$

$$\longrightarrow g(\zeta)g^{(k)}(\zeta) + a_{k+2}(z_0). \tag{4.2}$$

It is easy to see that $g(\zeta)g^{(k)}(\zeta)+a_{k+2}(z_0)\not\equiv 0$. If this is not the case, then we can deduce that g is a constant as we have done in Section 3, a contradiction.

Now we have shown that $g(\xi_0)g^{(k)}(\xi_0)+a_{k+2}(z_0)=0$ and $g(\zeta)g^{(k)}(\zeta)+a_{k+2}(z_0)\not\equiv 0$. Hence by (4.2) and Hurwitz's theorem, there exist points ζ_j such that $\zeta_j\to\xi_0$ and

$$f_j(z_j+\rho_j\zeta_j)\left\{\sum_{m=0}^{k} a_m(z_j+\rho_j\zeta_j) f_j^{(k-m)}(z_j+\rho_j\zeta_j)\right.$$
$$\left.+a_{k+1}(z_j+\rho_j\zeta_j)\right\} + a_{k+2}(z_j+\rho_j\zeta_j) = 0$$

for sufficiently large j. Thus we see from the hypotheses of Theorem 2 that $|g_j^{(k)}(\zeta_j)| = \rho_j^{\frac{k}{2}}|f_j^{(k)}(z_j+\rho_j\zeta_j)| \le \rho_j^{\frac{k}{2}}M$, and thus $|g^{(k)}(\xi_0)| = \lim_{j\to\infty}|g_j^{(k)}(\zeta_j)| = 0$, which contradicts $g(\xi_0)g^{(k)}(\xi_0) = -a_{k+2}(z_0)\neq 0$. The proof of Theorem 2 is complete.

References

[1] A. Alotaibi, *On the zeros of $af(f^{(k)})^n-1$ for $n\ge 2$*, Comput. Methods Funct. Theory **4** (2004), no. 1, 227–235.

[2] W. Bergweiler and A. Eremenko, *On the singularities of the inverse to a meromorphic function of finite order*, Rev. Mat. Iberoamericana **11** (1995), no. 2, 355–373.

[3] M. Buck, *A theorem on homogeneous differential polynomials*, Results Math. Online First; DOI 10.1007/s00025-012-0234-1(2012).

[4] H. H. Chen and M. L. Fang, *On the value distribution of f^nf'*, Sci. China, Ser. A. **38** (1995), 789–798.

[5] J. Clunie, *On a result of Hayman*, J. London Math. Soc. **42** (1967), 389–392.

[6] W. Doeringer, *Exceptional values of differetial polynomials*, Pacific J. Math. **98** (1982), no. 1, 55–62.

[7] W. K. Hayman, *Research Problems in Function Theory*, Athlone Press, London, 1967.

[8] X. J. Huang and Y. X. Gu, *On the value distribution of $f^2f^{(k)}$*, J. Aust. Math. Soc. **78** (2005), no. 1, 17–26.

[9] J. K. Langley, *The zeros of $ff''-b$*, Results Math. **44** (2003), no. 1-2, 130–140.

[10] E. Mues, *Über ein problem von Hayman*, Math. Z. **164** (1979), no. 3, 239–259.

[11] X. C. Pang and L. Zalcman, *On theorems of Hayman and Clunie*, New Zealand J. Math. **28** (1999), no. 1, 71–75.

[12] J. L. Schiff, *Normal Families*, Springer, New York, Berlin, Heidelberg, 1993.

[13] J. P. Wang, *On the zeros of $f^n(z)f^{(k)}(z)-c(z)$*, Complex Var. Elliptic Equ. **48** (2003), no. 8, 695–703.

[14] C. C. Yang and P. C. Hu, *On the value distribution of $ff^{(k)}$*, Kodai Math. J. **19** (1996), no. 2, 157–167.

[15] C. C. Yang, L. Yang, and Y. F. Wang, *On the zeros of $f(f^{(k)})^n-a$*, Chin. Sci. Bull. **38** (1993), no. 24, 2125–2128.

[16] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Dordrecht, Beijing, New York, Kluwer Academic Publishers and Science Press, 2003.

[17] L. Zalcman, *Normal families: New perspectives*, Bull. Amer. Math. Soc. (N.S.) **35** (1998), no. 3, 215–230.

[18] C. P. Zeng, *Normality and shared values with multiple zeros*, J. Math. Anal. Appl. **394** (2012), no. 2, 683–686.

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