J. Appl. Math. & Informatics Vol. **32**(2014), No. 3 - 4, pp. 547 - 554
http://dx.doi.org/10.14317/jami.2014.547

SOME PROPERTIES OF SOLUTIONS FOR A SIXTH-ORDER PARABOLIC EQUATION IN ONE SPATIAL DIMENSION

XIAOPENG ZHAO

ABSTRACT. In this paper, we consider the existence and uniqueness of global weak solution for a sixth-order classical surface-diffusion equation in one spatial dimension. Moreover, the regularity and blow-up of solutions are also studied.

AMS Mathematics Subject Classification : 35K55, 49A22.
Key words and phrases : Sixth-order diffusion equation, existence, regularity, blow-up.

1. Introduction

In the study of a thin, solid film grown on a solid substrate, in order to describe the continuum evolution of the film free surface, there arise a classical surface-diffusion equation (see [1])

$$v_n = \mathcal{D}\Delta_S\mu = \mathcal{D}\Delta_S(\mu_\gamma + \mu_w) = \mathcal{D}\Delta_S(\tilde{\gamma}_{\alpha\beta}C_{\alpha\beta} + \nu\Delta^2 u + \mu_w), \tag{1}$$

where v_n is the normal surface velocity, $\mathcal{D} = D_S S_0 \Omega_0 V_0/(RT)^{23}$ (D_s is the surface diffusivity, S_0 is the number of atoms per unit area on the surface, Ω_0 is the atomic volume, V_0 is the molar volume of lattice cites in the film, R is the universal gas constant and T is the absolute temperature), Δ_S is the surface Laplace operator, ν is the regularization coefficient that measures the energy of edges and corners, $C_{\alpha\beta}$ is the surface curvature tensor and μ_w being an exponentially decaying function of u that has a singularity at $u \to 0$ (see [1]).

In the small-slop approximation, in the particular cases of high-symmetry orientations of a crystal with cubic symmetry, neglect the exponentially decaying, consider the 1D case, then the evolution equation (1) for the film thickness can be written in the following form

$$\frac{\partial u}{\partial t} = \mathcal{D}D^2[\sigma D^2 u + \mu D^4 u - a|Du|^2 D^2 u], \tag{2}$$

Received October 5, 2013. Revised March 5, 2014. Accepted March 17, 2014.

(see [1]). Moreover, from a mathematical point of view, we will consider the nonlinear parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} = \gamma D^6 u + kD^4 u - \alpha D^2(|Du|^{p-2} D^2 u), \text{ in } Q_T, \\ u(x,t) = D^2 u(x,t) = D^4 u(x,t) = 0, \ x = 0, 1, \\ u(x,0) = u_0(x), \end{cases} \tag{3}$$

where $Q_T = (0,1) \times (0,T)$ and $p > 2$, $\gamma, k > 0$, α are constants.

In this paper, we consider some properties of solutions for problem (3). This paper is organized as follows. In the next section, we establish the existence of global weak solution in the space $H^{6,1}(Q_T)$. In Section 3, we consider the regularity of the solution for problem (3). In the last section, we consider the blow-up of solutions for the above problem.

In the following, the letters C, C_i, $(i = 0, 1, 2, \cdots)$ will always denote positive constants different in various occurrences.

2. Global weak solution

In this section, we consider the existence and uniqueness of global weak solutions of the problem (3).

Theorem 2.1. *Assume that $\alpha > 0$, $p \geq 4$, $u_0 \in H^3(0,1)$ with $D^i u_0(0,t) = D^i u_0(1,t) = 0$ $(i = 0, 2)$), then for all $t \in (0,T)$, there exists a unique solution $u(x,t)$ such that*

$$u(x,t) \in L^2(0,T; H^6(0,1)) \bigcap L^\infty(0,T; H^3(0,1)).$$

Proof. Multiplying the equation of (3) by u and integrating with respect to x over $(0,1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2 dx + \gamma \int_0^1 |D^3 u|^2 dx + \alpha \int_0^1 |Du|^{p-2}(D^2 u)^2 dx = k \int_0^1 |D^2 u|^2 dx.$$

Noticing that

$$\begin{aligned} k\int_0^1 |D^2 u|^2 dx = & - k\int_0^1 DuD^3 u dx \leq \frac{k^2}{\gamma}\int_0^1 |Du|^2 dx + \frac{\gamma}{4}\int_0^1 |D^3 u|^2 dx \\ = & - \frac{k^2}{\gamma}\int_0^1 uD^2 u dx + \frac{\gamma}{4}\int_0^1 |D^3 u|^2 dx \\ \leq & \frac{k}{2}\int_0^1 |D^2 u|^2 dx + \frac{k^3}{2\gamma^2}\int_0^1 u^2 dx + \frac{\gamma}{4}\int_0^1 |D^3 u|^2 dx, \end{aligned}$$

Hence, a simple calculation shows that

$$\frac{d}{dt}\int_0^1 u^2 dx + \gamma \int_0^1 |D^3 u|^2 dx \leq \frac{2k^3}{\gamma^2}\int_0^1 u^2 dx. \tag{4}$$

Gronwall's inequality implies that

$$\sup_{0<t<T}\int_0^1 u^2 dx \leq C, \text{ and } \iint_{Q_T} |D^3 u|^2 dxdt \leq C. \tag{5}$$

The energy function is

$$F(t)=\frac{\gamma}{2}\int_0^1|D^2u|^2dx-\frac{k}{2}\int_0^1|Du|^2dx+\frac{\alpha}{p(p-1)}\int_0^1|Du|^pdx.$$

Integrations by parts and (3) yield

$$\begin{aligned}\frac{dF(t)}{dt}&=\int_0^1[\gamma D^4u+kD^2u-\frac{\alpha}{p-1}D(|Du|^{p-2}Du)]u_tdx\\&=\int_0^1[\gamma D^4u+kD^2u-\frac{\alpha}{p-1}D(|Du|^{p-2}Du)]\\&\quad\cdot D^2[\gamma D^4u+kD^2u-\frac{\alpha}{p-1}D(|Du|^{p-2}Du)]dx\\&=-\int_0^1|D[\gamma D^4u+kD^2u-\frac{\alpha}{p-1}D(|Du|^{p-2}Du)]|^2dx\le 0.\end{aligned}$$

Therefore

$$\begin{aligned}F(t)=&\frac{\gamma}{2}\int_0^1|D^2u|^2dx-\frac{k}{2}\int_0^1|Du|^2+\frac{\alpha}{p(p-1)}\int_0^1|Du|^pdx\\\le& F(0)=\frac{\gamma}{2}\int_0^1|D^2u_0|^2dx-\frac{k}{2}\int_0^1|Du_0|^2+\frac{\alpha}{p(p-1)}\int_0^1|Du_0|^pdx.\end{aligned}$$

That is

$$\frac{\gamma}{2}\int_0^1|D^2u|^2dx\le\frac{\gamma}{2}\int_0^1|D^2u_0|^2+\frac{k}{2}\int_0^1|Du|^2dx+\frac{\alpha}{p(p-1)}\int_0^1|Du_0|^pdx.$$

It then follows from (5) that

$$\int_0^1|Du|^2dx\le\frac{\gamma}{2k}\int_0^1|D^2u|^2dx+\frac{k}{2\gamma}\int_0^1u^2dx\le\frac{\gamma}{2k}\int_0^1|D^2u|^2dx+C.$$

Summing the above two inequalities together, we get

$$\sup_{0<t<T}\int_0^1|D^2u|^2dx\le C \text{ and } \sup_{0<t<T}\int_0^1|Du|^2dx\le C. \tag{6}$$

By (5), (6) and Sobolev's embedding theorem, we have

$$\left(\int_0^1|u|^qdx\right)^{\frac{1}{q}}\le C\|u\|_{H^1}\le C,\ \forall q\in(0,+\infty]. \tag{7}$$

$$\left(\int_0^1|Du|^qdx\right)^{\frac{1}{q}}\le C\|u\|_{H^2}\le C,\ \forall q\in(0,+\infty]. \tag{8}$$

Again multiplying the equation of (3) by D^6u and integrating with respect to x over $(0,1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1|D^3u|^2dx+\gamma\int_0^1|D^6u|^2dx=k\int_0^1|D^5u|^2dx+\alpha\int_0^1D^2(|Du|^{p-2}D^2u)D^6udx.$$

By Nirenberg's inequality, we get

$$C\int_0^1 |D^2u|^6dx \le C'\left(\int_0^1 |D^6u|^2dx\right)^{\frac14}\left(\int_0^1 |D^2u|^2dx\right)^{\frac{11}{4}} \le \frac{\gamma}{10}\int_0^1 |D^6u|^2dx + C_1.$$
$$C\int_0^1 |D^2u|^4dx \le C'\left(\int_0^1 |D^6u|^2dx\right)^{\frac14}\left(\int_0^1 |D^2u|^2dx\right)^{\frac{7}{4}} \le \frac{\gamma}{10}\int_0^1 |D^6u|^2dx + C_2.$$
$$C\int_0^1 |D^3u|^4dx \le C'\left(\int_0^1 |D^6u|^2dx\right)^{\frac58}\left(\int_0^1 |D^2u|^2dx\right)^{\frac{11}{8}} \le \frac{\gamma}{10}\int_0^1 |D^6u|^2dx + C_3.$$

and

$$C\int_0^1 |D^4u|^2dx \le C'\left(\int_0^1 |D^6u|^2dx\right)^{\frac12}\left(\int_0^1 |D^2u|^2dx\right)^{\frac12} \le \frac{\gamma}{10}\int_0^1 |D^6u|^2dx + C_4. \tag{9}$$

Using (8) and above four inequalities, we derive that

$$\begin{aligned}
&\alpha\int_0^1 D^2(|Du|^{p-2}D^2u)D^6udx\\
\le&\alpha(p-2)(p-3)\int_0^1 |Du|^{p-4}|D^2u|^3D^6udx\\
&+3\alpha(p-2)\int_0^1 |Du|^{p-3}D^2uD^3uD^6udx + \alpha\int_0^1 |Du|^{p-2}D^4uD^6udx\\
\le&\alpha(p-2)(p-3)\sup_{x\in[0,1]}|Du|^{p-4}\cdot\int_0^1 |D^2u|^3D^6udx\\
&+3\alpha(p-2)\sup_{x\in[0,1]}|Du|^{p-3}\cdot\int_0^1 D^2uD^3uD^6udx\\
&+\sup_{x\in[0,1]}|Du|^{p-2}\cdot\alpha\int_0^1 D^4uD^6udx\\
\le&C(\int_0^1 |D^2u|^6dx + \int_0^1 |D^2u|^4dx + \int_0^1 |D^3u|^4dx + \int_0^1 |D^4u|^2dx) + \frac{\gamma}{5}\int_0^1 |D^6u|^2dx\\
\le&\frac{2\gamma}{5}\int_0^1 |D^6u|^2dx + C.
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
k\int_0^1 |D^5u|^2dx \le& C'\left(\int_0^1 |D^6u|^2dx\right)^{\frac34}\left(\int_0^1 |D^2u|^2dx\right)^{\frac14}\\
\le&\frac{\gamma}{10}\int_0^1 |D^6u|^2dx + C_5.
\end{aligned} \tag{10}$$

Then, summing up, we get

$$\frac{d}{dt}\int_0^1 |D^3u|^2dx + \gamma\int_0^1 |D^6u|^2dx \le C. \tag{11}$$

Hence

$$\sup_{0<t<T}\int_0^1 |D^3u|^2dx \le C \text{ and } \iint_{Q_T} |D^6u|^2dxdt \le C. \tag{12}$$

Therefore, by (9), (10) and (12), we immediately obtain

$$\iint_{Q_T} |D^4u|^2 dxdt \le C \text{ and } \iint_{Q_T} |D^5u|^2 dxdt \le C. \tag{13}$$

The a priori estimates (5)-(6) and (12)-(13) complete the proof of global existence of a $u(x,t) \in L^2(0,T;H^6(0,1)) \bigcap L^\infty(0,T;H^3(0,1))$.

Since the proof of uniqueness of global solution is so easy, we omit it here. Then, we complete the proof. $\square$

3. Regularity

The following Lemma (see [4]) will be used to prove the main result of this section.

Lemma 3.1. *Assume that* $\sup|f| < +\infty$, $a(x,t) \in C^{\kappa,\frac{\kappa}{6}}(\bar{Q}_T)$, $0 < \alpha < 1$, *and there exist two constants* a_0, b_0, A_0, B_0 *such that* $0 < a_0 \le a(x,t) \le A_0$, $0 < b_0 \le b(x,t) \le B_0$ *for all* $(x,t) \in Q_T$. *If* u *is a smooth solution for the following linear problem*

$$\begin{aligned}
&\frac{\partial u}{\partial t} + D^3(a(x,t)D^3u) + D^3(b(x,t)Du) = D^3 f,\ (x,t) \in Q_T,\\
&Du(x,t)|_{x=0,1} = D^3u(x,t)|_{x=0,1} = D^5u(x,t)|_{x=0,1} = 0,\ t \in [0,T],\\
&u(x,0) = u_0(x),\ x \in [0,1],
\end{aligned}$$

then, for any $\delta \in (0,\frac{1}{2})$, *there is a constant* C *depending on* a_0, b_0, A_0, B_0, δ, T, $\iint_{Q_T} u^2 dxdt$ *and* $\iint_{Q_T} |D^3u|^2 dxdt$, *such that*

$$|u(x_1,t_1) - u(x_2,t_2)| \le C(1 + \sup|f|)(|x_1 - x_2|^\delta + |t_1 - t_2|^{\frac{\delta}{6}}).$$

Now, we turn our discussion to the regularity of solutions.

Theorem 3.2. *Assume that* $p \ge 6$, $u_0 \in C^{6+\kappa}[0,1]$, $(0 < \kappa < 1)$, *then for any smooth initial value* u_0, *problem (3) admits a unique classical solution* $u(x,t) \in C^{6+\kappa,1+\frac{\kappa}{6}}(\bar{Q}_T)$.

Proof. By (5) and (8), we have

$$|u(x_1,t) - u(x_2,t)| \le C|x_1 - x_2|^\kappa,\ \kappa \in (0,1).$$

Integrating the equation of (3) with respect to x over $(y, y + (\Delta t)^{\frac{1}{6}}) \times (t_1,t_2)$, where $0 < t_1 < t_2 < T$, $\Delta t = t_2 - t_1$, we see that

$$\begin{aligned}
&\int_y^{y+(\Delta t)^{\frac{1}{6}}} [u(z,t_2) - u(z,t_1)]dz\\
=&\int_{t_1}^{t_2} [\gamma D^5u(y',s) + kD^3u(y',s) - \alpha D(|Du(y',s)|^{p-2}D^2u(y',s))\\
&- \gamma D^5u(y,s) - kD^3u(y,s) + \alpha D(|Du(y,s)|^{p-2}D^2u(y,s))]ds,
\end{aligned} \tag{14}$$

where $y' = y + (\Delta t)^{\frac{1}{6}}$. For simplicity, set

$$\begin{aligned} N(y,s) =& \gamma D^5 u(y',s) + kD^3 u(y',s) - \alpha D[|Du(y',s)|^{p-2} D^2 u(y',s)] \\ & - \gamma D^5 u(y,s) - kD^3 u(y,s) + \alpha D[|Du(y,s)|^{p-2} D^2 u(y,s)]. \end{aligned}$$

Then, (14) is converted into

$$(\Delta t)^{\frac{1}{6}} \int_0^1 [u(y + \theta(\Delta t)^{\frac{1}{6}}, t_2) - u(y + \theta(\Delta t)^{\frac{1}{6}}, t_1)] d\theta = \int_{t_1}^{t_2} N(y,s) ds.$$

Integrating the above equality with respect to y over $(x, x + (\Delta t)^{\frac{1}{6}}$, we derive that

$$(\Delta t)^{\frac{1}{3}} (u(x^*, t_2) - u(x^*, t_1)) = \int_{t_1}^{t_2} \int_x^{x+(\Delta t)^{\frac{1}{6}}} N(y,s) dy ds.$$

Here, we have used the mean value theorem, where $x^* = y^* + \theta^*(\Delta t)^{\frac{1}{6}}$, $y^* \in (x, x + (\Delta t)^{\frac{1}{6}})$, $\theta^* \in (0,1)$. Then, by Hölder's inequality and (8), (13), we end up with

$$|u(x^*, t_2) - u(x^*, t_1)| \le C(\Delta t)^{\frac{\kappa}{6}}, \ \alpha \in (0,1).$$

Similar to the discussion above , we have

$$|Du(x_1,t_1) - Du(x_2,t_2)| \le C(|x_1 - x_2|^{\frac{1}{2}} + |t_1 - t_2|^{\frac{1}{12}}). \tag{15}$$

and

$$|D^2 u(x_1,t_1) - D^2 u(x_2,t_2)| \le C(|x_1 - x_2|^{\frac{1}{2}} + |t_1 - t_2|^{\frac{1}{12}}). \tag{16}$$

We shall consider the Hölder estimate of $D^2 u$ based on Lemma 3.1. Suppose that $w = D^2 u - D^2 u_0$, then w satisfies the following problem

$$\begin{aligned} & \frac{\partial w}{\partial t} - D^3(a(x,t) D^3 w) + D^3(b(x,t) Dw) = D^3 f, \\ & w(x,t) = D^2 w(x,t) = D^4 w(x,t) = 0, \ x = 0, 1, \\ & w(x,0) = 0, \ x \in [0,1], \end{aligned} \tag{17}$$

where $a(x,t) = \gamma$, $b(x,t) = k$ and

$$f(x,t) = -\gamma D^5 u_0 - kD^3 u_0 + \alpha D(|Du|^{p-2} D^2 u)].$$

Define the linear spaces

$$X = \{u \in C^{1+\kappa, \frac{1+\kappa}{6}}(\bar{Q}_T); u|_{x=0,1} = D^2 u|_{x=0,1} = 0, u(x,0) = u_0(x)\}$$

and the associated operator $T : X \to X, u \to v$, where v is determined by the following linear problem

$$\begin{cases} v_t - \gamma D^6 v + (\alpha |Du|^{p-2} - k) D^4 v + 3\alpha(p-2)|Du|^{p-2} D^2 u D^3 v \\ +\alpha(p-2)(p-3)|Du|^{p-4} D^2 v = 0, (x,t) \in \Omega \times (0,T), \\ v(x,t)|_{x=0,1} = D^2 v(x,t)|_{x=0,1} = D^4 v(x,t)|_{x=0,1} = 0, \\ v(x,0) = v_0(x). \end{cases} \tag{18}$$

From the classical parabolic theory (see[3, 5]), we know that the above problem admits a unique solution in the space $C^{6+\kappa, 1+\frac{\kappa}{6}}(\bar{Q}_T)$. Thus, the operator T is well defined. It follows from the embedding theorem that the operator T is a compact operator. If $u = \sigma T u$ holds for some $\sigma \in (0, 1]$, then by the previous arguments, we know that there exists a constant C which is independent of u and σ, such that $\|u\|_{C^{6+\kappa,1+\frac{\kappa}{6}}(\bar{Q}_T)} \le C$. Then, it follows from the Leray-Schauder fixed point theorem that the operator T admits a fixed point u, which is the desired solution of problem (3). Furthermore, by the above arguments, we know that u is a classical solution. □

4. Blow-up

In the previous section, we have seen that the solution of problem (3) is globally classical, provided that $\alpha > 0$. The following theorem shows that the solution of the problem (3) blows up at a finite time for $\alpha < 0$ and $F(0) \le 0$.

Theorem 4.1. *Assume $u_0 \not\equiv 0$, $p > 2$, $\alpha < 0$ and $F(0) \le 0$, then the solution of problem (3) must blow up at a finite time, namely, for some $T^* > 0$,*

$$\lim_{t\to T^*} \|u(t)\| = +\infty.$$

Proof. Without loss of generality, we assume that $\int_0^1 u_0 dx = 0$. Otherwise, we may replace u by $v = u - M$, where $M = \int_0^1 u_0 dx$. For the energy functional $F(t)$,a direct calculation yields that $F'(t) \le 0$, which implies that $F(t) \le F(0)$. Let ω be the unique solution of the problem

$$\begin{cases} D^2\omega = u, \\ D\omega|_{x=0,1} = 0, \\ \int_0^1 \omega dx = 0. \end{cases}$$

Based on the equation of (3), we immediately obtain $\int_0^1 u(x,t)dx = \int_0^1 u_0(x)dx$, then, such function as ω is exists, which satisfies

$$\int_0^1 |D\omega|^2 dx \le \int_0^1 u^2 dx. \tag{19}$$

Multiplying the equation of (3) by ω and integrating with respect to x over $(0, 1)$, integrating by parts and using the boundary value conditions, we deduce that

$$\begin{aligned} \frac{d}{dt}\int_0^1 |D\omega|^2 dx =& -\frac{\alpha}{p-1}\int_0^1 |Du|^p dx + k\int_0^1 |Du|^2 dx - \gamma\int_0^1 |D^2u|^2 dx \\ \ge& -\frac{\alpha}{p-1}\int_0^1 |Du|^p dx + \frac{2\alpha}{p(p-1)}\int_0^1 |Du|^p dx - 2F(0) \\ =&\frac{\alpha}{p-1}\left(\frac{2\alpha}{p} - 1\right)\int_0^1 |Du|^p dx - 2F(0) \\ \ge&\frac{\alpha}{p-1}\left(\frac{2\alpha}{p} - 1\right)\int_0^1 |Du|^p dx. \end{aligned} \tag{20}$$

By Poincaré's inequality and the embedding of L^p space, we get

$$\left(\int_0^1 |D\omega|^2 dx\right)^{\frac{p}{2}} \leq C\left(\int_0^1 |u|^2 dx\right)^{\frac{p}{2}} \leq C\left(\int_0^1 |Du|^2 dx\right)^{\frac{p}{2}} \leq C\int_0^1 |Du|^p dx.$$

It then follows from (20) that

$$\frac{d}{dt}\int_0^1 |D\omega|^2 dx \geq \frac{\alpha(2\alpha-p)C}{p(p-1)}\left(\int_0^1 |D\omega|^2 dx\right)^{\frac{p}{2}}. \tag{21}$$

A direct integration of (21), we obtain

$$\left(\int_0^1 |D\omega|^2 dx\right)^{\frac{p}{2}-1} \geq \frac{1}{\left(\int_0^1 |D\omega_0|^2 dx\right)^{1-\frac{p}{2}} - \kappa t},$$

where $\kappa = \frac{\alpha(2\alpha-p)(p-2)C}{2p(p-1)}$. Noticing that $u_0 \not\equiv 0$, then $\int_0^1 |D\omega_0|^2 dx \not\equiv 0$. Combining (19) and above inequality, setting $T^* = \frac{1}{\kappa}(\int_0^1 |D\omega_0|^2 dx)^{1-\frac{p}{2}}$, we get u must blow up in a finite time T^*. $\square$

References

1. A. A. Golovin, M. S. Levine, T. V. Savina, S. H. Davis, *Faceting instability in the presernce of wetting interactions: A mechanism for the formation of quantum dots*, Phys. Rev. B, 70(2004), 235342.
2. A. Jüngel, J. Milisić, *A sixth-order nonlinear parabolic equation for quantum systems*, SIAM J. Math. Anal., 41(4)(2009), 1472C1490.
3. O. A. Ladyzhenskaja, V. A. Solonikov, N. N.Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*. Amer. Math. Soc., Providence, RI, 1968.
4. C. Liu, *Regularity of solutions for a sixth order nonlinear parabolic equation in two space dimensions*, Annales Polonici Mathematici, In press.
5. J. Yin, *On the Cahn-Hilliard equation with nonlinear principal part, J. Partial Differential Equations*, 7(1994), 77-96.

Xiaopeng Zhao received M.Sc. from Jilin University and Ph.D at Jilin University. Since 2013 he has been at Jiangnan University. His research interests include properties of higher-order nonlinear parabolic equations and numerical solutions for PDE.

School of Science, Jiangnan University, Wuxi 214122, China.
`e-mail: zhaoxiaopeng@sina.cn`