# ANALYSIS OF A FOURTH ORDER SCHEME AND APPLICATION OF LOCAL DEFECT CORRECTION METHOD 

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#### Abstract

This paper provides a new application similar to the Local Defect Correction (LDC) technique to solve Poisson problem $-u^{\prime \prime}(x)=$ $f(x)$ with Dirichlet boundary conditions. The exact solution is supposed to have high activity in some region of the domain. LDC is combined with a fourth order compact scheme which is recently developed in Abbas (Num. Meth. Partial differential equations, 2013). Numerical tests illustrate the interest of this application.


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## 1. Introduction

The Local Defect Correction (LDC) method was introduced by W. Hackbusch [8] for solving elliptic boundary value problem. LDC is a domain decomposition technique in which the local domain fully overlaps the global one. It is a generic iterative algorithm of multiscale type for the resolution of discrete problem. Two or more grids of calculation can be used. We refer to [6] for applications to problems in combustion and numerical simulations of the flow and heat transfer in a glass tank. An analysis of the LDC technique in combination with finite difference discretizations is presented in [4]. In [7] the method is extended to include adaptivity, multilevel refinement, domain decomposition and regredding. The LDC method is combined with finite volume discretizations in [5] and finite elements discretizations in $[1,2]$.
Let's briefly outline the basic version of the LDC technique. Consider the elliptic boundary value problem

$$
\left\{\begin{align*}
L u & =f, \text { in } \Omega,(a)  \tag{1}\\
u & =g, \text { on } \partial \Omega,(b)
\end{align*}\right.
$$

[^0]where $L$ is a linear elliptic differential operator, $f$ and $g$ are the source term and Dirichlet boundary condition, respectively. To discretize (1), we first choose a global coarse grid (grid spacing $H$ ), which we denote by $\Omega^{H}$. Let $L^{H}$ be a discrete operator approximating the continuous operator $L$. An initial approximation $u_{i}^{H}, i=0$, on $\Omega^{H}$ is obtained by solving the system
\[

$$
\begin{equation*}
L^{H} u_{i}^{H}=f^{H} . \tag{2}
\end{equation*}
$$

\]

In (2), the vector $f^{H}$ contains the source term $f$ and the Dirichlet boundary condition $g$ as well. Suppose that $L^{H}$ is invertible. Moreover, suppose that the exact solution $u(x, y)$ of (1) has a high activity region in some (small) part of the domain. We select a subdomain $\Omega_{l} \subset \Omega$ such that the high activity region of $u$ is contained in $\Omega_{l}$. The subdomain $\Omega_{l}$ is discretized by a local fine grid (grid spacing $h$ ). We denote it by $\Omega_{l}^{h}$. The fine grid $\Omega_{h}^{l}$ is built such that $\Omega_{l}^{H} \cap \Omega_{l} \subset \Omega_{l}^{h}$. This means that the coarse grid points that lie in the region of refinement are also points of $\Omega_{l}^{h}$. In order to formulate a discrete problem on $\Omega_{l}^{h}$, we should define artificial boundary conditions on $\Gamma$, the interface between $\Omega_{l}$ and $\Omega \backslash \Omega_{l}$, (Figure 1). Actually, we use an interpolation operator $P^{h, H}$ to obtain artificial boundary conditions on $\Gamma$. The operator $P^{h, H}$ gives the values of the fine grid points on $\Gamma^{h}$ by interpolating the values of the coarse grid points on the interface $\Gamma^{H}$.
On $\partial \Omega_{l} \backslash \Gamma$, we obtain boundary conditions using the Dirichlet boundary conditions $g$ in $(1)_{b}$. On the fine grid $\Omega_{l}^{h}$ we consider the following discrete problem,


Figure 1. Discretization in one dimension. The big points represent the coarse grid $\Omega^{H}$, the small points represent the fine domain $\Omega_{l}^{h}$.
at iteration $i=0$,

$$
\begin{equation*}
L_{l}^{h} u_{l, i}^{h}=f_{l}^{h}-\left.B_{l, \Gamma}^{h} u_{l, i}^{h}\right|_{\Gamma} \tag{3}
\end{equation*}
$$

where

$$
\begin{equation*}
\left.u_{l, i}^{h}\right|_{\Gamma}=\left.P^{h, H} u_{i}^{H}\right|_{\Gamma} . \tag{4}
\end{equation*}
$$

In (3), $i$ represents an index of iteration and $B_{l, \Gamma}^{h}$ is a square matrix representing the dependence of the fine grid solution on the artificial boundary conditions on $\Gamma$. In (3), the matrix $L_{l}^{h}$ (assumed to be invertible) is a discrete approximation of the operator $L$ on the subdomain $\Omega_{l}$. In the right term in (3), $f_{l}^{h}$ contains the
source term $f$ and the Dirichlet boundary condition $g$ on $\partial \Omega_{l} \backslash \Gamma$ given in (1). Notice that $\Gamma \cap \partial \Omega$ can be empty as in the figure 1. If we know the values of the defect $d^{H}=L^{H}\left(\left.u\right|_{\Omega^{H}}\right)-f^{H}$, we can use it to find more accurate approximation on the coarse grid. This can be obtained by replacing $d^{H}$ in the right hand side of (2). However, we do not know the exact solution of the exact continuous problem then we cannot calculate $d^{H}$. We will use $u_{l, 0}^{h}$ calculated on the fine grid to approximate $d^{H}$. Indeed, for all $i$, index of iteration, the function $w_{i}^{H}, i=0$, is the function on the coarse grid defined by:

$$
w_{i}^{H}(x, y)= \begin{cases}u_{l, i}^{h}(x, y), & (x, y) \in \Omega_{l}^{H},  \tag{5}\\ u_{i}^{H}(x, y), & (x, y) \in \Omega^{H} \backslash \Omega_{l}^{H}\end{cases}
$$

For each index of iteration $i$ we define $d_{i}^{H}$ by

$$
\begin{equation*}
d_{i}^{H}=L^{H} w_{i}^{H}-f^{H} \tag{6}
\end{equation*}
$$

Actually, $d_{i}^{H}$ provides an estimate of the local discrestization error at all points


Figure 2. The small points consist of points of $\Omega^{h}$, the big ones are the points of the safety region $\Omega_{\epsilon}^{H}$.
of $\Omega_{l}^{H}$. We observed numerically it is better to use the approximation (6) on a proper subset of $\Omega_{l}^{H}$ only. In particular, the points of the coarse grid near $\partial \Omega_{l}$ should be excluded. This leads to introduce a "Safety region" denoted by $\Omega_{\epsilon}^{H}$ (Figure 2). The estimation of the local error of discretization of the coarse grid is placed in the right hand side of the equations corresponding to coarse grid points belonging to $\Omega_{\epsilon}^{H}$ only. Then we apply the correction step on the coarse $\operatorname{grid}$ to find $u_{i}^{H}, i=1$,

$$
L^{H} u_{i}^{H}= \begin{cases}f^{H}(x, y)+d_{i-1}^{H}(x, y), & (x, y) \in \Omega_{\epsilon}^{H}  \tag{7}\\ f^{H}(x, y), & (x, y) \in \Omega^{H} \backslash \Omega_{\epsilon}^{H}\end{cases}
$$

As (7) contains estimations of the local error of the discretization on the coarse grid, we expect $u_{i+1}^{H}$ to be a more accurate approximation than $u_{i}^{H}$. Then we obtain better boundary conditions on $\Gamma$ and a solution on the fine local grid by (3) with $i=1$. By performing the iterations on the index $i$, we obtain the following algorithm:

## Algorithm 1.1. - Initialization

-Solve the problem(2) on the coarse grid $\Omega^{H}$ with $i=0$. We obtain the vector $u_{0}^{H}$.
-Solve the problem(3) on the fine local grid $\Omega_{l}^{h}$ with $i=0$. We obtain the vector $u_{l, 0}^{h}$.

- Iteration $i=1,2 \ldots$
- Compute $W_{i-1}^{H}$ by (5).
- Compute $d_{i-1}^{H}$ by (6).
- Re-solve the problem (7) on the coarse grid $\Omega^{H}$. We obtain the vector $u_{i}^{H}$.
- Solve the problem (3) on the local fine grid $\Omega_{l}^{h}$. We obtain the vector $u_{l, i}^{h}$.

In practice, one iteration is enough to obtain a satisafactory approximation on the composite grid $\Omega^{H, h}$ which is the grid formed by the union of the coarse grid and the fine grid, $\Omega^{H, h}=\Omega^{H} \cup \Omega_{l}^{h}$. The algorithm 1.1 is an elementary version of the LDC method. Several generalizations are possible:

- Use of several fine grids. Notice that the local problems are independent each other and can be solved simultaneously. Recursive refinement where a local fine grid can be a coarse grid for another local fine grid.
- Use of different discretizations. We refer to [6] for a detailed analysis of the behavior of the convergence for the Poisson problem solved with classic five points finite difference scheme.


## 2. Main results

2.1. Analysis of a fourth order scheme. We recall and analyse a new scheme called hermitian Box-scheme (HB-scheme), for the Poisson problem in one dimension based on the previous work [12]. This scheme appears to be fourth order accurate in practice. It has been successfully extended to two dimensions in [9] and three dimensions in [10]. Let's recall the principle of this scheme. Consider the one-dimensional Poisson problem on the interval $\Omega=(a, b)$ with length $L=b-a$,

$$
\left\{\begin{array}{l}
-u^{\prime \prime}(x)=f(x), \quad a<x<b  \tag{8}\\
u(a)=g_{a}, u(b)=g_{b}
\end{array}\right.
$$

Equation (8) is recast in mixed form:

$$
\left\{\begin{array}{l}
v^{\prime}(x)+f(x)=0,  \tag{9}\\
v(x)-u^{\prime}(x)=0, \\
u(a)=g_{a}, u(b)=g_{b}
\end{array}\right.
$$

We lay out on $\Omega$ a regular grid $x_{j}=a+j h, 0 \leq j \leq N$ with stepsize $h=L / N$, (Figure 3). The unknowns are denoted by $u_{j} \approx u\left(x_{j}\right)$ and $u_{x, j} \approx u^{\prime}\left(x_{j}\right)$. The vectors $U, U_{x} \in \mathbb{R}^{N-1}$ stand for the unknowns at internal points,

$$
\begin{equation*}
U=\left[u_{1}, u_{2}, \ldots, u_{N-1}\right]^{T} ; \quad U_{x}=\left[u_{x, 1}, \ldots, u_{x, N-1}\right]^{T} \tag{10}
\end{equation*}
$$

In analogy to the original box-scheme [13], the HB-scheme is deduced from the integration of the two equations $(9)_{a, b}$ on a box $\left.K_{j}=\right] x_{j-1}, x_{j+1}$ [ of length $2 h$.


Figure 3. Grid in dimension 1. The two boundary points $x_{0}, x_{N}$ (denoted by "०"). The points $x_{i}, i=1, \cdots, N-1$ are the interior points (denoted by " $\bullet$ ").

Suppose given the averaged values of the source term $f(x)$ on the box $K_{j}$,

$$
\begin{equation*}
\Pi^{0} f_{j}=\frac{1}{2 h} \int_{K_{j}} f(x) d x, \quad 1 \leq j \leq N-1 \tag{11}
\end{equation*}
$$

In practice, (11) can be approximated using Simpson formula

$$
\begin{equation*}
\Pi^{0} f_{j} \approx \frac{1}{6} f_{j-1}+\frac{2}{3} f_{j}+\frac{1}{6} f_{j+1}, 1 \leq j \leq N-1 \tag{12}
\end{equation*}
$$

The conservation equation $(9)_{a}$ becomes

$$
\begin{equation*}
-\frac{u_{x, j+1}-u_{x, j-1}}{2 h}=\Pi^{0} f_{j}, \quad 1 \leq j \leq N-1 \tag{13}
\end{equation*}
$$

Second, the equation $(9)_{b}$ is integrated on the box $K_{j}$. This yields

$$
\begin{equation*}
\frac{1}{2 h} \int_{x_{j-1}}^{x_{j+1}} v(x) d x=\frac{u\left(x_{j+1}\right)-u\left(x_{j-1}\right)}{2 h}, \quad 1 \leq j \leq N-1 \tag{14}
\end{equation*}
$$

Approximating the integral in the left-hand side of (14) by Simpson formula suggests the following fourth-order hermitian approximation

$$
\begin{equation*}
\frac{1}{6} u_{x, j-1}+\frac{2}{3} u_{x, j}+\frac{1}{6} u_{x, j+1}=\frac{u_{j+1}-u_{j-1}}{2 h}, \quad 1 \leq j \leq N-1 \tag{15}
\end{equation*}
$$

Equations (15) require approximations of the derivatives on the boundaries. Here, we consider the following third-order approximations

$$
\left\{\begin{array}{l}
\frac{1}{3} u_{x, 0}+\frac{2}{3} u_{x, 1}=\frac{1}{h}\left(\frac{1}{6} u_{2}+\frac{2}{3} u_{1}-\frac{5}{6} u_{0}\right)  \tag{16}\\
\frac{1}{3} u_{x, N}+\frac{2}{3} u_{x, N-1}=\frac{1}{h}\left(\frac{5}{6} u_{N}-\frac{2}{3} u_{N-1}-\frac{1}{6} u_{N-2}\right)
\end{array}\right.
$$

Relations (16) are obtained using Taylor expansions. In summary, the equations (13), (15), (16) with Dirichlet boundary conditions translate to the following

HB-scheme: Find $u=\left(u_{i}\right)_{0 \leq i \leq N}$ and $v=\left(v_{i}\right)_{0 \leq i \leq N}$ solution of:

$$
\left\{\begin{array}{l}
-\frac{u_{x, j+1}-u_{x, j-1}}{2 h}=\Pi^{0} f_{j}, \quad 1 \leq j \leq N-1  \tag{17}\\
\frac{1}{3} u_{x, 0}+\frac{2}{3} u_{x, 1}=\frac{1}{h}\left(\frac{1}{6} u_{2}+\frac{2}{3} u_{1}-\frac{5}{6} u_{0}\right), \\
\frac{1}{6} u_{x, j-1}+\frac{2}{3} u_{x, j}+\frac{1}{6} u_{x, j+1}=\frac{u_{j+1}-u_{j-1}}{2 h}, 1 \leq j \leq N-1, \\
\frac{1}{3} u_{x, N}+\frac{2}{3} u_{x, N-1}=\frac{1}{h}\left(\frac{5}{6} u_{N}-\frac{2}{3} u_{N-1}-\frac{1}{6} u_{N-2}\right), \\
u_{0}=g_{a}, \quad u_{N}=g_{b}
\end{array}\right.
$$

This HB-scheme is found numerically to be fourth order accurate as it is shown in the numerical tables.

Lemma 2.1. Suppose that $\Pi^{0} f_{j}$ is approximated by Simpson formula:

$$
\begin{equation*}
\Pi^{0} f_{j} \approx \sigma_{x} f_{j}=\frac{1}{6} f_{j-1}+\frac{2}{3} f_{j}+\frac{1}{6} f_{j+1}, 1 \leq j \leq N-1 \tag{18}
\end{equation*}
$$

and $\left(u_{j}\right)_{j \in \mathbb{Z}},\left(u_{x, j}\right)_{j \in \mathbb{Z}}$ are two sequences verifying

$$
\begin{cases}-\frac{u_{x, j+1}-u_{x, j-1}}{2 h}=\sigma_{x} f_{j}, & j \in \mathbb{Z}  \tag{19}\\ \sigma_{x} u_{x, j}=\frac{u_{j+1}-u_{j-1}}{2 h}, & j \in \mathbb{Z}\end{cases}
$$

Then $\left(u_{j}\right)_{j \in \mathbb{Z}}$ verifies

$$
\begin{equation*}
-\frac{u_{j+2}+u_{j-2}-2 u_{j}}{4 h^{2}}=\frac{1}{36} f_{j-2}+\frac{2}{9} f_{j-1}+\frac{1}{2} f_{j}+\frac{2}{9} f_{j+1}+\frac{1}{36} f_{j+2} . \tag{20}
\end{equation*}
$$

Proof. Denote by $E_{j}, j \in \mathbb{Z}$, the $j$-th equation of $(19)_{a}$. Using the sum

$$
\begin{equation*}
\frac{1}{6} E_{j-1}+\frac{2}{3} E_{j}+\frac{1}{6} E_{j+1}=\frac{1}{6} \sigma_{x} f_{j-1}+\frac{2}{3} \sigma_{x} f_{j}+\frac{1}{6} \sigma_{x} f_{j+1} \tag{21}
\end{equation*}
$$

and $(19)_{b}$, we find that the left term of (21) verifies

$$
\left\{\begin{align*}
\frac{1}{6} E_{j-1}+\frac{2}{3} E_{j}+\frac{1}{6} E_{j+1} & =-\frac{1}{6} \delta_{x} u_{x, j-1}-\frac{2}{3} \delta_{x} u_{x, j}-\frac{1}{6} \delta_{x} u_{x, j+1}  \tag{22}\\
& =-\frac{u_{j+2}+u_{j-2}-2 u_{j}}{4 h^{2}}
\end{align*}\right.
$$

The right term of (21) verifies

$$
\left\{\begin{aligned}
\frac{1}{6} \sigma_{x} f_{j-1}+\frac{2}{3} \sigma_{x} f_{j}+\frac{1}{6} \sigma_{x} f_{j+1}= & \frac{1}{6}\left(\frac{1}{6} f_{j-2}+\frac{2}{3} f_{j-1}+\frac{1}{6} f_{j}\right) \\
& +\frac{2}{3}\left(\frac{1}{6} f_{j-1}+\frac{2}{3} f_{j}+\frac{1}{6} f_{j+1}\right) \\
& +\frac{1}{6}\left(\frac{1}{6} f_{j}+\frac{2}{3} f_{j+1}+\frac{1}{6} f_{j+2}\right) \\
= & \frac{1}{36} f_{j-2}+\frac{2}{9} f_{j-1}+\frac{1}{2} f_{j}+\frac{2}{9} f_{j+1}+\frac{1}{36} f_{j+2}
\end{aligned}\right.
$$

where the equality (20) for $j \in \mathbb{Z}$.
We refer to [11] for a detailed proof of the following lemma concerning the maximum principles of the standard three point Laplacian scheme.
Lemma 2.2. Let $u=\left(u_{i}\right)_{0 \leq i \leq N}$. Suppose that $-\delta_{x}^{2} u_{i} \leq 0,1 \leq i \leq N-1$, then

$$
\begin{equation*}
\max _{1 \leq i \leq N-1} u_{i} \leq \max \left(u_{0}, u_{N}\right) \quad \text { (Maximum principle) } \tag{23}
\end{equation*}
$$

Similarly, if $-\delta_{x}^{2} u_{i} \geq 0,1 \leq i \leq N-1$, then

$$
\begin{equation*}
\min _{1 \leq i \leq N-1} u_{i} \geq \min \left(u_{0}, u_{N}\right) \quad \text { (Minimum principle). } \tag{24}
\end{equation*}
$$

The scheme (17) verifies the following discrete maximum and minimum principles.
Proposition 2.1 (Principles of maximum and minimum). Let $u=\left(u_{i}\right)_{0 \leq i \leq N}$ be the solution of $H B$-scheme (17). Suppose that $f_{i} \leq 0, \forall 1 \leq i \leq N-1$, then

$$
\begin{equation*}
\max _{1 \leq i \leq N-1} u_{i} \leq \max \left\{u_{0}, u_{N}\right\} \quad \text { (Maximum principle). } \tag{25}
\end{equation*}
$$

Similarly, if $f_{i} \geq 0, \forall 1 \leq i \leq N-1$, $u$ verifies

$$
\begin{equation*}
\min _{1 \leq i \leq N-1} u_{i} \geq \min \left\{u_{0}, u_{N}\right\} \quad \text { (Minimum principle). } \tag{26}
\end{equation*}
$$

This means that $u$ attains its maxima and minima on the boundary points $x_{0}$ and $x_{N}$.
Proof. Suppose that $f_{i} \leq 0, \forall 1 \leq i \leq N-1$ then

$$
\begin{equation*}
\Pi^{0} f_{i} \leq 0, \quad 1 \leq i \leq N-1 \tag{27}
\end{equation*}
$$

By (20),

$$
\begin{equation*}
-\frac{u_{i+2}+u_{i-2}-2 u_{i}}{4 h^{2}} \leq 0, \quad \forall 2 \leq i \leq N-2 . \tag{28}
\end{equation*}
$$

If $N$ is even, using the maximum principle of Lemma 2.2 we obtain

$$
\left\{\begin{array}{l}
\max _{1 \leq k \leq \frac{N-2}{2}} u_{2 k} \leq \max \left\{u_{0}, u_{N}\right\},  \tag{29}\\
\max _{1 \leq k \leq \frac{N-2}{2}} u_{2 k+1} \leq \max \left\{u_{1}, u_{N-1}\right\} .
\end{array}\right.
$$

In the same manner if $N$ is odd, we obtain

$$
\left\{\begin{array}{l}
\max _{1 \leq k \leq \frac{N-1}{2}} u_{2 k} \leq \max \left\{u_{0}, u_{N-1}\right\}  \tag{30}\\
\max _{1 \leq k \leq \frac{N-1}{2}} u_{2 k+1} \leq \max \left\{u_{1}, u_{N}\right\}
\end{array}\right.
$$

It results for all $N$,

$$
\begin{equation*}
\max _{2 \leq i \leq N-2} u_{i} \leq \max \left\{u_{0}, u_{1}, u_{N-1}, u_{N}\right\} \tag{31}
\end{equation*}
$$

Then to prove (25) it is enough to prove that

$$
\begin{equation*}
\max \left\{u_{1}, u_{N-1}\right\} \leq \max \left\{u_{0}, u_{N}\right\} \tag{32}
\end{equation*}
$$

Suppose that

$$
\begin{equation*}
\max \left\{u_{1}, u_{N-1}\right\}=u_{1} \tag{33}
\end{equation*}
$$

We have

$$
\begin{equation*}
-\delta_{x} u_{x, j}=\Pi^{0} f_{j} \leq 0, \quad \forall 1 \leq j \leq N-1 \tag{34}
\end{equation*}
$$

This gives $u_{x, 0} \leq u_{x, 2}$ and $u_{x, 1} \leq u_{x, 3}$. The HB-scheme (17) verifies

$$
\left\{\begin{array}{l}
\frac{1}{3} u_{x, 0}+\frac{2}{3} u_{x, 1}=\frac{2}{3} \frac{u_{1}-u_{0}}{h}+\frac{1}{3} \frac{u_{2}-u_{0}}{2 h}  \tag{35}\\
\frac{1}{6} u_{x, 0}+\frac{2}{3} u_{x, 1}+\frac{1}{6} u_{x, 2}=\frac{u_{2}-u_{0}}{2 h}
\end{array}\right.
$$

Moreover,

$$
\frac{1}{3} u_{x, 0}+\frac{2}{3} u_{x, 1} \leq \frac{1}{6} u_{x, 0}+\frac{2}{3} u_{x, 1}+\frac{1}{6} u_{x, 2} \leq \frac{u_{2}-u_{0}}{2 h}
$$

Therefore,

$$
\frac{2}{3} \frac{u_{1}-u_{0}}{h}+\frac{1}{3} \frac{u_{2}-u_{0}}{2 h} \leq \frac{u_{2}-u_{0}}{2 h}
$$

But we have $u_{2} \leq u_{0}$, then

$$
\begin{equation*}
\frac{2}{3} \frac{u_{1}-u_{0}}{h} \leq \frac{2}{3} \frac{u_{2}-u_{0}}{2 h} \leq 0 \tag{36}
\end{equation*}
$$

therefore

$$
\begin{equation*}
u_{1} \leq u_{0} \tag{37}
\end{equation*}
$$

where (32). Suppose that

$$
\begin{equation*}
\max \left\{u_{1}, u_{N-1}\right\}=u_{N-1} \tag{38}
\end{equation*}
$$

The equations (17) give

$$
\left\{\begin{array}{l}
\frac{1}{3} u_{x, N}+\frac{2}{3} u_{x, N-1}=\frac{2}{3} \frac{u_{N}-u_{N-1}}{h}+\frac{1}{3} \frac{u_{N}-u_{N-2}}{2 h},  \tag{39}\\
\frac{1}{6} u_{x, N-2}+\frac{2}{3} u_{x, N-1}+\frac{1}{6} u_{x, N}=\frac{u_{N}-u_{N-2}}{2 h}
\end{array}\right.
$$

In addition, using (34), $u_{x, N-2} \leq u_{x, N}$, therefore

$$
\begin{equation*}
\frac{1}{3} u_{x, N}+\frac{2}{3} u_{x, N-1} \geq \frac{1}{6} u_{x, N-2}+\frac{2}{3} u_{x, N-1}+\frac{1}{6} u_{x, N}=\frac{u_{N}-u_{N-2}}{2 h}, \tag{40}
\end{equation*}
$$

as $u_{N-1} \geq u_{N-3}$, and $u_{N} \geq u_{N-2}$, we get

$$
\begin{equation*}
\frac{2}{3} \frac{u_{N}-u_{N-1}}{h} \geq \frac{2}{3} \frac{u_{N}-u_{N-2}}{2 h} \geq 0 \tag{41}
\end{equation*}
$$

We deduce

$$
\begin{equation*}
u_{N-1} \leq u_{N} \tag{42}
\end{equation*}
$$

where

$$
\begin{equation*}
\max \left\{u_{1}, u_{N-1}\right\} \leq \max \left\{u_{0}, u_{N}\right\} \tag{43}
\end{equation*}
$$

Finally,

$$
\begin{equation*}
\max _{1 \leq i \leq N-1} u_{i} \leq \max \left\{u_{0}, u_{N}\right\} \tag{44}
\end{equation*}
$$

where (25). In the same manner we can prove the minimum principle (26).
Corollary 2.1 (existence and uniqueness of solution). The HB-scheme (17) has unique solution $(u, v) \in l_{h, 0}^{2} \times l_{h}^{2}$.

Proof. We have (17) a linear system with $2 N$ equations and $2 N$ unknowns. To prove the existence and uniqueness of the solution, it is enough to prove that $\Pi^{0} f_{i}=0,1 \leq i \leq N-1$, implies $u=v=0$. Let $u=\left(u_{i}\right)_{0 \leq i \leq N}$ and $v=\left(v_{i}\right)_{0 \leq i \leq N}$ such that

$$
\begin{cases}-\delta_{x} v_{j}=0, & 1 \leq j \leq N-1,  \tag{45}\\ \frac{1}{3} v_{N}+\frac{1}{3} v_{N-1}=\frac{2}{3} \frac{u_{N}-u_{N-1}}{h}+\frac{1}{3} \frac{u_{N}-u_{N-2}}{2 h}, & j=N, \\ \frac{1}{6} v_{j-1}+\frac{2}{3} v_{j}+\frac{1}{6} v_{j+1}=\frac{u_{j+1}-u_{j-1}}{2 h}, & 1 \leq j \leq N-1, \\ \frac{1}{3} v_{0}+\frac{2}{3} v_{1}=\frac{2}{3} \frac{u_{1}-u_{0}}{h}+\frac{1}{3} \frac{u_{2}-u_{0}}{2 h}, & j=0, \\ u_{0}=0, u_{N}=0\end{cases}
$$

Using the discrete maximum principle of proposition 2.1, we obtain

$$
\begin{equation*}
\max _{1 \leq i \leq N-1} u_{i} \leq \max \left\{u_{0}, u_{N}\right\} \tag{46}
\end{equation*}
$$

then

$$
\begin{equation*}
u_{i}=0, \quad \forall 1 \leq i \leq N-1 \tag{47}
\end{equation*}
$$

which proves that the matrix associated to the HB-scheme (17) has nul kernel. As it is square matrix then it is invertible, where the existence and uniqueness of the solution of HB-scheme.
2.2. Matrix form of hermitian Box-scheme. We summarize the finitedifference and matrix notations used in the sequel.

- The tridiagonal matrix $T \in \mathbb{M}_{N-1}(\mathbb{R})$ is

$$
T=\left[\begin{array}{ccccc}
2 & -1 & 0 & \ldots & 0  \tag{48}\\
-1 & 2 & -1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ldots & \vdots \\
0 & \ldots & -1 & 2 & -1 \\
0 & \ldots & 0 & -1 & 2
\end{array}\right]
$$

- The Simpson matrix $P_{s} \in \mathbb{M}_{N-1}(\mathbb{R})$ is

$$
\begin{equation*}
P_{s}=I-T / 6, \tag{49}
\end{equation*}
$$

where $I$ is the identity matrix of order $N-1$.

- The antisymmetric matrix $K \in \mathbb{M}_{N-1}(\mathbb{R})$ given by

$$
K=\left[\begin{array}{ccccc}
0 & 1 & 0 & \ldots & 0  \tag{50}\\
-1 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ldots & \vdots \\
0 & \ldots & -1 & 0 & 1 \\
0 & \ldots & 0 & -1 & 0
\end{array}\right]
$$

- Denoting $\left(e_{i}\right)_{1 \leq i \leq N-1}$ the canonical basis of $\mathbb{R}^{N-1}$, the matrices $F_{1}, F_{2} \in$ $\mathbb{M}_{N-1}(\mathbb{R})$ are defined by

$$
\left\{\begin{array}{l}
F_{1}=e_{1} e_{1}^{T}+e_{N-1} e_{N-1}^{T}  \tag{51}\\
F_{2}=-e_{1} e_{1}^{T}+e_{N-1} e_{N-1}^{T}
\end{array}\right.
$$

Let us turn now to the matrix form of the scheme. We use the notation $U_{L}=$ $u_{0}, U_{R}=u_{N}, U_{x, L}=u_{x, 0}, U_{x, R}=u_{x, N}$ for the boundary values.

Proposition 2.2. For all linear approximation of the derivatives on the boundary $U_{x, L}, U_{x, R}$ in terms of $U, U_{x}$ and the values on the boundary $U_{L}, U_{R}$, there exist matrices $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{M}_{N-1}(\mathbb{R})$ such that

$$
\begin{equation*}
e_{1} U_{x, L}+e_{N-1} U_{x, R}=\frac{1}{h} \mathcal{A} U-\mathcal{B} U_{x}+\frac{1}{h} \mathcal{C}\left(e_{1} U_{L}+e_{N-1} U_{R}\right) \tag{52}
\end{equation*}
$$

Proof. Any linear approximation of $u_{x, 0}$ in terms of $u=\left(u_{i}\right)_{0 \leq i \leq N}, \quad u_{x}=$ $\left(u_{x, i}\right)_{0 \leq i \leq N}$ and $u_{0}, u_{N}$ can be written in general form as:

$$
\begin{equation*}
u_{x, 0}=\sum_{i=1}^{i=N-1} \alpha_{i}^{0} u_{i}+\sum_{j=1}^{N-1} \beta_{j}^{0} u_{x, j}+\gamma_{0}^{0} u_{0}+\gamma_{N}^{0} u_{N} \tag{53}
\end{equation*}
$$

with $\alpha_{i}^{0}, \beta_{j}^{0}, \gamma_{0}^{0}, \gamma_{N}^{0} \in \mathbb{R}$.
Similarly,

$$
\begin{equation*}
u_{x, N}=\sum_{i=1}^{N-1} \alpha_{i}^{N} u_{i}+\sum_{j=1}^{N-1} \beta_{j}^{N} u_{x, j}+\gamma_{0}^{N} u_{0}+\gamma_{N}^{N} u_{N} \tag{54}
\end{equation*}
$$

with $\alpha_{i}^{N}, \beta_{j}^{N}, \gamma_{0}^{N}, \gamma_{N}^{N} \in \mathbb{R}$. Multiplying (53) by $e_{1}$ and (54) by $e_{N-1}$ we get

$$
\left\{\begin{array}{l}
e_{1} u_{x, 0}=\sum_{i=1}^{N-1} \alpha_{i}^{0} e_{1} u_{i}+\sum_{j=1}^{N-1} \beta_{j}^{0} e_{1} u_{x, j}+\gamma_{0}^{0} e_{1} u_{0}+\gamma_{N}^{0} e_{1} u_{N}  \tag{55}\\
e_{N-1} u_{x, N}=\sum_{i=1}^{N-1} \alpha_{i}^{N} e_{N-1} u_{i}+\sum_{j=1}^{N-1} \beta_{j}^{N} e_{N-1} u_{x, j}+\gamma_{0}^{N} e_{N-1} u_{0}+\gamma_{N}^{N} e_{N-1} u_{N}
\end{array}\right.
$$

Adding the equations (55) together

$$
\left\{\begin{align*}
e_{1} U_{x, L}+e_{N-1} U_{x, R}= & \sum_{i=1}^{N-1}\left(\alpha_{i}^{0} e_{1}+\alpha_{i}^{N} e_{N-1}\right) U_{i}+\sum_{j=1}^{N-1}\left(\beta_{j}^{0} e_{1}+\beta_{j}^{N} e_{N-1}\right) U_{x, j}  \tag{56}\\
& +\left(\gamma_{0}^{0} e_{1}+\gamma_{0}^{N} e_{N-1}\right) U_{L}+\left(\gamma_{N}^{0} e_{1}+\gamma_{N}^{N} e_{N-1}\right) U_{R} .
\end{align*}\right.
$$

Using the fact that $U_{i}=e_{i}^{T} U, U_{x, j}=e_{j}^{T} U_{x}, U_{L}=e_{1}^{T} e_{1} U_{L}$, and $U_{R}=$ $e_{N-1}^{T} e_{N-1} U_{R}$, we can write (56) as

$$
\left\{\begin{align*}
e_{1} U_{x, L}+e_{N-1} U_{x, R}= & \sum_{i=1}^{N-1}\left(\alpha_{i}^{0} e_{1} e_{i}^{T}+\alpha_{i}^{N} e_{N-1} e_{i}^{T}\right) U  \tag{57}\\
& +\sum_{j=1}^{N-1}\left(\beta_{j}^{0} e_{1} e_{j}^{T}+\beta_{j}^{N} e_{N-1} e_{j}^{T}\right) U_{x} \\
& +\left(\gamma_{0}^{0} e_{1} e_{1}^{T}+\gamma_{0}^{N} e_{N-1} e_{1}^{T}\right) e_{1} U_{L} \\
& +\left(\gamma_{N}^{0} e_{1} e_{N-1}^{T}+\gamma_{N}^{N} e_{N-1} e_{N-1}^{T}\right) e_{N-1} U_{R}
\end{align*}\right.
$$

Finally to complete the proof, let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in M_{N-1}(\mathbb{R})$ be the matrices

$$
\left\{\begin{array}{l}
\mathcal{A}=h \sum_{i=1}^{N-1}\left(\alpha_{i}^{0} e_{1} e_{i}^{T}+\alpha_{i}^{N} e_{N-1} e_{i}^{T}\right),  \tag{58}\\
\mathcal{B}=\sum_{j=1}^{N-1}\left(\beta_{j}^{0} e_{1} e_{j}^{T}+\beta_{j}^{N} e_{N-1} e_{j}^{T}\right), \\
\mathcal{C}=h\left(\gamma_{0}^{0} e_{1} e_{1}^{T}+\gamma_{0}^{N} e_{N-1} e_{1}^{T}+\gamma_{N}^{0} e_{1} e_{N-1}^{T}+\gamma_{N}^{N} e_{N-1} e_{N-1}^{T}\right)
\end{array}\right.
$$

We claim that (16) translates into matrix form as [9]

$$
\begin{equation*}
e_{1} U_{x, L}+e_{N-1} U_{x, R}=\frac{1}{h}\left(\mathcal{A} U-h \mathcal{B} U_{x}+\mathcal{C}\left(e_{1} U_{L}+e_{N-1} U_{R}\right)\right) \tag{59}
\end{equation*}
$$

where the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are given by

$$
\left\{\begin{array}{l}
\mathcal{A}=-2 F_{2}+\frac{1}{2}\left(e_{1} e_{2}^{T}-e_{N-1} e_{N-2}^{T}\right),  \tag{60}\\
\mathcal{B}=2 F_{1}, \\
\mathcal{C}=\frac{5}{2} F_{2},
\end{array}\right.
$$

and the vector of derivative is given in terms of $U$ and the boundary conditions by

$$
\begin{equation*}
U_{x}=\frac{1}{h} \mathcal{D} U+\frac{1}{h} \mathcal{E}\left(e_{1} U_{L}+e_{N-1} U_{R}\right), \tag{61}
\end{equation*}
$$

where the matrices $\mathcal{D}$ and $\mathcal{E}$ are

$$
\left\{\begin{array}{l}
\mathcal{D}=\frac{1}{2}\left(P_{s}-\frac{1}{6} \mathcal{B}\right)^{-1}\left(K-\frac{1}{3} \mathcal{A}\right)  \tag{62}\\
\mathcal{E}=\frac{1}{2}\left(P_{s}-\frac{1}{6} \mathcal{B}\right)^{-1}\left(F_{2}-\frac{1}{3} \mathcal{C}\right)
\end{array}\right.
$$

Using (59) and (61), $U_{x}$ can be eliminated which gives the expression of the HB-scheme in the sole unknown $U$ as

$$
\begin{equation*}
\frac{1}{h^{2}} \mathcal{H} U=F-\frac{1}{h^{2}} \mathcal{G}\left(e_{1} U_{L}+e_{N-1} U_{R}\right) \tag{63}
\end{equation*}
$$

where $F=\left[\Pi^{0} f_{1}, \cdots, \Pi^{0} f_{N-1}\right]^{T}\left(\Pi^{0} f_{j}\right.$ is given in (11)). The matrices $\mathcal{H}, \mathcal{G} \in$ $\mathbb{M}_{N-1}(\mathbb{R})$ are

$$
\left\{\begin{array}{l}
\mathcal{H}=-\frac{1}{4}\left(K-F_{2} \mathcal{B}\right)\left(P_{s}-\frac{1}{6} \mathcal{B}\right)^{-1}\left(K-\frac{1}{3} \mathcal{A}\right)-\frac{1}{2} F_{2} \mathcal{A}  \tag{64}\\
\mathcal{G}=-\frac{1}{4}\left(K-F_{2} \mathcal{B}\right)\left(P_{s}-\frac{1}{6} \mathcal{B}\right)^{-1}\left(F_{2}-\frac{1}{3} \mathcal{C}\right)-\frac{1}{2} F_{2} \mathcal{C}
\end{array}\right.
$$

If needed, the gradient is recovered as a postprocessing by (61).
2.3. Two-grid refinement algorithm. Here, a multiscale technique is combined with the hermitian Box-scheme. We consider the problem

$$
\left\{\begin{array}{l}
-u^{\prime \prime}(x)=f(x), \quad x \in \Omega=(a, b)  \tag{65}\\
u(a)=g_{a}, u(b)=g_{b}
\end{array}\right.
$$

The domain $\Omega$ is discretized with stepsize $H=1 / N$ such that the discrete coarse points are $x_{i}^{H}=a+i H, 0 \leq i \leq N$. The coarse grid is $\Omega^{H}$. The unknowns vectors are $U^{H}$ and $U_{x}^{H} \in \mathbb{R}^{(N-1)}$ such that $U_{i}^{H} \approx u\left(x_{i}^{H}\right)$ and $U_{x, i}^{H} \approx u^{\prime}\left(x_{i}^{H}\right)$. To avoid any confusion between variables, we perform a change of notations for the sequel

$$
\begin{cases}H=h_{c}, & (\text { Coarse grid size) }  \tag{66}\\ h=h_{f}, & \text { (Fine grid size) }\end{cases}
$$

The hermitian Box-scheme corresponding to the problem (65) has the following matrix form

$$
\begin{equation*}
\frac{1}{h_{c}^{2}} \mathcal{H} U_{i}^{h_{c}}=F-\frac{1}{h_{c}^{2}} \mathcal{G}\left(e_{1} g_{a}+e_{N-1} g_{b}\right), \tag{67}
\end{equation*}
$$

with $F=\left[\Pi^{0} f_{1}, \Pi^{0} f_{2}, \ldots, \Pi^{0} f_{N-1}\right]^{T}$. By comparing with (2), let

$$
\left\{\begin{align*}
L^{h_{c}} & =\frac{1}{h_{c}^{2}} \mathcal{H}  \tag{68}\\
f^{h_{c}} & =F-\frac{1}{h_{c}^{2}} \mathcal{G}\left(e_{1} g_{a}+e_{N-1} g_{b}\right)
\end{align*}\right.
$$

By resolving (67) we obtain the solution vector $U_{i}^{h_{c}}, i=0$, on $\Omega^{h_{c}}$ and the global vector solution $\widetilde{U}_{i}^{h_{c}} \in \mathbb{R}^{(N+1)}$

$$
\begin{equation*}
\widetilde{U}_{i}^{h_{c}}=\left[g_{a}, U_{i, 1}^{h_{c}}, U_{i, 2}^{h_{c}}, \ldots, U_{i, N-1}^{h_{c}}, g_{b}\right]^{T} \tag{69}
\end{equation*}
$$

The domain $\Omega_{l}$ has been choosen such that $\Omega_{l}=\left(a_{1}, b_{1}\right)$, with $a_{1}, b_{1}$ two fixed scalars between $a$ and $b, a \leq a_{1}<b_{1} \leq b$. The choice of $a_{1}$ and $b_{1}$ is adapted such that for all $N$ there exist $i_{0}$ and $j_{0}$ between 1 and $N-1$ such that

$$
\begin{equation*}
a_{1}=x^{h_{c}}\left(i_{0}\right), b_{1}=x^{h_{c}}\left(j_{0}\right) \tag{70}
\end{equation*}
$$

that is to say $a_{1}$ and $b_{1}$ are two points of the coarse grid $\Omega^{h_{c}}$. In order that $\Omega_{l}^{h}$ has the same number of points as $\Omega^{H}$, we choose $a_{1}$ and $b_{1}$ such that $b_{1}-a_{1}=\frac{b-a}{2}$. Suppose that we know the values $g_{a 1}=u\left(a_{1}\right)$ and $g_{b_{1}}=u\left(b_{1}\right)$ where $u(x)$ is the exact solution. Hence, on the domain $\Omega_{l}$ we can write

$$
\left\{\begin{array}{l}
-u_{l}^{\prime \prime}=f_{l}, \quad \text { in } \Omega_{l}  \tag{71}\\
u_{l}\left(a_{1}\right)=g_{a_{1}}, u_{l}\left(b_{1}\right)=g_{b_{1}}
\end{array}\right.
$$

with $u_{l}=\left.u\right|_{\Omega_{l}}$ and $f_{l}=\left.f\right|_{\Omega_{l}}$. The domain $\Omega_{l}$ is discretized with stepsize $h_{f}=\frac{h_{c}}{2}$ such that the discrete fine points are $x_{i}^{h_{f}}, x_{i}^{h_{f}}=a_{1}+i h_{f}, 0 \leq i \leq N_{f}$, with $N_{f}=\frac{b_{1}-a_{1}}{h_{f}}$. Notice the equality between $N$ and $N_{f}$ when $b_{1}-a_{1}=\frac{b-a}{2}$. The fine grid is $\Omega_{l}^{h_{f}}$. The unknown vectors are $U_{l}^{h_{f}}$ and $U_{l, x}^{h_{f}} \in \mathbb{R}^{(N-1)}$ such that $U_{l, i}^{h_{f}} \approx u_{l}\left(x_{i}^{h_{f}}\right)$ and $U_{l, x, i}^{h_{f}} \approx u_{l}^{\prime}\left(x_{i}^{h_{f}}\right), 1 \leq i \leq N-1$. The interface between $\Omega_{l}$ and $\Omega$ is $\partial \Omega_{l}=\left\{a_{1}, b_{1}\right\}$. Based on the matrix form (63), the hermtian Box-scheme corresponding to (71) is

$$
\begin{equation*}
\frac{1}{h_{f}^{2}} \mathcal{H} U_{i}^{h_{f}}=F_{l}-\frac{1}{h_{f}^{2}} \mathcal{G}\left(e_{1} g_{a_{1}}+e_{N-1} g_{b_{1}}\right), \tag{72}
\end{equation*}
$$

with $F_{l}=\left[\Pi^{0} f_{1}, \Pi^{0} f_{2}, \ldots, \Pi^{0} f_{N-1}\right]^{T} \in \mathbb{R}^{N-1}$. The problem is that we do not know the values $g_{a_{1}}$ and $g_{b_{1}}$ on $\partial \Omega_{l}$ in (71). To approach these values, we replace it by artificial boundary conditions

$$
\left\{\begin{array}{l}
g_{a_{1}} \approx \widetilde{U}_{0, i_{0}}^{h_{c}},  \tag{73}\\
g_{b_{1}} \approx \widetilde{U}_{0, j_{0}}^{h_{c}}
\end{array}\right.
$$

with $i_{0}$ and $j_{0}$ are the indices of the points $a_{1}$ and $b_{1}$ in the coarse grid defined in (70). Here, in particular, the artificial boundary $\partial \Omega_{l}=\left\{a_{1}, b_{1}\right\}$ is composed of two points of the coarse grid $\Omega^{h_{c}}$ then there is no need to interpolate therefore the matrix $B_{l, \Gamma}^{h_{c}}$ is zero. By comparing (72) and (3), let

$$
\left\{\begin{array}{l}
L_{l}^{h_{f}}=\frac{1}{h_{f}^{2}} \mathcal{H}  \tag{74}\\
f_{l}^{h_{f}}=F_{l}-\frac{1}{h_{f}^{2}} \mathcal{G}\left(e_{1} \widetilde{U}_{i_{0}}^{h_{c}}+e_{N-1} \widetilde{U}_{j_{0}}^{h_{c}}\right)
\end{array}\right.
$$

By solving (72) with $i=0$ we get

$$
\begin{equation*}
\frac{1}{h_{f}^{2}} \mathcal{H} U_{l, i}^{h_{f}}=F_{l}-\frac{1}{h_{f}^{2}} \mathcal{G}\left(e_{1} \widetilde{U}_{i, i_{0}}^{h_{c}}+e_{N-1} \widetilde{U}_{i, j_{0}}^{h_{c}}\right), \tag{75}
\end{equation*}
$$

then we obtain the approximation $U_{l, 0}^{h_{f}}$.
Following (5) the vector $W_{i}^{h_{c}}=\left(W_{i, j}^{h_{c}}\right)_{1 \leq j \leq N-1}$ verifies

$$
W_{i}^{h_{c}}= \begin{cases}U_{l, i}^{h_{c}}, & \text { in } \Omega_{l}^{h_{c}}  \tag{76}\\ U_{i}^{h_{c}}, & \text { in } \Omega^{h_{c}} \backslash \Omega_{l}^{h_{c}}\end{cases}
$$

The defect vector $d_{i}^{h_{c}} \in \mathbb{R}^{(N-1)}$ is calculated by

$$
\begin{equation*}
d_{i}^{h_{c}}=\frac{1}{h_{c}^{2}} \mathcal{H} W_{i}^{h_{c}}-f^{h_{c}} \tag{77}
\end{equation*}
$$

Finally, the algorithm is the following:

## Algorithm 2.1. •Initialization

- Solve the problem (67) on the coarse grid $\Omega^{h_{c}}$. The vector solution is denoted by $U_{0}^{h_{c}} \in \mathbb{R}^{(N-1)}$.
- Solve the problem (75) on the local grid $\Omega_{l}^{h_{f}}$. The vector solution is denoted by $U_{l, 0}^{h_{f}} \in \mathbb{R}^{(N-1)}$.
- Iteration $i=1,2, \ldots$
- Compute the vector $W_{i-1}^{h_{c}}$ by (76).
- Compute the defect vector $d_{i-1}^{h_{c}}$ by (77).
- Solve on the coarse grid

$$
\frac{1}{h_{c}^{2}} \mathcal{H}\left(U_{i}^{h_{c}}\right)_{j}= \begin{cases}f_{j}^{h_{c}}+d_{i-1, j}^{h_{c}}, & x_{j}^{h_{c}} \in \Omega_{\epsilon}^{h_{c}},  \tag{78}\\ f_{j}^{h_{c}}, & x_{j}^{h_{c}} \in \Omega^{h_{c}} \backslash \Omega_{\epsilon}^{h_{c}}\end{cases}
$$

with $\epsilon>0$. The solution vector is denoted by $U_{i}^{h_{c}} \in \mathbb{R}^{(N-1)}$.

- Solve the problem (75) on the fine grid. The vector solution is denoted by $U_{l, i}^{h_{f}} \in \mathbb{R}^{(N-1)}$.
2.4. Numerical results. In this part, we display some numerical results proving the interest of the technique presented in the last section. The calculation is performed using Matlab. The CPU time is calculated by the functions tic and toc. Notice that the CPU time can be extremely reduced using FFT or another programming language [3]. The algorithm used so far is the algorithm 2.1 after only one iteration. We intend to study in another paper the behavior of the convergence of this algorithm. In the numerical tables, $u_{e x}, u_{x, e x}$ are the exact solution and derivative, $u, u_{x}$ are the computed solution and derivative on the uniform grid and $u^{h_{f}, h_{c}}, u_{x}^{h_{f}, h_{c}}$ are the computed solution and derivative on the composite domain. We use the discrete norm $L^{\infty}$ to estimate the error:

$$
\begin{equation*}
\left\|u_{e x}-u\right\|_{\infty}=\max _{i=1, . ., N-1}\left|u_{e x}\left(x_{i}\right)-u_{i}\right| \tag{79}
\end{equation*}
$$

The convergence rate is calculated using the formula

$$
\begin{equation*}
\text { Conv. rate }=\log _{2}\left(e_{N / 2} / e_{N}\right) \tag{80}
\end{equation*}
$$

where $e_{N}$ is the error obtained on a mesh of size $N$.

Table 1. Error and convergence rate for Test 1 on uniform grid

| $h_{f}$ | $\left\\|u_{e x}-u\right\\|_{\infty}$ | $\left\\|u_{x, e x}-u_{x}\right\\|_{\infty}$ | Grid points | Time(s.) |
| :--- | :--- | :--- | :--- | :--- |
| $1 / 128$ | $1.289(-4)$ | $3.778(-3)$ | 128 | 0.02 |
| Conv. rate | 4.00 | 4.00 |  |  |
| $1 / 256$ | $7.832(-6)$ | $2.391(-4)$ | 256 | 0.07 |
| Conv. rate | 4.00 | 4.00 |  |  |
| $1 / 512$ | $4.861(-7)$ | $1.482(-5)$ | 512 | 0.50 |
| Conv. rate | 4.00 | 4.00 |  |  |
| $1 / 1024$ | $3.033(-8)$ | $9.246(-7)$ | 1024 | 8.11 |
| Conv. rate | 4.00 | 4.00 |  |  |
| $1 / 2048$ | $1.887(-9)$ | $5.775(-8)$ | 2048 | 48.21 |

Table 2. Error and convergence rate for Test 1 on composite grid

| $h_{c}$ | $\left\\|u_{e x}-u^{h_{f}, h_{c}}\right\\|_{\infty}$ | $\left\\|u_{x, e x}-u_{x}^{h_{f}, h_{c}}\right\\|_{\infty}$ | Grid points | Time(s.) |
| :--- | :--- | :--- | :--- | :--- |
| $1 / 64$ | $1.289(-4)$ | $3.778(-3)$ | 96 | 0.01 |
| $1 / 128$ | $7.832(-6)$ | $2.391(-4)$ | 192 | 0.02 |
| $1 / 256$ | $4.861(-7)$ | $1.482(-5)$ | 384 | 0.07 |
| $1 / 512$ | $3.033(-8)$ | $9.246(-7)$ | 768 | 2.35 |
| $1 / 1024$ | $1.887(-9)$ | $5.775(-8)$ | 1536 | 26.11 |

Test 1: We consider the Gaussian function $u_{e x}(x)=\exp \left(-500\left(x-\frac{1}{2}\right)^{2}\right)$. This function has a high activity region around $x=\frac{1}{2}$. The results obtained on the composite grid are compared with those computed by the HB-scheme on the uniform grid with fine stepsize $h_{f}$. Observe in tables 1 and 2 the same accuracy obtained by the two approaches with less points on the composite grid.
Test 2: Suppose that the exact solution is $u_{e x}(x)=\frac{1}{2}\left[\tanh 50\left(\left(x-\frac{1}{2}\right)\right)+1\right]$ on $\Omega=(0,1)$. This function has a deep gradient around $x=\frac{1}{2}$. We choose $\Omega_{l}=\left(\frac{1}{4}, \frac{3}{4}\right)$ and $\Omega_{\epsilon}=\left(\frac{1}{4}+\epsilon, \frac{3}{4}-\epsilon\right)$ with $\epsilon=\frac{1}{12}$. The numerical results are given in tables 3 and 4 .

## 3. Conclusion

This paper provides a new application of the multiscale technique combined with a new high order scheme called HB-scheme. The high order accuracy of this scheme is particularly interesting in applications where physical fields have to be calculated as an outcome of a potential solution of a Poisson problem (electromagnetism, gravitation). The main features of this work are the high order of precision on the boundaries and the calculation of the derivatives. This

Table 3. Error and convergence rate for Test 2 on uniform grid

| $h_{f}$ | $\left\\|u_{e x}-u\right\\|_{\infty}$ | $\left\\|u_{x, e x}-u_{x}\right\\|_{\infty}$ | Grid points | Time(s.) |
| :--- | :--- | :--- | :--- | :--- |
| $1 / 256$ | $3.481(-5)$ | $3.500(-3)$ | 256 | 0.08 |
| Conv. rate | 4.02 | 4.08 |  |  |
| $1 / 512$ | $2.082(-6)$ | $2.061(-4)$ | 512 | 0.59 |
| Conv. rate | 4.01 | 4.02 |  |  |
| $1 / 1024$ | $1.291(-7)$ | $1.269(-5)$ | 1024 | 8.19 |
| Conv. rate | 4.00 | 4.00 |  |  |
| $1 / 2048$ | $8.062(-9)$ | $7.904(-7)$ | 2048 | 48.01 |

Table 4. Error and convergence rate of Test 2 on composite grid

| $h_{c}$ | $\left\\|u_{e x}-u^{h_{f}, h_{c}}\right\\|_{\infty}$ | $\left\\|u_{x, e x}-u_{x}^{h_{f}, h_{x}}\right\\|_{\infty}$ | Grid points | Time(s.) |
| :--- | :--- | :--- | :--- | :--- |
| $1 / 128$ | $3.481(-5)$ | $3.500(-3)$ | 192 | 0.02 |
| $1 / 256$ | $2.082(-6)$ | $2.061(-4)$ | 384 | 0.07 |
| $1 / 512$ | $1.291(-7)$ | $1.269(-5)$ | 768 | 2.16 |
| $1 / 1024$ | $8.062(-9)$ | $7.904(-7)$ | 1536 | 25.49 |

application illustrates the gain in memory and CPU time using this technique as it is shown in the numerical tables. For instance, we have obtained for Test 1 the same precision using 3072 points instead of 4096 . This gain is expected to be more interesting in higher dimensions. A generalization of this application to higher dimensions using cubic spline interpolations, based on the previous results in $[9,10]$ will be a part of future work.

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