J. Appl. Math. & Informatics Vol. **32***(2014), No. 3 - 4, pp. 539 - 546*
http://dx.doi.org/10.14317/jami.2014.539

SOME ANALYSES ON A PROPOSED METHOD OF THE OPTIMAL NETWORK SELECTION PROBLEM

JONG SEUL LIM

Abstract. This paper introduces the approximation and a proposed method to deal the optimum location network selection problem such that the total cost is minimized. For the proposed method, we derived a feasible solution and the variance. To compare the performances of the approximation and the proposed method, computer simulation is also implemented. The result showed the solutions being optimum with 74% for the proposed method and 57% for the approximation. When the solutions is not optimum, maximum and average deviations are below 4% and 2% respectively. The results indicate a slightly better performance of the proposed method in a certain case.

AMS Mathematics Subject Classification : 68M20.
Key word and phrases : Sources, Approximation, Optimum solutions.

1. Introduction

We consider the situation that m given items are to be assigned to n given locations, such that every item must be assigned to one and only one location, but more than one item may be assigned to the same location.

The cost of assigning the ith item to the jth location is p_{ij}, and that of assigning at least one item to the jth location is c_j, $i = 1, 2, ..., m$, and $j = 1, 2, ..., n$.

To get the optimum solutions of this type of the problem, traditionally the approximation [1] which is reasonably efficient has been being applied.

In the first assigns all the items to the same locations, selects the one which minimizes the total cost and then proceeds recursively, selecting one additional location at a time. For example, suppose that $k < n$ locations is tested to see whether further reduction is possible of the value of the objective function (1), when it is also selected and the m items are assigned accordingly.

Received October 18, 2013. Revised December 14, 2013. Accepted December 17, 2013.

If these are such locations, then the procedure selects the one for which the reduction is the largest. Iteration terminates when no further reduction is possible. At each stage of iteration, it considers all possible ways of adding the unselected location to those already selected, and of deleting the latter ones, and then course of action which reduces the most the value of the objective function. Define

$$x_{ij} = \begin{cases} 1, & \text{if ith item is received from jth source,} \\ 0, & \text{if otherwise; and} \end{cases} \tag{1}$$

$$y_j \ = \text{Minimum } \{\sum_{i=1}^{m} \ x_{ij}, 1\}, \ i = 1, ..., m, j = 1, 2, ..., n. \tag{2}$$

Then, the optimum location selection problem can be defined as

$$\text{Minimize } C \ = \ \sum_{i=1}^{m}\sum_{j=1}^{m} x_{ij}p_{ij} + \sum_{j=1}^{n} c_j y_j, \tag{3}$$

$$\text{Subject to} \quad \sum_{j=1}^{n} x_{ij}, x_{ij} \leq y_j,$$

where C is the total cost incurred from being assigned, x_{ij}, $y_j = 0, 1$, for all $i = 1, ..., m$, and $j = 1, ..., n$.

2. The Approximation

In order to see the difference between the approximation and the proposed method, we describe the former given in [1] with minor modifications. Let $I \ = \ 1, 2, ..., m$, the set of the m given items; $J \ = \ 1, 2, ..., n$, the set of the n given locations;

$$J(k) \ = \ \text{the set of locations selected at the end of the } kth \text{ iteration.} \tag{4}$$

Initially, define $J(1) \ = \ J_1$, where J_1 minimizes the following with respect to all $j \in J$,

$$\sum_{i=1}^{m} x_{ij}p_{ij} \ + \ c_j. \tag{5}$$

For $k \ = \ 2, 3, ..., n$, and $j \in J$, define

$$u_i^k \ = \ \min_{j \in J(k-1)} p_{ij}, \text{ and}$$

$$\rho_j(u^k) \ = \ \sum_{i=1}^{m} \max(u_i^k - p_{ij}, 0) - c_j, \text{ for } j \notin J(k-1). \tag{6}$$

Then find $j_k \notin J(k-1)$ such that

$$\rho_{j_k}(u^k) \ = \ \rho_k \ = \ \max_{j \in J(k-1)} \rho_j(u^k), \tag{7}$$

and define $J(k) = J(k-1)Uj_k$, where U_{jk} denotes the union of sets. The iteration procedure terminates either as soon as $k = 1, 2, ..., n$ is found for which $\rho_k \geq 0$, or $\rho_k > 0$, for all $k = 1, 2, ..., n$. In the first case, the optimum solution is

$$y_j = \begin{cases} 1, & \text{if } j \in J(k-1), \\ 0, & \text{if } j \notin J(k-1); \text{ and for every } i \in I, \end{cases} \tag{8}$$

$$x_{ij} = \begin{cases} 1, & \text{if } j \text{ is such that } p_{ij} = u_i^k, \\ 0, & \text{if otherwise.} \end{cases} \tag{9}$$

In the second case, the optimum solution may be obtained from (9) by setting $k = n+1$. From (6), (7), and (9), we see that at the end of each iteration, every item i is assigned to the location for which p_{ij} is minimized among all the locations that are selected, and that the corresponding u_i^k can be further reduced to offset the additional cost c_j.

3. A Feasible Solution And The Variance

A feasible solution to the problem may be represented by a vector $s = (x_{ij}, y_j, i = 1, ..., m, \text{ and } j = 1, ..., n)$. However, those for which (2) does not hold need not be considered. Thus we may define the set of all feasible solutions as

$$S = \text{The space of all } s = (x_{ij}, y_j, i = 1, ..., m, \text{and } j = 1, ..., n) \tag{10}$$

which satisfy (1) and (2). The problem is similar to those of optimum location selections ([1], [2], [5]). For any $k = 1, ..., n$, and any subset J_k of k elements out of set $\{1, 2, ..., n\}$, define

$$S(k) = \left\{ All\ s \in S : \sum_{j=1}^{n} y_j = k \right\} \text{and} \tag{11}$$

$$S(J_k) = \{All \quad s \quad \epsilon \quad S : y_j = 1\ if\ and\ only\ if\ j\ \epsilon\ J_k\}.$$

In other words, $S(k)$ represents all the possible ways of procuring the items from exactly k sources, and $S(J_k)$, from k given sources. The numbers of feasible solutions in S, $S(k)$, and $S(J_k)$ are respectively n^m, N_k, and $A(k, m)$, which are given by [4] as

$$A(k,m) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^m, \quad \text{and } N_k = \binom{n}{k} A(k,m). \tag{12}$$

From (3) and (10), for any solution s in S, denote the corresponding total cost by

$$C(s) \;=\; \sum_{i=1}^{m} \quad \sum_{j=1}^{n} p_{ij}x_{ij}(s) \;+\; \sum_{j=1}^{n} c_j y_j(s)$$

The mean is given as

$$\mu_k = \frac{1}{\binom{n}{k}A(k,m)} \;\left\{ \sum_{J_k} \sum_{s\epsilon S(J_k)} \left[\sum_{i=1}^{m} \; \sum_{j=1}^{n} p_{ij}x_{ij}(s) \;+ \sum_{j=1}^{n} c_j y_j(s) \right] \right\} \qquad (13)$$

Then we can find

$$E([C(s)]^2) \;=\; \frac{1}{\binom{n}{k}A(k,m)} \left(\sum_{J_k} \sum_{s\epsilon S(J_k)} \left\{ \sum_{i} \sum_{j} \; p_{ij} \; x_{ij} \; (s) \;+\; \sum_{j} \; c_j \; y_j \; (s) \right\}^2 \right)$$

$$= \frac{1}{\binom{n}{k}A(k,m)} \; \left(\sum_{J_k} \; \sum_{s\epsilon S(J_k)} \; \left\{ \sum_{i} \sum_{j} \sum_{u} \sum_{v} \; p_{ij} \; x_{ij}(s) \; p_{uv} \; x_{uv} \; (s) \right.\right.$$

$$\left.\left. + \, 2 \sum_{i} \sum_{j} \sum_{u} \; p_{ij} \; x_{ij}(s) \; c_u \; y_u \; (s) \;+\; \sum_{u} \sum_{v} \; c_u c_v y_u \; (s) y_v(s) \right\} \right) \qquad (14)$$

The first summation in (14) can be broken down as follows:

$$\sum_{J_k} \; \sum_{S(J_k)} \sum_{\substack{i \\ i \neq u}} \sum_{\substack{j \\ j \neq v}} \sum_{u} \sum_{v} p_{ij}x_{ij}(s)p_{uv}x_{uv}(s) \;=\; \sum_{J_k} \; \sum_{S(J_k)} \sum_{i} \sum_{j} p_{ij}^2 x_{ij}(s), \quad (15)$$

$$\sum_{J_k} \; \sum_{S(J_k)} \sum_{\substack{i \\ i \neq u}} \sum_{\substack{j \\ j \neq v}} \sum_{u} \sum_{v} p_{ij}x_{ij}(s)p_{uv}x_{uv}(s) \;=\; 0, \qquad (16)$$

$$\sum_{J_k} \; \sum_{S(J_k)} \sum_{\substack{i \\ i \neq u}} \sum_{\substack{j \\ j \neq v}} \sum_{u} \sum_{v} p_{ij}x_{ij}(s)p_{uv}x_{uv}(s)$$

$$= \sum_{J_k} \; \sum_{S(J_k)} \; \sum_{i \neq u} \; \sum_{j} p_{ij}p_{uj}x_{ij}(s)x_{uj}(s), \qquad (17)$$

$$\sum_{J_k} \; \sum_{S(J_k)} \sum_{\substack{i \\ i \neq u}} \sum_{\substack{j \\ j \neq v}} \sum_{u} \sum_{v} p_{ij}x_{ij}(s)p_{uv}x_{uv}(s) \qquad (18)$$

Then

$$(15) \;=\; \binom{n-1}{k-1} \{A(k-1, m-1) \;+\; A(k, m-1)\} \; \sum_{i=1}^{m} \; \sum_{j=1}^{n} \; p_{ij}^2 \qquad (19)$$

For given i, j, and u in (17), the number of solutions s in $S(J_k)$ for which $x_{ij}(s)x_{uj}(s) \neq 0$ is:

$$A(k-1, m-2) + \sum_{i=1}^{m-k-1} \; \binom{m-2}{i} A(k-1, m-2-i) = A(k-1, m-2) + A(k, m-2)$$

Therefore

$$(17) \;=\sum_{J_k}\sum_{j\epsilon J_k}\{A(k-1,m-2)+A(k,m-2)\}\sum_{i\neq u}p_{ij}p_{uj}$$

$$=\;(\sum_{i\neq u}\sum_{j}\;p_{ij}p_{uj})\;\{A(k-1,m-2)\;+\;A(k,m-2)\}\;\binom{n-1}{k-1} \tag{20}$$

$$(18)\;=\;\sum_{J_k}\sum_{S(J_k)}\sum_{i\neq u}\sum_{j\neq v}p_{ij}\;p_{uv}x_{ij}(s)x_{uv}(s)$$

$$=\;\sum_{J_k}\sum_{S(J_k)}\sum_{i\neq u}\sum_{\substack{j\neq v\\ j,v\epsilon J_k}}p_{ij}\;p_{uv}\;x_{ij}(s)x_{uv}(s)$$

$$=\;\sum_{J_k}\sum_{\substack{j\neq v\\ j,v\epsilon J_k}}\sum_{S(J_k)}\sum_{i\neq u}p_{ij}p_{uv}x_{ij}\;(s)x_{uv}(s) \tag{21}$$

For given $i\neq u$ and $j\neq v$, the number of solutions s in $S(J_k)$ is: $A(k-2,m-2)+\sum_{\varphi=1}^{m-k}\binom{m-2}{\varphi}A(k-2,m-2-\varphi)+\sum_{\varphi=1}^{m-k}\binom{m-2}{\varphi}A(k-2,m-2-\varphi)$

$$+\sum_{\varphi+t=2}^{m-k}\binom{m-2}{\varphi}\binom{m-2-\varphi}{t}A(k-2,m-2-\varphi-t)$$

$$=\;A(k-2,m-2)\;+\;A(k-1,m-2)\;+A(k-1,m-2)+A(k,m-2). \tag{22}$$

From (21) and (22), we see that

$$(18)=\sum_{J_k}\sum_{\substack{j\neq v\\ j,v\epsilon J_k}}\sum_{\substack{i\neq u\\ i,u=1}}p_{ij}p_{uv}\{A(k-2,m-2)+2A(k-1,m-2)+A(k,m-2)\}$$

$$=\;\{A(k-2,m-2)+2A(k-1,m-2)+A(k,m-2)\}\;\binom{m-2}{k-2}\sum_{\substack{i,u=1\\ i\neq u}}\sum_{\substack{j,v=1\\ j\neq v}}p_{ij}p_{uv}$$

$$=\;\left\{\sum_{i,u=1}^{m}\sum_{j,v=1}^{n}\;p_{ij}p_{uv}\right\}\;\{1/k(k-1)\}\{A(k,m)-A(k,m-1)\} \tag{23}$$

Now

$$\sum_{i\neq u}\sum p_{ij}p_{uv}=\sum_{i=1}^{m}\sum_{j=1}^{n}p_{ij}(p_{\cdot j}-p_{ij})\;=\;\sum_{j}\;p^2{}_{\cdot j}-\sum_{i}\sum_{j}p_{ij}^2, \tag{24}$$

$$\sum_{i,u=1}^{m}\sum_{j,v=1}^{n}p_{ij}p_{uv}=\sum_{i=1}^{m}\sum_{j=1}^{n}p_{ij}(p_{\cdot\cdot}-p_{i\cdot}-p_{\cdot j}+p_{ij})\;=$$

$$p_{\cdot\cdot}^2+\sum_{i}p_{i\cdot}^2\;+\;\sum_{j}\;p_{\cdot j}^2+\sum_{i}\sum_{j}p_{ij}^2,\;\text{where}\;p_{\cdot\cdot}\;=\sum_{i}\sum_{j}. \tag{25}$$

Based on (14), (19) through (25), the first summation can be written as

$$\left\{\frac{1}{n-1}\binom{n}{k}A(k,m)-\frac{1}{n-1}\binom{n}{k}A(k,m-1)\right\}\sum_i\sum_j p_{ij}^2$$

$$+\left\{\frac{1}{n-2}\binom{n}{k}A(k,m-1)-\frac{1}{n(n-1)}\binom{n}{k}A(k,m)\right\}\sum_j p_{.j}^2$$

$$+\frac{1}{n(n-1)}\left[\{A(k,m-1)-A(k,m)\}\sum_j p_{i.}^2+\{A(k,m)-A(k,m-1)\}\right]p_{..}^2$$

Then we can obtain the variance σ_k^2 as

$$\sigma_k^2 = (1-r_{km})\sum_{i=1}^{m} V_i + (r_{km}-1/n)V_J + 2\,(1-k/n)\,V_{JF}\mid k\,(1-k/n)\,V_F, \quad (26)$$

where $p_i. = \sum_n^{j=1} p_{ij},\ p_{ij} = \sum_m^{i=1} p_{ij},\ \overline{p} = \sum_n^{j=1} p_{.j}/n,$

$$\overline{p}_{i.}\ p_{i.}/n, \overline{p} = \overline{p}_{.}/m, \overline{c} = \sum_n^{j=1} c_j/n,\ r_{km} = A(k,m-1)\,/\,A(k,m),$$

$$V_i = \sum_n^{j=1}(p_{ij}-\overline{p}_{i.})^2/(n-1),\ V_J = \sum_n^{j=1}(p_{.j}-\overline{p}_{.})^2/(n-1),$$

$$V_{JF} = \sum_n^{j=1}(c_j-\overline{c})\,(p_{.j}-\overline{p}_{.})/(n-1),\ V_F = \sum_n^{j=1}(c_j-\overline{c})^2/(n-1).$$

4. The Proposed Method

With the derivations in the previous sections and using the feasible solution and the variance, the iterations are implemented. At the first and second iterations, the procedure is the same as the approximation derived in the section 2. If ρ_2 in (7) is non-positive, then iterations terminate. Otherwise for $k = 3, 4, ..., n$, every $i \in I$, and $j \in J(k-1)$, define

$$w_{ij}^k = u_i^k, \text{ if } p_{ij} \neq u_i^k,\ w_{ij}^k = \min_{j'\in J(k-1)-j} p_{ij'}, \text{ if otherwise;}$$

$$\text{and } \tau_j(u^k) = -\sum_{i=1}^{m}\max(w_{ij}^k - p_{ij}, 0) + c_j \quad (27)$$

In other words, if location j is to be deleted, those items which are assigned to it at the $(k-1)th$ iteration must be reassigned to the next possible location which will be retained. Let

$$\lambda(u^k) = \begin{cases} \rho_j(u^k) & \text{if } j \notin J(k-1) \\ \tau_j(u^k) & \text{if } j \in J(k-1) \end{cases} \tag{28}$$

Find j_k such that

$$\lambda_{j_k}(u^k) = \lambda = \max_{j \in J} \lambda_j(u^k) \tag{29}$$

Then define

$$J_k = \begin{cases} J(k-1)Uj_k & \text{if } j \notin J(k-1) \\ J(k-1) - j_k & \text{if } j \in J(k-1). \end{cases} \tag{30}$$

The iteration procedure terminate either as soon as a $k = 3, 4, ...,$ is found for which $\rho_k \leq 0$, or $\rho_k > 0$ for $J(k) = J$. In the first case, the optimum solution is the same as (9). In the second case, $y_j = 1$ for all $j \in J$, and for every $i \in I$, $x_{ij} = 1$, if p_{ij} is minimized with respect to all the locations in J.

Using the above derivations for the proposed method, the optimum solution of the related problem can be found by direct enumeration. One of the optimum solution is the minimum cost obtained.

Due to the complexity of the problem, iterations by computer simulation are implemented. These iterations produce the minimum costs from the approximation in the section 2 and those from the proposed method in the section 4. With the minimum costs obtained, we can compare and judge performance of the methods.

5. Conclusions

We showed the derivation of a feasible solution and the variance of the cost function. Also we tested the performance of the proposed method derived in this paper by the simulation for 100 problems with a different random number generator. The p_{ij}'s were drawn from a uniform distributions in the (0,150) interval and a lower bound fixed at 20 and and an upper bound ranging from 30 to 100. Both the proposed method and the approximation are applied, and the optimum solutions were found by direct search. To 50 of those problems, the proposed method is also applied where the initial solutions are based on all locations. The results of the experiments are optimum 74% from the proposed method, while 57% from the approximation. Also the proposed method was tested on 50 problems using two different types of initial solutions. Single location and all possible locations. The results indicate a slightly better performance of the proposed method when the initial solutions are based on consideration of all locations. However, the tests were limited in scope and no general rules can be from those results.

References

1. Cornuejols, Gerard, Fisher, and L. Marshall, *Location of Bank Accounts of Optimize Float: An Analytic Study of Exact and Approximate Algorithms*, Manage Science, Vol.23 (1997), No.8. 789-810.
2. Cramer and Harold, *Mathematical Methods of Statistics* , Princeton University Press, Princeton, NJ, 1999.
3. D. Ryu and H. Kim, *Convergence of Linear Processes with The Negatively Dependent random Variables*, J. Appl. Math. & Informatics Vol.20 (2006), No.1-2, 603-610.
4. Feller and William, *An Introduction to Probability Theory and Its Applications*, Vol. I, Third Edition John Wiley and Sons, Inc., New York, NY 1991.
5. J. Johannes and R. Schenk, *Adaptive Estimation of Linear Functionals in Functional Linear Models*, Mathematical Methods of Statistics, Vol. 21 (2012), No.3, 189-214.
6. Kuehn, A. Alfred, Hamburger, and J. Michael, *A Heuristic Program for Locating Warehouse*, Management Science, Vol. 9 (1993), No.4, 643-666.

Jong Seul Lim received the BS degree from Seoul National University and the Ph.D degree in communications and operations engineering from Polytechnic University, New York, in 1988. He worked with AT&T Bell Laboratories and developed the computer network and cellular communication systems. After then, he joined SK Telecom Corp. Since 1993, he has worked with Sunmoon University, Korea as a professor.

Information Communication Engineering Department, Sunmoon University, Asan, Chungnam, Korea.
e-mail: jslsky7@hotmail.com