Bull. Korean Math. Soc. **51** (2014), No. 3, pp. 613–619
http://dx.doi.org/10.4134/BKMS.2014.51.3.613

WHEN AN $\mathscr{S}$-CLOSED SUBMODULE IS A DIRECT SUMMAND

Yongduo Wang and Dejun Wu

Abstract. It is well known that a direct sum of CLS-modules is not, in general, a CLS-module. It is proved that if $M = M_1 \oplus M_2$, where M_1 and M_2 are CLS-modules such that M_1 and M_2 are relatively ojective (or M_1 is M_2-ejective), then M is a CLS-module and some known results are generalized.

1. Introduction

CS-modules play important roles in rings and categories of modules and their generalizations have been studied extensively by many authors recently. In [3], Goodearl defined an *$\mathscr{S}$-closed* submodule of a module M is a submodule N for which M/N is nonsingular. Note that $\mathscr{S}$-closed submodules are always closed. In general, closed submodules need not be $\mathscr{S}$-closed. For example, 0 is a closed submodule of any module M, but 0 is $\mathscr{S}$-closed in M only if M is nonsingular. As a proper generalization of CS-modules, Tercan introduced the concept of CLS-modules. Following [8], a module M is called a *CLS-module* if every $\mathscr{S}$-closed submodule of M is a direct summand of M. In this paper, we continue the study of CLS-modules. Some preliminary results on CLS-modules are given in Section 1. In Section 2, direct sums of CLS-modules are studied. It is shown that if $M = M_1 \oplus M_2$, where M_1 and M_2 are CLS-modules such that M_1 and M_2 are relatively ojective, then M is a CLS-module and some known results are generalized. Tercan [8] proved that if a module $M = M_1 \oplus M_2$ where M_1 and M_2 are CS-modules such that M_1 is M_2-injective, then M is a CS-module if and only if $Z_2(M)$ is a CS-module. It is shown that Tercan's claim is not true in Section 3.

Throughout this paper, R is an associative ring with identity and all modules are unital right R-modules. We use $N \leq M$ to indicate that N is a submodule of M. Let M be a module and $S \leq M$. S is *essential* in M (denoted by $S \leq_e M$) if for any $T \leq M, S \cap T = 0$ implies $T = 0$. A module M is *CS* if for any submodule N of M, there exists a direct summand K of M such that

Received June 2, 2011.
2010 *Mathematics Subject Classification.* 16D10.
Key words and phrases. CLS-module, ejective module, ojective module.

$N \leq_e K$. A submodule K of M is closed in M if K has no proper essential extension in M, i.e., whenever L is a submodule of M such that K is essential in L, then $K = L$. It is well known that M is CS if and only every closed submodule is a direct summand of M. $Z(M)(Z_2(M))$ denotes the (second) singular submodule of M. For standard definitions we refer to [3].

2. Preliminary results

Lemma 2.1 ([8, Lemma 7])**.** *Any direct summand of a CLS-module is a CLS-module.*

Proposition 2.2. *A module M is a CLS-module if and only if for each $\mathscr{S}$-closed submodule K of M, there exists a complement L of K in M such that every homomorphism $f : K \oplus L \to M$ can be extended to a homomorphism $g : M \to M$.*

Proof. This is a direct consequence of [7, Lemma 2]. □

Following [1], a module M is *$\mathscr{G}$-extending* if for each submodule X of M there exists a direct summand D of M such that $X \cap D \leq_e X$ and $X \cap D \leq_e D$.

Proposition 2.3. *Let M be a $\mathscr{G}$-extending module. Then M is a CLS-module.*

Proof. Let N be an $\mathscr{S}$-closed submodule of M. There exists a direct summand D of M such that $N \cap D \leq_e N$ and $N \cap D \leq_e D$. Note that $D/(N \cap D)$ is both singular and nonsingular. Then $D = N \cap D$ and so $N = D$. Therefore, M is a CLS-module. □

In general, a CLS-module need not be a $\mathscr{G}$-extending module as the following example shows.

Example 2.4. Let K be a field and $V = K \times K$. Consider the ring R of 2×2 matrix of the form (a_{ij}) with $a_{11}, a_{22} \in K, a_{12} \in V, a_{21} = 0$ and $a_{11} = a_{22}$. Following [8, Example 14], R_R is a CLS module which is not a module with (C_{11}). Therefore, R_R is not a $\mathscr{G}$-extending module by [1, Proposition 1.6].

Applying Proposition 2.3, we will give some examples which are CLS modules, but not CS-modules as follows.

Example 2.5. Let M_1 and M_2 be abelian groups (i.e., $\mathbb{Z}$-modules) with M_1 divisible and $M_2 = \mathbb{Z}_{p^n}$, where p is a prime and n is a positive integer. Since $M = M_1 \oplus M_2$ is $\mathscr{G}$-extending by [1, Example 3.4], it is a CLS module by Proposition 2.3. But M is not CS, when M_1 is torsion-free. In particular, $\mathbb{Q} \oplus \mathbb{Z}_{p^n} \, (n \geq 2, p = \text{prime})$ is a CLS module, but not CS.

Example 2.6. Let M_1 be a $\mathscr{G}$-extending module with a finite composition series, $0 = X_0 \leq X_1 \leq \cdots \leq X_m = M_1$. Let $M_2 = X_m/X_{m-1} \oplus \cdots \oplus X_1/X_0$. Since $M = M_1 \oplus M_2$ is $\mathscr{G}$-extending by [1, Example 3.4], it is a CLS module by Proposition 2.3. But M is not CS in general. In particular, $M \oplus (U/V)$ is a CLS module, but not CS, where M is a uniserial module with unique composition series $0 \neq V \subset U \subset M$.

Proposition 2.7. *Let M be a nonsingular module. Then the following conditions are equivalent.*

(i) *M is a CS-module.*
(ii) *M is a $\mathscr{G}$-extending module.*
(iii) *M is a CLS-module.*

Proof. By [1, Proposition 1.8] and [8, Corollary 5]. □

Proposition 2.8. *Let M be a CLS-module and X be a submodule of M. If $Z(M/X) = 0$, then M/X is a CS-module.*

Proof. Since M is a CLS-module, X is a direct summand of M. Write $M = X \oplus X', X' \leq M$. Then M/X is a CS-module by Lemma 2.1 and Proposition 2.7. □

Corollary 2.9 ([1, Proposition 1.9])**.** *If M is $\mathscr{G}$-extending, $X \trianglelefteq M$, and $Z(M/X) = 0$, then M/X is a CS-module.*

Corollary 2.10 ([1, Corollary 3.11(i)])**.** *Let M be a $\mathscr{G}$-extending module. If D is a direct summand of M such that $Z(D) = 0$, then D is a CS-module.*

Proposition 2.11. *Let $K \leq_e M$ such that K is a CLS-module and for each $e^2 = e \in$End(K) there exists $\bar{e}^2 = \bar{e} \in$End(M) such that $\bar{e}|_K = e$. Then M is a CLS-module.*

Proof. Assume K is a CLS-module. Let X be an $\mathscr{S}$-closed submodule of M. Then $K = (X \cap K) \oplus K', K' \leq K$. Let $X \cap K = eK$, where $e^2 = e \in$End(K). By hypothesis, there exists $\bar{e}^2 = \bar{e} \in$End(M) such that $\bar{e}|_K = e$. Since $K \leq_e M$, $\bar{e}K \leq_e \bar{e}M$. Observe that $\bar{e}M \cap X \leq_e \bar{e}M$. But $\bar{e}M/(\bar{e}M \cap X)$ is nonsingular. Hence $\bar{e}M \leq X$. Thus $X = \bar{e}M$ as $\bar{e}K \leq_e X$. Therefore, M is a CLS-module. □

By analogy with the proof of [2, Corollary 3.14], we can obtain:

Corollary 2.12. *Let M be a module. If M is CLS, then so is the rational hull of M.*

3. Direct sums of CLS modules

It is well known that a direct sum of CLS-modules is not, in general, a CLS-module (see [8]). In this section, direct sums of CLS-modules are studied. It is shown that if $M = M_1 \oplus M_2$, where M_1 and M_2 are CLS-modules and M_1 and M_2 are relatively ojective, then M is a CLS-module and some known results are generalized. Tercan [8] proved that if a module $M = M_1 \oplus M_2$ where M_1 and M_2 are CS-modules such that M_1 is M_2-injective, then M is a CS-module if and only if $Z_2(M)$ is a CS-module. It is shown that Tercan's claim is not true in this section.

Let A, B be right R-modules. Recall that B is A-ojective [6] if and only if for any complement C of B in $A \oplus B, A \oplus B$ decomposes as $A \oplus B = C \oplus A' \oplus B'$

with $A' \leq A$ and $B' \leq B$. A and B are relatively ojective if A is B-ojective and B is A-ojective.

Lemma 3.1. *Let $M = A \oplus B$, where B is A-ojective and A is a CLS-module. If X is an $\mathscr{S}$-closed submodule of M such that $X \cap B = 0$, then M decomposes as $M = D \oplus A' \oplus B'$, where $A' \leq A, B' \leq B$.*

Proof. Let X be an $\mathscr{S}$-closed submodule of M with $X \cap B = 0$. Then M/X is nonsingular. Note that $X \cap A$ is an $\mathscr{S}$-closed submodule of A. Hence $X \cap A$ is a direct summand of A. Write $A = (X \cap A) \oplus A_1, A_1 \leq A$. By Lemma 2.1 and Proposition 2.7, A_1 is a CS-module. Let $K = (X \oplus B) \cap A$. Then $X \oplus B = K \oplus B$ and $K = (X \cap A) \oplus (K \cap A_1)$. There exists a closed submodule A_1' of A_1 such that $K \cap A_1 \leq_e A_1'$. Then A_1' is a direct summand of A_1. Write $A_1 = A_1' \oplus A_1'', A_1'' \leq A_1$. Now $X \oplus B = K \oplus B = (X \cap A) \oplus (K \cap A_1) \oplus B \leq_e (X \cap A) \oplus A_1' \oplus B$. Let $N = (X \cap A) \oplus A_1' \oplus B$. Then X is a complement of B in N. Now B is $(X \cap A) \oplus A_1'$-ojective by [6, Proposition 8]. By [6, Theorem 7], $N = X \oplus A' \oplus B'$, where $A' \leq (X \cap A) \oplus A_1'$ and $B' \leq B$. Therefore, $M = X \oplus A' \oplus A_1'' \oplus B'$, as required. □

Theorem 3.2. *Let $M = M_1 \oplus M_2$, where M_1 and M_2 are CLS-modules. If M_1 and M_2 are relatively ojective, then M is a CLS-module.*

Proof. Let X be an $\mathscr{S}$-closed submodule of M. If $X \cap M_1 = 0$, then X is a direct summand of M by Lemma 3.1. Let $X \cap M_1 \neq 0$. Then $X \cap M_1$ is a direct summand of M_1. Write $M_1 = (X \cap M_1) \oplus M_1', M_1' \leq M_1$. If $X \cap M_2 = 0$, then the result follows by Lemma 3.1. Let $X \cap M_2 \neq 0$. Then $X \cap M_2$ is a direct summand of M_2. Write $M_2 = (X \cap M_2) \oplus M_2', M_2' \leq M_2$. Then $X = (X \cap M_1) \oplus (X \cap M_2) \oplus (X \cap (M_1' \oplus M_2'))$. Note that M_1' and M_2' are CS-modules and M_1' and M_2' are relatively ojective, so $M_1' \oplus M_2'$ are a CS-module by [6, Theorem 7]. Hence $X \cap (M_1' \oplus M_2')$ is a direct summand of $M_1' \oplus M_2'$. Therefore, M is a CLS-module, as desired. □

Corollary 3.3 ([8, Theorem 10])**.** *Let R be a ring and M a right R-module such that $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is a finite direct sum of relatively injective modules $M_i, 1 \leq i \leq n$. Then M is a CLS-module if and only if M_i is a CLS-module for each $1 \leq i \leq n$.*

Let M_1 and M_2 be modules such that $M = M_1 \oplus M_2$. Recall that M_1 is M_2-ejective [1] if and only if for every submodule K of M with $K \cap M_1 = 0$ there exists a submodule M_3 of M such that $M = M_1 \oplus M_3$ and $K \cap M_3 \leq_e K$.

Lemma 3.4. *Let A_1 be a direct summand of A and B_1 a direct summand of B. If A is B-ejective, then A_1 is B_1-ejective.*

Proof. Write $M = A \oplus B$, $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$. First we prove that A_1 is B-ejective. Write $N = A_1 \oplus B$. Let X be a submodule of N with $X \cap A_1 = 0$. Then $X \cap A = 0$. Since A is B-ejective, there is a submodule C of M such that $M = A \oplus C$ and $X \cap C \leq_e X$. Hence $N = A_1 \oplus (N \cap (A_2 \oplus C))$.

Clearly, $X \cap (N \cap (A_2 \oplus C)) \leq_e X$. Therefore, A_1 is B-ejective. Next we prove that A is B_1-ejective. Write $L = A \oplus B_1$. Let Y be a submodule of L with $Y \cap A = 0$. Since A is B-ejective, there exists a submodule D of M such that $M = A \oplus D$ and $D \cap Y \leq_e Y$. Then $L = A \oplus (L \cap D)$. Clearly, $Y \cap (L \cap D) \leq_e Y$. Therefore, A is B_1-ejective. Thus A_1 is B_1-ejective. □

Theorem 3.5. *Let $M = M_1 \oplus M_2$, where M_1 and M_2 are CLS-modules. If M_1 is M_2-ejective, then M is a CLS-module.*

Proof. Let N be an $\mathscr{S}$-closed submodule of M. If $N \cap M_1 = 0$, then M_1 is nonsingular. Since M_1 is M_2-ejective, there is a submodule M_3 of M such that $M = M_1 \oplus M_3$ and $N \cap M_3 \leq_e N$. Note that $N/(N \cap M_3)$ is both singular and nonsingular. Hence $N = N \cap M_3$. Since $M_3 \cong M_2$, M_3 is a CLS-module. Clearly, M_3/N is nonsingular. Then N is a direct summand of M. Let $N \cap M_1 \neq 0$. Then $N \cap M_1$ is a direct summand of M_1. Write $M_1 = (N \cap M_1) \oplus M_1', M_1' \leq M_1$. Similarly, $M_2 = (N \cap M_2) \oplus M_2', M_2' \leq M_2$. Then $N = (N \cap M_1) \oplus (N \cap M_2) \oplus (N \cap (M_1' \oplus M_2'))$. Since M_1 is M_2-ejective, M_1' is M_2'-ejective by Lemma 3.4. Note that M_1' and M_2' are $\mathscr{G}$-extending modules. By [1, Theorem 3.1], $M_1' \oplus M_2'$ is $\mathscr{G}$-extending. Hence $N \cap (M_1' \oplus M_2')$ is a direct summand of $M_1' \oplus M_2'$. Therefore, M is a CLS-module, as desired. □

Corollary 3.6 ([8, Theorem 9])**.** *Let $M = M_1 \oplus M_2$, where M_1 and M_2 are CLS-modules. If M_1 is M_2-injective, then M is a CLS-module.*

Corollary 3.7. *Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ be a finite direct sum. If M_i is M_j-ejective for all $j > i$ and each M_i is a CLS-module, then M is a CLS-module.*

Proof. By analogy with the proof of [1, Corollary 3.2]. □

Corollary 3.8. *Let $M = M_1 \oplus M_2$. Then*

(i) *If M_1 is injective, then M is a CLS-module if and only if M_2 is a CLS-module.*
(ii) *If M_1 is a CLS-module and M_2 is semisimple, then M is a CLS-module.*

Corollary 3.9. *A module M is a CLS-module if and only if $M = Z_2(M) \oplus M', M' \leq M$, where $Z_2(M)$ and M' are CLS-modules.*

Proof. Let M be a CLS-module. Then $M = Z_2(M) \oplus M', M' \leq M$. By Lemma 2.1, $Z_2(M)$ and M' are CLS-modules. Conversely, if $M = Z_2(M) \oplus M', M' \leq M$, then M' is $Z_2(M)$-injective. Now the result follows by Theorem 3.5. □

Corollary 3.10. *Let $M = M_1 \oplus M_2$, where M_1 and M_2 are CS-modules. If M is nonsingular and M_1 is M_2-ejective, then M is a CS-module.*

Proof. By Proposition 2.7 and Theorem 3.5. □

Corollary 3.11. *Let $M = M_1 \oplus M_2$ be a direct sum of CS-modules M_1 and M_2, where M_2 is nonsingular. If M_1 is M_2-ejective and $Z_2(M_1)$ is M_2-injective, then M is a CS-module.*

Proof. By analogy with the proof of [8, Corollary 11]. □

Corollary 3.12 ([4, Theorem 4]). *Let $M = M_1 \oplus M_2$ be a direct sum of CS-modules M_1 and M_2, where M_2 is nonsingular. If M_1 is M_2-injective, then M is a CS-module.*

Corollary 3.13. *Let $M = M_1 \oplus M_2$ be a direct sum of CS-modules M_1 and M_2. If M_1 is M_2-ejective, $Z_2(M_1)$ is M_2-injective and $Z_2(M_2)$ is M_1-injective, then M is a CS-module if and only if $Z_2(M)$ is a CS-module.*

Proof. Let $Z_2(M)$ be a CS-module. Then $M = Z_2(M_1) \oplus Z_2(M_1) \oplus M_1' \oplus M_2'$, where $M_1' \leq M_1$ and $M_2' \leq M_2$. By [6, Theorem 1], $Z_2(M_1)$ is M_1'-injective and $Z_2(M_2)$ is M_2'-injective. Then $Z_2(M)$ is $M_1' \oplus M_2'$-injective. Since M_1 is M_2-ejective, $M_1' \oplus M_2'$ is a CS-module by Corollary 3.10. Hence M is a CS-module by [6, Theorem 1]. □

Corollary 3.14. *Let $M = M_1 \oplus M_2$ be a direct sum of CS-modules M_1 and M_2 such that M_1 is M_2-injective and $Z_2(M_2)$ is M_1-injective. Then M is a CS-module if and only if $Z_2(M)$ is a CS-module.*

We close this paper with the following.

A. Tercan [8, Corollary 13] showed that if a module $M = M_1 \oplus M_2$ where M_1 and M_2 are CS-modules such that M_1 is M_2-injective, then M is a CS-module if and only if $Z_2(M)$ is a CS-module. The following example shows that this claim is not true.

Example 3.15. Let $R = \mathbb{Z}$ and $M_{\mathbb{Z}} = \mathbb{Q} \oplus \mathbb{Z}_{p^n}(n \geq 2, p = \text{prime})$. We know that $\mathbb{Q}$ is $\mathbb{Z}_{p^n}$-injective and $\mathbb{Q}$, $\mathbb{Z}_{p^n}$ are uniform modules. Following by [1, Example 3.4], M is not CS. Next we show that $Z_2(M)$ is CS. Since $\mathbb{Q}_{\mathbb{Z}}$ is nonsingular, it is easy to see that $Z_2(M) = Z_2(\mathbb{Z}_{p^n})$. Since $\mathbb{Z}_{p^n}$ is CS, $Z_2(\mathbb{Z}_{p^n})$, as a direct summand of $\mathbb{Z}_{p^n}$, is CS.

References

[1] E. Akalan, G. F. Birkenmeier, and A. Tercan, *Goldie extending modules*, Comm. Algebra **37** (2009), no. 2, 663–683.

[2] G. F. Birkenmeier and A. Tercan, *When some complement of a submodule is a summand*, Comm. Algebra **35** (2007), no. 2, 597–611.

[3] K. R. Goodearl, *Ring Theory*, Marcel-Dekker, New York, 1976.

[4] A. Harmanci and P. F. Smith, *Finite direct sums of CS-modules*, Houston J. Math. **19** (1993), no. 4, 523–532.

[5] M. A. Kamal and B. J. Müller, *Extending modules over commutative domains*, Osaka J. Math. **25** (1988), no. 3, 531–538.

[6] S. H. Mohamed and B. J. Müller, *Ojective modules*, Comm. Algebra **30** (2002), no. 4, 1817–1827.

[7] P. F. Smith and A. Tercan, *Continuous and quasi-continuous modules*, Houston J. Math. **18** (1992), no. 3, 339–348.

[8] A. Tercan, *On CLS-modules*, Rocky Mountain J. Math. **25** (1995), no. 4, 1557–1564.

Yongduo Wang
Department of Applied Mathematics
Lanzhou University of Technology
Lanzhou 730050, P. R. China
E-mail address: ydwang@lut.cn

Dejun Wu
Department of Applied Mathematics
Lanzhou University of Technology
Lanzhou 730050, P. R. China
E-mail address: wudj2007@gmail.com