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GENERAL DECAY FOR A SEMILINEAR WAVE EQUATION WITH BOUNDARY FRICTIONAL AND MEMORY CONDITIONS

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Abstract. In this paper, we investigate the influence of boundary dissipations on decay property of the solutions for a semilinear wave equation with damping and memory condition on the boundary using the multiplier technique.

1. Introduction

Our goal in this paper is to study the general decay rates of the solution for a semilinear wave equation with boundary damping and memory conditions of the form:

$$u'' - \Delta u + h(\nabla u) = 0 \text{ in } \Omega \times (0,\infty), \tag{1.1}$$

$$u = 0 \text{ on } \Gamma_1 \times (0,\infty), \tag{1.2}$$

$$\frac{\partial u}{\partial \nu} + \int_0^t k(t-s,x)u'(s)ds + g(u') = 0 \text{ on } \Gamma_0 \times (0,\infty), \tag{1.3}$$

$$u(0) = u_0,\ \ u'(0) = u_1 \text{ in } \Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^n (n \geq 1)$ with a sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here Γ_0 and Γ_1 are closed and disjoint, ν is the unit outward normal to Γ, $u' = \frac{\partial u}{\partial t}$, $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i}$. h, k and g are given functions.

For the last several decades, the boundary stabilization of classical wave equations has generated much interest and consideration in the literature. Indeed, when $h = 0$ and $k(t,x) = 0$, the problem has been treated many authors [8, 14, 15] for linear cases and [7, 9, 16] for nonlinear ones. Aassila et al. [1] studied problem (1.1)-(1.4), with $h = 0$ and a kernel k of exponential decay, and established some decay results. The stability of systems with $h \neq 0$ is

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somewhat delicate due to the lack of dissipativity (see e.g. [5, 13]). Guesmia [5] investigated the stability to the problem (1.1)-(1.4) with $k = 0$ and source term by introducing a special inequality. Most of these works are concerned with the exponential and polynomial decay rates when the kernel function k decays exponentially or the dissipative function g has polynomial growth near the origin. On the other hand, most recently general decay results for various systems have been obtained under generalized conditions on the functions k or g. For the related problems, we refer [2, 3, 6, 9, 10, 11]. Messaoudi and Soufyane [12] considered the following wave equation

$$\begin{cases} u'' - \Delta u + f_1(u) = 0 \text{ in } \Omega \times (0, \infty), \\ u = 0 \text{ on } \Gamma_1 \times (0, \infty), \\ u = -\int_0^t f_2(t-s)\frac{\partial u}{\partial \nu}(s)ds \text{ on } \Gamma_0 \times (0, \infty), \\ u(0) = u_0, \ \ u'(0) = u_1 \text{ in } \Omega. \end{cases}$$

By establishing some relations between the relaxation function f_2 and the corresponding resolvent kernel, they proved a general decay result, which is more general than those usually found in the literature. Motivated by these results, in this work we prove general decay of energy to the problem (1.1)-(1.4) by applying the frameworks of [12] with some necessary modification due to the nature of the problem treated here. It is worth to mention that the results in this paper improve some of those given in [1, 5] from some aspects. The remaining part of this paper is organized as follows. In Section 2, we give some notations and material needed for our work and state main results. Section 3 is devoted to investigate the decay of the solution energy.

2. Statement of main results

In this section, we present some material needed in the proof of our result and state main result. For a Banach space X, $||\cdot||_X$ denotes the norm of X. Let us consider the Hilbert space $L^2(\Omega)$ and $L^2(\Gamma_0)$ endowed with the inner products

$$(u, v) = \int_\Omega u(x)v(x)dx, \quad \text{and } (u, v)_{\Gamma_0} = \int_{\Gamma_0} u(x)v(x)d\Gamma$$

and the corresponding norm $||u||^2_{L^2(\Omega)} = (u, u)$ and $||u||^2_{L^2(\Gamma_0)} = (u, u)_{\Gamma_0}$, respectively. For simplicity, we denote $||\cdot||_{L^2(\Omega)}$ and $||\cdot||_{L^2(\Gamma_0)}$ by $||\cdot||$ and $||\cdot||_{\Gamma_0}$, respectively.

We denote

$$V = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1\}.$$

Let x^0 be a fixed point in $\mathbb{R}^n$, $m = x - x^0$ and $R = \max\{|x - x^0| : x \in \overline{\Omega}\}$. Assume that there exists $0 < \delta < 1$ such that

$$\Gamma_0 = \{x \in \Gamma : m \cdot \nu \geq \delta > 0\} \quad \text{and } \Gamma_1 = \{x \in \Gamma : m \cdot \nu \leq 0\}. \tag{2.1}$$

Let λ_1 and λ be the smallest positive constants such that

$$||u||^2 \le \lambda ||\nabla u||^2 \quad \text{and } ||u||_{\Gamma_0}^2 \le \lambda_1 ||\nabla u||^2, \ \ \forall u \in V. \tag{2.2}$$

Now, we state the assumptions for the problem (1.1)-(1.4).

($\mathbf{H_1}$) Let $h : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function such that ∇h is bounded and there exists $\beta > 0$ satisfying that

$$|h(x)| \le \beta |x| \quad \text{for all } x \in \mathbb{R}^n. \tag{2.3}$$

($\mathbf{H_2}$) Assume that $g : \mathbb{R} \to \mathbb{R}$ is a C^1 function verifying $sg(s) > 0$ for $s \neq 0$ and there exist positive constants μ_1 and μ_2 such that $\mu_2 \ge \mu_1$ and

$$\mu_1 |s| \le |g(s)| \le \mu_2 |s| \quad \text{for all } s \in \mathbb{R}. \tag{2.4}$$

($\mathbf{H_3}$) Let $k \in C^2(\mathbb{R}_+; L^\infty(\Gamma_1))$ such that

$$k(0,x) > 0, \ \ k(t,x) \ge 0, \ \ k'(t,x) \le 0, \ \ k''(t,x) \ge -\zeta(t) k'(t,x), \tag{2.5}$$

where $k'(t,x)$ denotes the derivative with respect to t, $\zeta : \mathbb{R}_+ \to (0, \infty)$ is a nonincreasing function satisfying $\zeta(t) \ge \zeta_0$ for some $\zeta_0 > 0$.

We denote $K(t) = ||k(t)||_{L^\infty(\Gamma_1)}$ for $t \ge 0$.

Examples. (1) Let

$$k(t,x) = k(t) = \frac{e^{-t}}{(e+t)(\ln(e+t))^p}, \ \text{ where } \ \ p > 0.$$

Then simple calculations yield that k satisfies all the conditions in (H_3) with

$$\zeta(t) = 1 + \frac{\ln(e+t) + p}{(e+t)\ln(e+t)}.$$

(2) $k(t) = \frac{e^{-t}}{(1+t)^p}$, where $p > 0$, is also an example verifying (H_3) with

$$\zeta(t) = 1 + \frac{p}{1+t} + \frac{p}{(1+t)(1+t+p)}.$$

According to previous results existing in the literature (see e.g. [4]), we can state the following existence result of the solution subject to (1.1)-(1.4) under the conditions (H_1)-(H_4):

Theorem 2.1. *For the initial data* $(u_0, u_1) \in V \times L^2(\Omega)$, *the problem* (1.1)-(1.4) *has a unique weak solution* u *in the class*

$$u \in C(0,T;V) \cap C^1(0,T;L^2(\Omega)).$$

Furthermore, $(u_0, u_1) \in (V \cap H^2(\Omega)) \times V$ *with the compatibility conditions* $\frac{\partial u_0}{\partial \nu} + g(u_1) = 0$ *on* Γ_0, *the problem* (1.1)-(1.4) *has a unique solution* u *in the class*

$$u \in L^\infty(0,T;V \cap H^2(\Omega)), \ \ u' \in L^\infty(0,T;V), \ \ u'' \in L^\infty(0,T;L^2(\Omega)).$$

Our main result is the following.

Theorem 2.2. *If β and $K(0) = ||k(0)||_{L^\infty(\Gamma_1)}$ are sufficiently small, there exist $C_0 > 0$ and $\omega > 0$ such that*

(i) *if $u_0 = 0$ on Γ_0, then $E(t) \leq C_0 E(0) e^{-\omega \int_0^t \zeta(s)ds}$,*

(ii) *otherwise, $E(t) \leq C_0 \Big[E(0) + ||u_0||^2_{\Gamma_0} \int_0^t K^2(s) e^{\omega \int_0^s \zeta(\tau)d\tau} ds\Big] e^{-\omega \int_0^t \zeta(s)ds}$,*

where

$$E(t) = \frac{1}{2}||u'||^2 + \frac{1}{2}||\nabla u||^2 + \frac{1}{2}\int_{\Gamma_0} k(t,x)|u(t,x)|^2 d\Gamma - \frac{1}{2}\int_{\Gamma_0} k'\Box u d\Gamma. \tag{2.6}$$

Remark 2.1. Exponential decay, given in earlier literature (see e.g. [1, 5]), is a special case of Theorem 2.2. Indeed, if $k(t,x) = k(t) = e^{-at}$ for some $a > 0$, $\zeta(t)$ equals to the constant $a > 0$ and Theorem 2.2 implies exponential decay estimates.

3. Asymptotic behavior of solutions

In this section we shall prove the decay rates in Theorem 2.2. From now on, we shall omit x and t in all functions of x and t if there is no ambiguity. c denotes an arbitrary positive constant independent of t and x, which may be different from line to line and even in the same line.

Integration by parts yields

$$\begin{aligned} &\int_0^t k(t-s,x)u'(s,x)ds \\ &= k(0,x)u(t,x) - k(t,x)u_0(x) + \int_0^t k'(t-s,x)u(s,x)ds \\ &= k(t,x)u(t,x) - k(t,x)u_0(x) + \int_0^t k'(t-s,x)(u(s,x)-u(t,x))ds. \end{aligned} \tag{3.1}$$

Multiplying (1.1) by u', which makes sense because $u'(t) \in L^\infty(0,T;V)$, we obtain from (1.2), (1.3) and (3.1) the identity

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(||u'||^2 + ||\nabla u||^2\Big) \\ &= -(h(\nabla u), u') - (g(u'), u')_{\Gamma_0} - (k(0)u - k(t)u_0 + \int_0^t k'(t-s)u(s)ds, u')_{\Gamma_0}. \end{aligned} \tag{3.2}$$

On the other hand, a direct calculation gives

$$\begin{aligned} &\int_0^t k'(t-s,x)u(s,x)u'(t,x)ds \\ &= -\frac{1}{2}k'(t,x)|u(t,x)|^2 + \frac{1}{2}k''\Box u \\ &\quad -\frac{1}{2}\frac{d}{dt}\Big(k'\Box u - k(t,x)|u(t,x)|^2 + k(0,x)|u(t,x)|^2\Big), \end{aligned}$$

where $k'\Box u = \int_0^t k'(t-s,x)|u(t,x)-u(s,x)|^2 ds$. Applying this to (3.2), we get

$$\frac{d}{dt}E(t) = -(h(\nabla u),u') - (g(u'),u')_{\Gamma_0} + \int_{\Gamma_0} k(t,x)u_0(x)u'(t,x)d\Gamma + \frac{1}{2}\int_{\Gamma_0} k'(t,x)|u(t,x)|^2 d\Gamma - \frac{1}{2}\int_{\Gamma_0} k''\Box u d\Gamma. \tag{3.3}$$

Use (2.3), (2.4), (2.5) and Young inequality to obtain

$$\frac{d}{dt}E(t) \le \frac{\beta}{2}||u'||^2 + \frac{\beta}{2}||\nabla u||^2 + \frac{\mu_1}{2}||u'||^2_{\Gamma_0} + \frac{1}{2\mu_1}\int_{\Gamma_0} k^2(t,x)|u_0(x)|^2 d\Gamma + \frac{1}{2}\int_{\Gamma_0} k'(t,x)|u(t,x)|^2 d\Gamma - \frac{1}{2}\int_{\Gamma_0} k''\Box u d\Gamma. \tag{3.4}$$

First, let us define the perturbed modified energy by

$$L(t) = ME(t) + \Psi(t), \tag{3.5}$$

where

$$\Psi(t) = 2(u', m\cdot\nabla u) + (n-1)(u',u). \tag{3.6}$$

It is easy to check that

$$L(t) \sim E(t) \quad \text{for appropriately large } M > 0. \tag{3.7}$$

Proposition 3.1. *For sufficiently small $\beta > 0$, $K(0) > 0$ and large $M > 0$, there exist positive constants α_1, α_2 and α_3 verifying*

$$\frac{d}{dt}L(t) \le -\alpha_1 E(t) + \alpha_2 \int_{\Gamma_0} k^2(t,x)|u_0(x)|^2 d\Gamma - \alpha_3 \int_{\Gamma_0} k'\Box u d\Gamma. \tag{3.8}$$

Proof. Using (1.1)-(1.4), we have

$$\begin{aligned}\Psi'(t) = &\int_{\Gamma_0}(m\cdot\nu)|u'|^2 d\Gamma - ||u'||^2 + (\frac{\partial u}{\partial\nu}, 2m\cdot\nabla u + (n-1)u)_{\Gamma_0} \\ &+ (2m\cdot\nabla u, \frac{\partial u}{\partial \nu})_{\Gamma_1} - \int_{\Gamma}(m\cdot\nu)|\nabla u|^2 d\Gamma - ||\nabla u||^2 \\ &- (2m\cdot\nabla u + (n-1)u, h(\nabla u)).\end{aligned} \tag{3.9}$$

Noting that

$$u = 0,\ \ \frac{\partial u}{\partial x_i} = \nu_i \frac{\partial u}{\partial \nu}\ \ (i = 1,\dots,n) \quad\text{and}\quad m\cdot\nu \le 0 \text{ on } \Gamma_1,$$

we have

$$\begin{aligned}\Psi'(t) \le &\int_{\Gamma_0}(m\cdot\nu)|u'|^2 d\Gamma - ||u'||^2 + (\frac{\partial u}{\partial\nu}, 2m\cdot\nabla u + (n-1)u)_{\Gamma_0} \\ &- \int_{\Gamma_0}(m\cdot\nu)|\nabla u|^2 d\Gamma - ||\nabla u||^2 - (2m\cdot\nabla u + (n-1)u, h(\nabla u)).\end{aligned} \tag{3.10}$$

Young inequality, (2.2) and (2.3) give

$$(\frac{\partial u}{\partial \nu}, 2m\cdot\nabla u + (n-1)u)_{\Gamma_0} \le \eta||\nabla u||^2_{\Gamma_0} + \eta||\nabla u||^2 + C_\eta||\frac{\partial u}{\partial \nu}||^2_{\Gamma_0} \tag{3.11}$$

and

$$(2m\cdot\nabla u + (n-1)u, h(\nabla u)) \le \beta(2R+(n-1)\lambda)||\nabla u||^2, \tag{3.12}$$

where $C_\eta > 0$ is a constant depending on η, δ, R, n and λ_1.

Substituting these into (3.10) and using (2.1), we see that

$$\begin{aligned}\Psi'(t) \le & \int_{\Gamma_0}(m\cdot\nu)|u'|^2 d\Gamma - ||u'||^2 - (\delta-\eta)\int_{\Gamma_0}|\nabla u|^2 d\Gamma \\ & - (1-\eta-\beta(2R+(n-1)\lambda))||\nabla u||^2 + C_\eta||\frac{\partial u}{\partial \nu}||^2_{\Gamma_0}.\end{aligned} \tag{3.13}$$

From (3.4) and (3.13), it follows that

(3.14)

$$\begin{aligned}\frac{d}{dt}L(t) =\ & ME'(t) + \Psi'(t) \\ \le\ & \frac{M\beta}{2}||u'||^2 + \frac{M\beta}{2}||\nabla u||^2 - M(\mu_1 - \frac{1}{2})||u'||^2_{\Gamma_0} \\ & + \frac{M}{2}\int_{\Gamma_0}k^2(t,x)|u_0(x)|^2 d\Gamma \\ & + \frac{M}{2}\int_{\Gamma_0}k'(t,x)|u(t,x)|^2 d\Gamma - \frac{M}{2}\int_{\Gamma_0}k''\Box u d\Gamma \\ & + \int_{\Gamma_0}(m\cdot\nu)|u'|^2 d\Gamma - ||u'||^2 - (\delta-\eta)\int_{\Gamma_0}|\nabla u|^2 d\Gamma \\ & - (1-\eta-\beta(2R+(n-1)\lambda))||\nabla u||^2 + C_\eta||\frac{\partial u}{\partial \nu}||^2_{\Gamma_0} \\ \le\ & -(1-\frac{M\beta}{2})||u'||^2 - (1-\frac{M\beta}{2}-\eta-\beta(2R+(n-1)\lambda))||\nabla u||^2 \\ & - \frac{\mu_1 M}{2}||u'||^2_{\Gamma_0} + \frac{M}{2\mu_1}\int_{\Gamma_0}k^2(t,x)|u_0(x)|^2 d\Gamma + \int_{\Gamma_0}(m\cdot\nu)|u'|^2 d\Gamma \\ & - (\delta-\eta)\int_{\Gamma_0}|\nabla u|^2 d\Gamma + C_\eta||\frac{\partial u}{\partial \nu}||^2_{\Gamma_0}.\end{aligned}$$

Eq.(3.1), Cauchy-Swartz inequality and (H_3) infer that

(3.15)

$$\begin{aligned}||\frac{\partial u}{\partial \nu}||^2_{\Gamma_0} =\ & \Big||k(t)u(t) - k(t)u_0 + \int_0^t k'(t-s)(u(s)-u(t))ds - g(u')\Big||^2_{\Gamma_0} \\ \le\ & c\Big[\int_{\Gamma_0}k^2(t,x)|u(t,x)|^2 d\Gamma + \int_{\Gamma_0}k^2(t,x)|u_0(x)|^2 d\Gamma \\ & - \int_{\Gamma_0}k(0,x)k'\Box u d\Gamma + ||g(u')||^2_{\Gamma_0}\Big]\end{aligned}$$

$$\le c\Big[K^2(0)\int_{\Gamma_0}|u(t,x)|^2d\Gamma+\int_{\Gamma_0}k^2(t,x)|u_0(x)|^2d\Gamma$$
$$-K(0)\int_{\Gamma_0}k'\Box ud\Gamma+\mu_2^2||u'||_{\Gamma_0}^2\Big].$$

Combining (3.14) and (3.15), we get

$$\frac{d}{dt}L(t)\le -c_1||u'||^2-c_2||\nabla u||^2-c_3||u'||_{\Gamma_0}^2-(\delta-\eta)\int_{\Gamma_0}|\nabla u|^2d\Gamma$$
$$+(\frac{M}{2\mu_1}+C_\eta)\int_{\Gamma_0}k^2(t,x)|u_0(x)|^2d\Gamma-C_\eta K(0)\int_{\Gamma_0}k'\Box ud\Gamma, \tag{3.16}$$

where $c_1=1-\frac{M\beta}{2}$, $c_2=1-\frac{M\beta}{2}-\eta-\beta(2R+(n-1)\lambda)-C_\eta K^2(0)$, $c_3=\frac{\mu_1M}{2}-R-\mu_2^2$.

Taking η such that $0<\eta<\delta$ and $M>0$ large enough such that $c_3>0$, and then choosing β and $K(0)$ sufficiently small so that $c_1>0$ and $c_2>0$, we complete the proof of Proposition 3.1. □

Continuity of the proof of Theorem 2.2. Multiplying (3.8) by $\zeta(t)$ and using (2.5) and (3.4), we obtain

(3.17)
$$\zeta(t)\frac{d}{dt}L(t)\le -\alpha_1\zeta(t)E(t)+\alpha_2\zeta(t)\int_{\Gamma_0}k^2(t,x)|u_0|^2d\Gamma-\alpha_3\zeta(t)\int_{\Gamma_0}k'\Box ud\Gamma$$
$$\le -\alpha_1\zeta(t)E(t)+\alpha_2\zeta(t)K^2(t)\int_{\Gamma_0}|u_0|^2d\Gamma+\alpha_3\int_{\Gamma_0}k''\Box ud\Gamma$$
$$\le -\alpha_1\zeta(t)E(t)+\alpha_2\zeta(t)K^2(t)\int_{\Gamma_0}|u_0|^2d\Gamma$$
$$+\alpha_3\Big(-2E'(t)-\mu_1||u'||_{\Gamma_0}^2+\beta||u'||^2+\beta||\nabla u||^2$$
$$+\frac{1}{\mu_1}\int_{\Gamma_0}k^2(t,x)|u_0|^2d\Gamma\Big).$$

Noting that $\zeta'(t)\le 0$ and $\zeta(t)\ge\zeta_0$, it follows that

$$\frac{d}{dt}\Big(\zeta(t)L(t)+2\alpha_3E(t)\Big)$$
$$\le -\alpha_1\zeta(t)E(t)+cK^2(t)\int_{\Gamma_0}|u_0|^2d\Gamma+\alpha_3\beta||u'||^2+\alpha_3\beta||\nabla u||^2$$
$$\le -(\alpha_1-2\alpha_3\beta/\zeta_0)\zeta(t)E(t)+cK^2(t)\int_{\Gamma_0}|u_0|^2d\Gamma. \tag{3.18}$$

Define

$$\mathcal{L}(t)=\zeta(t)L(t)+2\alpha_3E(t),$$

then it is easy to show that $\mathcal{L}(t)$ is equivalent to $E(t)$ by using (3.7) and the fact ζ is nonincreasing. Thus, choosing $\beta>0$ sufficiently small again such that

$\alpha_1 - 2\alpha_3\beta/\zeta_0 > 0$, we arrive at

$$\mathcal{L}'(t) \le -\alpha\zeta(t)\mathcal{L}(t) + cK^2(t)||u_0||^2_{\Gamma_0} \quad \text{for some } \alpha > 0. \tag{3.19}$$

Case (i) : If $u_0 = 0$ on Γ_0, (3.19) becomes

$$\mathcal{L}'(t) \le -\alpha\zeta(t)\mathcal{L}(t). \tag{3.20}$$

Integration this over $(0, t)$ gives

$$\mathcal{L}(t) \le \mathcal{L}(0)e^{-\alpha\int_0^t \zeta(s)ds}. \tag{3.21}$$

Case (ii) : If $u_0 \ne 0$ on Γ_0, we put

$$\mathcal{F}(t) := \mathcal{L}(t) - c||u_0||^2_{\Gamma_0} e^{-\alpha\int_0^t \zeta(s)ds} \int_0^t K^2(s)e^{\alpha\int_0^s \zeta(\tau)d\tau}ds. \tag{3.22}$$

Then, it holds that

$$\mathcal{F}'(t) \le -\alpha\zeta(t)\mathcal{F}(t), \tag{3.23}$$

and hence

$$\mathcal{F}(t) \le \mathcal{F}(0)e^{-\alpha\int_0^t \zeta(s)ds}. \tag{3.24}$$

This yields that

$$\mathcal{L}(t) \le \Big(\mathcal{L}(0) + c||u_0||^2_{\Gamma_0} \int_0^t K^2(s)e^{\alpha\int_0^s \zeta(\tau)d\tau}ds\Big)e^{-\alpha\int_0^t \zeta(s)ds}. \tag{3.25}$$

Consequently, the equivalent relations of $\mathcal{L}$, L, and E and (3.21), (3.25) yield the results in Theorem 2.2. □

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