Bull. Korean Math. Soc. **51** (2014), No. 3, pp. 659–666
http://dx.doi.org/10.4134/BKMS.2014.51.3.659

T-NEIGHBORHOODS IN VARIOUS CLASSES OF ANALYTIC FUNCTIONS

Saeid Shams, Ali Ebadian, Mahta Sayadiazar, and Janusz Sokół

Abstract. Let $\mathcal{A}$ be the class of analytic functions f in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ with the normalization conditions $f(0) = f'(0) - 1 = 0$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $\delta > 0$ are given, then the T_δ-neighborhood of the function f is defined as

$$TN_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A} : \sum_{n=2}^{\infty} T_n |a_n - b_n| \le \delta \right\},$$

where $T = \{T_n\}_{n=2}^{\infty}$ is a sequence of positive numbers. In the present paper we investigate some problems concerning T_δ-neighborhoods of functions in various classes of analytic functions with $T = \left\{2^{-n}/n^2\right\}_{n=2}^{\infty}$. We also find bounds for $\delta_T^*(A, B)$ defined by

$$\delta_T^*(A, B) = \inf \left\{ \delta > 0 : B \subset TN_\delta(f) \ \text{ for all } \ f \in A \right\},$$

where A, B are given subsets of $\mathcal{A}$.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ with the normalization conditions $f(0) = f'(0) - 1 = 0$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then the T_δ-neighborhood of the function f is defined as

$$TN_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A} : \sum_{n=2}^{\infty} T_n |a_n - b_n| \le \delta \right\}, \tag{1.1}$$

where δ is a positive number and $T = \{T_n\}_{n=2}^{\infty}$ is a sequence of positive numbers. St. Ruscheweyh in [14] considered $T = \{n\}_{n=2}^{\infty}$ and showed that if $f \in \mathcal{C}$, then $TN_{1/4}(f) \subset \mathcal{S}^*$, where $\mathcal{C}, \mathcal{S}^*$ denote the well known classes of convex and starlike functions, respectively. In [4, 5, 6, 7, 10, 11, 12, 17, 18] other authors investigated some interesting results concerning neighborhoods of several classes of analytic functions. Some of the relations between the neighborhoods for a certain class of analytic functions was described by S. Shams et al. [15].

Received January 18, 2013; Revised July 10, 2013.
2010 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C50, 40A05.
Key words and phrases. analytic functions, univalent, starlike, convex, close-to-convex, concave functions, neighborhood, T_δ-neighborhood, T-factor.

Also U. Bednarz and J. Sokół in [7] considered $T = \{\frac{1}{n^2(n-1)}\}_{n=2}^{\infty}$ and investigated T_δ-neighborhood for various subclasses of analytic functions. Motivated by the above results, we consider in this paper T_δ-neighborhood (1.1) with $T = \{2^{-n}n^{-2}\}_{n=2}^{\infty}$. We use this sequence because it is sufficiently strongly convergent to 0, which is necessary for the series considered here to be convergent. Notice that $\sum_{n=1}^{\infty} 2^{-n}n^{-2} = \pi^2/12 - (\log 2)^2/2$ and it is the value of dilogarithm at 1/2, [13].

The convolution or Hadamard product of the functions f and g of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad |z| < 1,$$

is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Definition 1.1 ([2])**.** Let us consider the functions f that are meromorphic and univalent in $\mathbb{U}$, holomorphic at 0 and have the expansion $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. If, in addition, the complement of $f(\mathbb{U})$ with respect to $\mathbb{C}$ is convex, then f is called a concave univalent function. The class of all concave functions is denoted by $\mathcal{C}o$.

It is well known [1], that if $f \in \mathcal{C}o$, then $|a_n| \geq 1$ for all $n > 1$ and equality holds if and only if $f(z) = z/(1 - \mu z)$, $|\mu| = 1$ (see [1, 3]). The authors in [2] considered the class $\mathcal{C}o(p) \subset \mathcal{C}o$ consisting of all concave functions that have a pole at the point p and are analytic in $|z| < |p|$. They proved that if $f \in \mathcal{C}o(1)$, then

$$\left| a_n - \frac{n+1}{2} \right| \leq \frac{n-1}{2} \quad \text{for} \quad n \geq 2, \tag{1.2}$$

and equality holds only for the function f_θ defined by

$$f_\theta(z) = \frac{2z - (1 - e^{i\theta})z^2}{2(1 - z)^2}, \quad |z| < 1.$$

It is well known that if $f \in \mathcal{C}o(1)$, then the complement of $f(\mathbb{U})$ can be represented as the union of a set of mutually disjoint half-lines (the end point of one half-line can lie on the another half-line), so $f(\mathbb{U})$ is a linearly accessible domain in the strict sense (see [8, 16]).

The authors in [7] also showed that $\mathcal{C}o(1) \subset \mathcal{K}$, where $\mathcal{K}$ is the set of close-to-convex functions.

2. Main results

Throughout this section T will always be the sequence given by

$$T = \{T_n\}_{n=2}^{\infty} = \left\{2^{-n}n^{-2}\right\}_{n=2}^{\infty}, \tag{2.1}$$

unless otherwise stated.

Theorem 2.1. *If* $f, g \in \mathcal{A}$ *are of the form* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ *with* $|a_n| \le n$ *and* $|b_n| \le n$ *for* $n = 2, 3, 4, \ldots$, *then* $g \in TN_{\log\{4/e\}}(f)$, *where* T *is given in* (2.1). *The number* $\log\{4/e\}$ *is the best possible.*

Proof. A simple calculation shows that

$$\sum_{n=1}^{\infty} \frac{z^n}{n2^n} = \int_0^z \sum_{n=1}^{\infty} \frac{\zeta^{n-1}}{2^n}\, \mathrm{d}\zeta = \int_0^z \frac{1/2}{1-\zeta/2}\, \mathrm{d}\zeta = \log \frac{1}{1-z/2}, \quad |z| < 2, \tag{2.2}$$

so we have

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = \log 2, \tag{2.3}$$

and then

$$\sum_{n=2}^{\infty} T_n |a_n - b_n| \le \sum_{n=2}^{\infty} \frac{2n}{n^2 2^n} = 2 \sum_{n=2}^{\infty} \frac{1}{n2^n} = 2 \log 2 - 1 = \log\{4/e\}.$$

For the functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + \sum_{n=2}^{\infty} n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n = z - \sum_{n=2}^{\infty} n z^n$$

we have

$$\sum_{n=2}^{\infty} T_n |a_n - b_n| = 2 \sum_{n=2}^{\infty} \frac{1}{n2^n} = \log\{4/e\}.$$

Therefore, the number $\log\{4/e\}$ cannot be replaced by a smaller one and it is the best possible. □

It is well known that $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$ (see [9]), where $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ denote the classes of univalent, starlike, convex and close-to-convex functions, respectively. Also, if $f \in \mathcal{S}^*$, then $|a_n| \le n$, $n = 2, 3, \ldots$, while if $f \in \mathcal{C}$, then $|a_n| \le 1$, $n = 2, 3, \ldots$.

Therefore we obtain the following corollary.

Corollary 2.2. *If* $f \in \mathcal{S}$, *then we have*

$$\mathcal{S} \subset TN_{\log\{4/e\}}(f),$$

where T *is given in* (2.1).

The constant $\log\{4/e\} \approx 0.386$ seems not to be the best possible. An interesting open problem is to find the smallest constant ϱ such that for each $f \in \mathcal{S}$

$$\mathcal{S} \subset TN_{\varrho}(f),$$

where T is given in (2.1). For the Koebe function $f(z) = z/(1-z)^2$ and $g(z) = -f(-z)$ we have $f, g \in \mathcal{S}$ and

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + \sum_{n=2}^{\infty} n z^n, \; g(z) = z + \sum_{n=2}^{\infty} b_n z^n = z + \sum_{n=2}^{\infty} (-1)^{n-1} n z^n$$

so by (2.2)

$$\sum_{n=2}^{\infty} T_n |a_n - b_n| = \sum_{k=1}^{\infty} \frac{4k}{(2k)^2 2^{2k}} = \log\{4/3\}.$$

Therefore, the number ϱ cannot be smaller than $\log\{4/3\}$. We conjecture that $\varrho = \log\{4/3\} = 0.28768\cdots$.

Corollary 2.3. *Let $f \in \mathcal{C}$. Then $\mathcal{S} \subset TN_\beta(f)$ with*

$$\beta = \log\{2/e\} + \frac{\pi^2}{12} - \frac{(\log 2)^2}{2} = 0.275\cdots. \tag{2.4}$$

Proof. At first, note that

$$f_2(x) = -\int_1^x \frac{\log t}{t-1} \mathrm{d}t, \quad x \in [0,2],$$

is the dilogarithm. From the tables of dilogarithms we have

$$f_2(x) = \sum_{k=1}^{\infty} (-1)^k \frac{(x-1)^k}{k^2}, \quad x \in [0,2], \tag{2.5}$$

$$f_2(x) + f_2(1-x) = -\log\{x\} \cdot \log\{1-x\} + \pi^2/6, \tag{2.6}$$

$$f_2(1+x) - f_2(x) = -\log\{x\} \cdot \log\{x+1\} - \pi^2/12 - f_2(x^2)/2. \tag{2.7}$$

Therefore, using (2.5) and (2.6) we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^2 2^n} = f_2(1/2) = \frac{\pi^2}{12} - \frac{(\log 2)^2}{2}. \tag{2.8}$$

If

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S},$$

then $|a_n| \le 1$, $|b_n| \le n$ and by (2.3), (2.8) we have

$$\sum_{n=2}^{\infty} T_n |a_n - b_n| \le \sum_{n=2}^{\infty} \frac{n+1}{n^2 2^n} = \sum_{n=2}^{\infty} \frac{1}{n 2^n} + \sum_{n=2}^{\infty} \frac{1}{n^2 2^n} = \log\{2/e\} + f_2(1/2)$$
$$= 0.275\cdots.$$

□

In a similar way as in Corollary 2.2, the constant $0.275\cdots$ given in Corollary 2.3 is also not sharp but if the class $\mathcal{S}$ is replaced by the much larger class of all normalized analytic functions f such that $|a_n(f)| \le n$ for $n \ge 2$, then (2.4) becomes sharp. The best possible constant in the case $f \in \mathcal{S}$ is not known. We conjecture that the sharp constant is attained by the functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = \frac{z}{(1-z)^n} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n = \frac{z}{1+z}.$$

It is clear that $f \in \mathcal{S}$ and $g \in \mathcal{C}$. Moreover,

$$\sum_{n=2}^{\infty} T_n |a_n - b_n| = \sum_{n=2}^{\infty} \frac{n+1}{2^n n^2} - \sum_{n=2}^{\infty} \frac{1+(-1)^{n-1}}{2^n n^2}$$

$$= \log\{2/e\} + f_2(1/2) - \sum_{k=1}^{\infty} \frac{2}{2^{2k+1}(2k+1)^2}. \tag{2.9}$$

From the tables of dilogarithms we have

$$\sum_{k=1}^{\infty} \frac{2}{2^{2k+1}(2k+1)^2} = \int_0^{1/2} \frac{1}{t} \log \frac{1+t}{1-t} \mathrm{d}t - 1 = f_2(1/2) - f_2(3/2) - 1.$$

By (2.7) we have

$$f_2(1/2) - f_2(3/2) = \frac{f_2(1/4)}{2} + \frac{\pi^2}{12} - \log\{2\} \cdot \log\{3/2\}.$$

Applying this in (2.9) we further get,

$$\begin{aligned} &\sum_{n=2}^{\infty} T_n |a_n - b_n| \\ = \ &\log\{2/e\} + f_2(1/2) - \left\{ \frac{f_2(1/4)}{2} + \frac{\pi^2}{12} - \log\{2\} \cdot \log\{3/2\} - 1 \right\} \\ = \ &\log\{2\} \cdot \log\left\{ 3e/(2\sqrt{2}) \right\} - \frac{f_2(1/4)}{2} = 0.24473\cdots, \end{aligned}$$

because $f_2(1/4) = 0.978469393\cdots$. Therefore, the smallest constant β such that $\mathcal{S} \subset TN_\beta(f)$ for each $f \in \mathcal{C}$ lies between $0.2447\cdots$ and $0.275\cdots$. We conjecture that it is the first number.

Theorem 2.4. *Let f, g_1, g_2 be of the form*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ g_1(z) = z + \sum_{n=2}^{\infty} c_n z^n, \ g_2(z) = z + \sum_{n=2}^{\infty} d_n z^n,$$

where $|a_n| \le n$, $|c_n| \le n$, $|d_n| \le n$, $n = 2, 3, \ldots$. Then

$$g_1 * g_2 \in TN_{\log 2}(f).$$

The number $\log 2$ is the best possible.

Proof. Since

$$(g_1 * g_2)(z) = z + \sum_{n=2}^{\infty} c_n d_n z^n,$$

then we have

$$\sum_{n=2}^{\infty} \frac{1}{n^2 2^n} |c_n d_n - a_n| \le \sum_{n=2}^{\infty} \frac{n^2+n}{n^2 2^n} = \log 2.$$

The functions

$$f(z) = z - \sum_{n=2}^{\infty} n z^n, \quad g_1(z) = g_2(z) = z + \sum_{n=2}^{\infty} n z^n$$

show that the number $\log\{2\}$ is the best possible. Therefore the proof is completed. □

Definition 2.5 ([7]). Let A and B be arbitrary subsets of the $\mathcal{A}$, and let T be a sequence of positive number, then $\delta_T^*(A,B)$ is defined by

$$\delta_T^*(A,B) = \inf\{\delta > 0 : B \subset TN_\delta(f) \text{ for all } f \in A\}.$$

Let us denote

$$T(f,g) = \sum_{n=2}^{\infty} T_n |a_n - b_n|. \tag{2.10}$$

Therefore, we can write

$$\begin{aligned}\delta_T^*(A,B) &= \inf\{\delta : T(f,g) < \delta \text{ for all } f \in A,\, g \in B\} \\ &= \sup\{T(f,g) : \ f \in A,\, g \in B\},\end{aligned}$$

where the condition $T(f,g) < \delta$ means that the series $T(f,g)$ is convergent and its sum is less than δ. Therefore, we see that $\delta_T^*(A,B) = \delta_T^*(B,A)$, and we will say that $\delta_T^*(A,B)$ is the T-factor with respect to the classes A and B. Making use of the above definition, Corollary 2.2 and the consideration below Corollary 2.2, we can state next corollary where $T = \{T_n\}_{n=2}^{\infty}$ is again of the form (2.1).

Corollary 2.6. *The T-factor with respect to the classes $\mathcal{S}$ and $\mathcal{S}$ satisfies the following inequality*

$$0.287\cdots = \log\{4/3\} \le \delta_T^*(\mathcal{S},\mathcal{S}) \le \log\{4/e\} = 0.386\cdots. \tag{2.11}$$

It is well known that the Koebe function and all its rotations belong to each of the classes $\mathcal{S}$, $\mathcal{S}^*$ and $\mathcal{K}$ (univalent, starlike and close-to-convex functions respectively), then Corollary 2.6 follows the next corollary.

Corollary 2.7. *Let A and B be one of the classes $\mathcal{S}$, $\mathcal{S}^*$ or $\mathcal{K}$. Then*

$$\log\{4/3\} \le \delta_T^*(A,B) \le \log\{4/e\}.$$

In the same way as above, we can express Corollary 2.3 in terms T-factor. It is done in the next result.

Corollary 2.8. *The T-factor with respect to the classes $\mathcal{C}$ of convex functions and $\mathcal{S}$ satisfies the following inequality*

$$0.24473\cdots \le \delta_T^*(\mathcal{C},\mathcal{S}) \le 0.275\cdots.$$

Remark 2.9. Now we consider the "central" function with respect to coefficient in the class $\mathcal{C}o(1)$ which is denoted by $f_c(z)$ and defined by

$$f_c(z) = \frac{1}{2}\left\{\frac{z}{1-z} + \frac{z}{(1-z)^2}\right\} = z + \sum_{n=1}^{\infty} \frac{n+1}{2} z^n, \quad |z| < 1. \tag{2.12}$$

In [7] the authors showed that $f_c \in \mathcal{C}o(1)$.

Theorem 2.10. *The following inclusion relation holds*

$$\mathcal{C}o(1) \subset TN_\delta(f_c),$$

where $\delta = \log\sqrt{2/e} + \pi^2/24 - (\log 2)^2/4 = 0.13769\cdots$.

Proof. Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}o(1)$, then from (1.2), and using (2.3) and (2.4) with $x = -1/2$, we obtain

$$\begin{aligned}\sum_{n=2}^{\infty} T_n \left|a_n - \frac{n+1}{2}\right| &\le \frac{1}{2}\sum_{n=2}^{\infty}\frac{n-1}{n^2 2^n} \\ &= \frac{1}{2}\left\{\log 2 - \frac{1}{2} + f_2(1/2) - \frac{1}{2}\right\} \\ &= \frac{1}{2}\left\{\log 2 - 1 + \frac{\pi^2}{12} - \frac{(\log 2)^2}{2}\right\} \\ &= 0.13769\cdots = \delta.\end{aligned}$$

□

Acknowledgment. The authors would like to express their sincerest thanks to the referees for a careful reading and various suggestions made for the improvement of the paper.

References

[1] F. G. Avkhadiev, Ch. Pommerenke, and K. J. Wirths, *On the coefficients of concave univalent functions*, Math. Nachr. **271** (2004), 3–9.

[2] ———, *Sharp inequalities for the coefficients of concave schlicht functions*, Comment. Math. Helv. **81** (2006), no. 4, 801–807.

[3] F. G. Avkhadiev and K. J. Wirths, *Convex holes produce lower bound for coefficients*, Complex Var. Theory Appl. **47** (2002), no. 7, 553–563.

[4] U. Bednarz, *Stability of the Hadamard product of k-uniformly convex and k-starlike functions in certain neighbourhood*, Demonstratio Math. **38** (2005), no. 4, 837–845.

[5] U. Bednarz and S. Kanas, *Stability of the integral convolution of k-uniformly convex and k-starlike functions*, J. Appl. Anal. **10** (2004), no. 1, 105–115.

[6] U. Bednarz and J. Sokół, *On the integral convolution of certain classes of analytic functions*, Taiwanese J. Math. **13** (2009), no. 5, 1387–1396.

[7] ———, *T-neighborhoods of analytic functions*, J. Math. Appl. **32** (2010), 25–32.

[8] A. Bielecki and Z. Lewandowski, *Sur une généralisation de quelques théorèmes de M. Biernacki sur les fonctions analytiques*, Ann. Polon. Math. **12** (1962), 65–70.

[9] P. L. Duren, *Univalent functions*, Springer Verlag, Grund. math. Wiss. **259**, New York, Berlin, Heidelberg, Tokyo, 1983.

[10] R. Fournier, *A note on neighbourhoods of univalent functions*, Proc. Amer. Math. Soc. **87** (1983), no. 1, 117–120.

[11] ———, *On neighbourhoods of univalent starlike functions*, Ann. Polon. Math. **47** (1986), no. 20, 189–202.
[12] ———, *On neighbourhoods of univalent convex functions*, Rocky Mountain J. Math. **16** (1986), no. 3, 579–589.
[13] L. Lewin, *Dilogarithms and Associated Functions*, Macdonald, London, 1958.
[14] St. Ruscheweyh, *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc. **81** (1981), no. 4, 521–527.
[15] S. Shams and S. R. Kulkarni, *Certain properties of the class of univalent functions defined by Ruscheweyh derivative*, Bull. Cal. Math. Soc. **97** (2005), no. 3, 223–234.
[16] T. Sheil-Small, *On linear accessibility and the conformal mapping of convex domains*, J. Analyse Math. **25** (1972), 259–276.
[17] T. Sheil-Small and E. M. Silvia, *Neighborhoods of analytic functions*, J. Analyse Math. **52** (1989), 210–240.
[18] J. Stankiewicz, *Neighbourhoods of meromorphic functions and Hadamard products*, Ann. Polon. Math. **46** (1985), 317–331.

Saeid Shams
Department of Mathematics
University of Urmia
Urmia, Iran
E-mail address: s.shams@urmia.ac.ir

Ali Ebadian
Department of Mathematics
Payame Noor University
Tehran, Iran
E-mail address: a.ebadian@urmia.ac.ir

Mahta Sayadiazar
Department of Mathematics
University of Urmia
Urmia, Iran
E-mail address: m.sayyadiazar@yahoo.com

Janusz Sokół
Department of Mathematics
Rzeszów University of Technology
Al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland
E-mail address: jsokol@prz.edu.pl