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RESULTS OF CERTAIN LOCAL COHOMOLOGY MODULES

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Abstract. Let R be a commutative Noetherian ring, I and J two ideals of R, and M a finitely generated R-module. We prove that

$$\mathrm{Ext}^i_R(R/I, H^t_{I,J}(M))$$

is finitely generated for $i = 0, 1$ where $t = \inf\{i \in \mathbb{N}_0 : H^i_{I,J}(M)$ is not finitely generated$\}$. Also, we prove that $H^i_{I+J}(H^t_{I,J}(M))$ is Artinian when $\dim(R/I + J) = 0$ and $i = 0, 1$.

1. Introduction

Throughout this paper, we assume that R is a commutative Noetherian ring with non-zero identity, I and J two ideals of R, and M a finitely generated R-module. Recently Takahashi, Yoshino and Yoshizawa in [18], introduce the module $H^i_{I,J}(M)$ as a generalization of the ordinary local cohomology module $H^i_I(M)$ that defined by Grothendieck in [5]. They considered (I, J)-torsion submodule $\Gamma_{I,J}(M)$ of M which consists of all elements x of M with $\mathrm{Supp}(Rx) \subseteq W(I, J)$, where $W(I, J) = \{\mathfrak{p} \in \mathrm{Spec}(R) : I^n \subseteq \mathfrak{p} + J$ for an integer $n \geq 1\}$. Furthermore, they defined the local cohomology functor $H^i_{I,J}$ with respect to (I, J) to be the i-th right derived functor of $\Gamma_{I,J}$. Notice that if $J = 0$, then $H^i_{I,J}$ coincides with the ordinary local cohomology functor H^i_I. In [6], Grothendieck conjectured that the module $\mathrm{Hom}_R(R/I, H^i_I(M))$ is finitely generated for all $i \geq 0$. Hartshorne provided a counter-example to this conjecture in [7]. However, this conjecture is true in many cases see for example [1], [3], [4], [11] and [12]. In [8], Huneke asked when the local cohomology module $H^i_I(M)$ is Artinian. In this regard see [9], [10], [13] and [17]. The main aim of this paper is to prove the following theorem.

Theorem 1.1. *Let t be a non-negative integer such that $H^i_{I,J}(M)$ is finitely generated for all $i < t$. Then the following statements hold*:

(a) $\mathrm{Ext}^j(R/I, H^t_{I,J}(M))$ *is finitely generated for* $j = 0, 1$.

(b) $H^i_{I+J}(H^t_{I,J}(M))$ *is Artinian when* $\dim(R/I + J) = 0$ *and* $i = 0, 1$.

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2. The results

Recall that $\tilde{W}(I,J)$ is called the set of ideals $\mathfrak{a}$ of R such that $I^n \subseteq \mathfrak{a}+J$ for some integer n (cf. [18]).

Lemma 2.1. *Let $\mathfrak{a}$ be an ideal in $\tilde{W}(I,J)$. Then $(0 :_M \mathfrak{a}) = (0 :_{\Gamma_{I,J}(M)} \mathfrak{a})$.*

Proof. Since $\Gamma_{I,J}(M) \subseteq M$, we have $(0 :_{\Gamma_{I,J}(M)} \mathfrak{a}) \subseteq (0 :_M \mathfrak{a})$. To prove the converse inclusion, take $x \in (0 :_M \mathfrak{a})$, then $\mathfrak{a} \subseteq \operatorname{Ann}(x)$ and there is an integer n with $I^n \subseteq \mathfrak{a}+J$. Thus $I^n \subseteq \operatorname{Ann}(x)+J$ and hence $x \in (0 :_{\Gamma_{I,J}(M)} \mathfrak{a})$, as required. □

The following theorem extends [4, Theorem A].

Theorem 2.2. *Let $\mathfrak{a}$ be an ideal in $\tilde{W}(I,J)$, t a non-negative integer, and N an R-module such that $\operatorname{Ext}^i_R(R/\mathfrak{a},N)$ is a finitely generated R-module for $i = t, t+1$. If $H^i_{I,J}(N)$ is finitely generated for all $i < t$, then $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^t_{I,J}(N))$ is finitely generated for $j = 0, 1$.*

Proof. We use induction on t. Let $t = 0$. Then the short exact sequence

$$(\dagger) \qquad 0 \longrightarrow \Gamma_{I,J}(N) \longrightarrow N \longrightarrow N/\Gamma_{I,J}(N) \longrightarrow 0$$

induces the following exact sequence

$$0\longrightarrow \operatorname{Hom}_R(R/\mathfrak{a},\Gamma_{I,J}(N))\longrightarrow \operatorname{Hom}_R(R/\mathfrak{a},N)\longrightarrow \operatorname{Hom}_R(R/\mathfrak{a},N/\Gamma_{I,J}(N))$$

$$(\dagger\dagger) \qquad \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a},\Gamma_{I,J}(N))\longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a},N).$$

By Lemma 2.1 and [18, Corollary 1.13], we have $\operatorname{Hom}_R(R/\mathfrak{a},N/\Gamma_{I,J}(N)) = 0$. Hence by our hypothesis and using $(\dagger\dagger)$ the result follows. Suppose that $t > 0$ and that the case $t-1$ is settled. Since $\Gamma_{I,J}(N)$ is finitely generated, the R-module $\operatorname{Ext}^i_R(R/\mathfrak{a},\Gamma_{I,J}(N))$ is finitely generated for all i. Now, by the exact sequence $(\dagger)$, $\operatorname{Ext}^i_R(R/\mathfrak{a},N/\Gamma_{I,J}(N))$ is finitely generated for $i = t, t+1$. On the other hand $\Gamma_{I,J}(N/\Gamma_{I,J}(N)) = 0$ and $H^i_{I,J}(N) \cong H^i_{I,J}(N/\Gamma_{I,J}(N))$ for all $i > 0$. Hence we may assume that $\Gamma_{I,J}(N) = 0$. Let E be an injective hull of N and put $L = E/N$. Thus $\Gamma_{I,J}(E) = 0$ and $\operatorname{Hom}_R(R/\mathfrak{a},E) = 0$. Consequently, $\operatorname{Ext}^i_R(R/\mathfrak{a},L) \cong \operatorname{Ext}^{i+1}_R(R/\mathfrak{a},N)$ and $H^i_{I,J}(L) \cong H^{i+1}_{I,J}(N)$ for all $i \geq 0$. Now the induction hypothesis yields that $\operatorname{Ext}^j_R(R/\mathfrak{a},H^{t-1}_{I,J}(L))$ is finitely generated for $j = 0,1$, and hence $\operatorname{Ext}^j_R(R/\mathfrak{a},H^t_{I,J}(N))$ is finitely generated for $j = 0,1$. □

The following corollary immediately follows by Theorem 2.2.

Corollary 2.3. *Let t be a non-negative integer such that $H^i_{I,J}(M)$ is finitely generated for all $i < t$. Then $\operatorname{Ext}^j_R(R/I, H^t_{I,J}(M))$ is finitely generated for $j = 0,1$.*

Recall that an R-module N is called Minimax if there is a finitely generated submodule L of N such that N/L is Artinian (cf. [19]). The class of Minimax modules includes all finitely generated and all Artinian modules. Moreover, it is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of R-modules (cf. [16] and [19]).

Proposition 2.4. *Let $\mathfrak{a}$ be an ideal in $\tilde{W}(I,J)$, t a non-negative integer, and N an R-module such that $\operatorname{Ext}^i_R(R/\mathfrak{a},N)$ is a Minimax R-module for $i=t,t+1$. If $H^i_{I,J}(N)$ is Minimax for all $i<t$, then $\operatorname{Ext}^j_R(R/\mathfrak{a},H^t_{I,J}(N))$ is Minimax for $j=0,1$.*

Proof. By using Minimax instead of finitely generated in the proof of Theorem 2.2 and using the same arguments the result follows. □

Proposition 2.5. *Let t be a non-negative integer such that $H^i_{I,J}(M)$ is Artinian for all $i<t$. Then $H^i_I(M)$ is Artinian for all $i<t$.*

Proof. Let $E^\bullet$ denote an injective resolution of the R-module M. By [18, Proposition 1.4], there is the following isomorphism of complexes

$$\Gamma_I(\Gamma_{I,J}(E^\bullet)) \cong \Gamma_I(E^\bullet).$$

Hence, by [15, Theorem 11.38], there is the Grothendieck spectral sequence

$$E_2^{p,q} := H^p_I(H^q_{I,J}(M)) \underset{p}{\Longrightarrow} H^{p+q}_I(M).$$

Since $E_i^{p,q}$ is a subquotient of $E_2^{p,q}$ for all $i\geq 2$, by [2, Theorem 6.1.2] we deduce that $E_i^{p,q}=0$ for all $i\geq 2, p\geq 1$, and $q<t$. There is a finite filtration of the module $H^j = H^j_I(M)$

$$(*) \qquad 0=\varphi^{j+1}H^j \subseteq \varphi^j H^j \subseteq \cdots \subseteq \varphi^1 H^j \subseteq \varphi^0 H^j = H^j_I(M)$$

such that $E_\infty^{i,j-i} \cong \varphi^i H^j/\varphi^{i+1}H^j$ for all $0\leq i\leq j$. Since $E_i^{p,q}\cong E_\infty^{p,q}$ for i sufficiently large, we have that $E_\infty^{p,q}=0$ for all $p\geq 1$ and $q<t$. Hence, from $(*)$ we conclude that $0=\varphi^{j+1}H^j=\varphi^j H^j=\cdots=\varphi^1 H^j$ and $E_\infty^{0,j}\cong H^j_I(M)$ for all $j<t$. On the other hand $E_\infty^{0,j}\cong E_2^{0,j}$ for all $j<t$. Hence, we get that $H^i_I(M)$ is Artinian for all $i<t$. □

Corollary 2.6. *The following holds*:

$$\inf\{i \mid H^i_{I,J}(M) \text{ is not Artinian}\} \leq \inf\{i \mid H^i_I(M) \text{ is not Artinian}\}.$$

Proof. This is immediate by Proposition 2.5. □

Theorem 2.7. *Let t be a non-negative integer such that $H^i_{I,J}(M)$ is finitely generated for all $i<t$. If $\dim R/I+J=0$, then $H^j_{I+J}(H^t_{I,J}(M))$ is Artinian for $j=0,1$.*

Proof. By [18, Proposition 1.4] and [15, Theorem 11.38], there is the Grothendieck spectral sequence

$$E_2^{p,q} := H_{I'}^p(H_{I,J}^q(M)) \underset{p}{\Longrightarrow} H_{I+J}^{p+q}(M).$$

For all $i \geq 2$ and $j = 0, 1$, we consider the exact sequence

$$(\star) \qquad 0 \longrightarrow \ker d_i^{j,t} \longrightarrow E_i^{j,t} \xrightarrow{d_i^{j,t}} E_i^{i+j,t-i+1}.$$

Since $E_i^{j,t} = \ker d_{i-1}^{j,t} / \operatorname{im} d_{i-1}^{j-i+1,t+i-2}$ and $E_i^{p,q} = 0$ for all $q < 0$, we may use $(\star)$ to obtain $E_{t+2}^{j,t} \cong E_{t+3}^{j,t} \cong \cdots \cong E_\infty^{j,t}$ and $E_{i+1}^{j,t} \cong \ker d_i^{j,t}$. There exists a finite filtration of the module $H^{t+j} = H_{I+J}^{t+j}(M)$,

$$0 = \varphi^{t+j+1} H^{t+j} \subseteq \varphi^{t+j} H^{t+j} \subseteq \cdots \subseteq \varphi^1 H^{t+j} \subseteq \varphi^0 H^{t+j} = H_{I+I'}^{t+j}(M)$$

such that $E_\infty^{i,t+j-1} = \varphi^i H^{t+j} / \varphi^{i+1} H^{t+j}$ for all $0 \leq i \leq t+j$. Therefore $\ker d_{t+1}^{j,t}$ is Artinian and notice that $E_i^{p,q}$ is a subquotient of $E_2^{p,q}$ for all $i \geq 2$. Hence by the exact sequence $(\star)$ the result follows, as required. □

The following corollary improves [17, Theorem 2.2].

Corollary 2.8. *If* $\dim(R/I+J) = 0$, *then* $H_{I+J}^j(H_{I,J}^1(M))$ *is Artinian for* $j = 0, 1$.

Proof. This is immediate by Theorem 2.7. □

Hartshorne [7] defined that an R-module N is I-cofinte, if $\operatorname{Supp}(N) \subseteq V(I)$ and $\operatorname{Ext}_R^i(R/I, N)$ is a finitely generated R-module for all i.

The following result extends [14, Proposition 3.15].

Proposition 2.9. *If* I' *is an ideal of* R *such that* $\operatorname{cd} I' = 1$ *and* $I \subseteq I'$, *then* $H_{I'}^i(H_{I,J}^j(M))$ *is* I'*-cofinite for all* i *and* j, *where* cd *is the cohomological dimension of* I' *in* R.

Proof. By [18, Proposition 1.4] and [15, Theorem 11.38], there is the Grothendieck spectral sequence

$$E_2^{p,q} := H_{I'}^p(H_{I,J}^q(M)) \underset{p}{\Longrightarrow} H_{I'}^{p+q}(M).$$

By using the same arguments as in the proof of Theorem 2.7, for all $i \geq 0$ and all $j \geq 0$, we obtain the exact sequence

$$0 \longrightarrow \ker d_2^{j,i} \longrightarrow E_2^{j,i} \xrightarrow{d_2^{j,i}} E_2^{j+2,i-1} \longrightarrow \operatorname{coker} d_2^{j+2,i-1} \longrightarrow 0.$$

Since $H_{I'}^j(H_{I,J}^i(M)) = E_2^{j,i} = 0$ for all $j \geq 2$ and all $i \geq 0$, we can assume that $j = 0, 1$. Hence we have $E_\infty^{j,i} \cong \ker d_2^{j,i} = E_2^{j,i}$ for all $i \geq 0$. On the other hand $E_\infty^{j,i} = 0$ for all $j \geq 2$ and so there is an exact sequence

$$0 \longrightarrow H_{I'}^1(H_{I,J}^{i-1}(M)) \longrightarrow H_{I'}^i(M) \longrightarrow H_{I'}^0(H_{I,J}^i(M)) \longrightarrow 0.$$

By our assumption $H_{I'}^i(M) = 0$ for all $i \neq 0, 1$, and therefore by [14, Corollary 3.16] the result follows, as required. □

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