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THE ORIENTABLE NUMBERS OF A GRAPH

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ABSTRACT. For a connected graph G, there are orientations of G have different hull numbers, geodetic numbers, and convexity numbers. The lower orientable hull number $h^-(G)$ is defined as the minimum hull number among all the orientations of G and the upper orientable hull number $h^+(G)$ as the maximum hull number among all the orientations of G. The lower and upper orientable geodetic numbers $g^-(G)$ and $g^+(G)$ are defined similarily. In this paper, We investigate characterizations of the orientable numbers and the conditions that the relation $h^-(G) \leq g^-(G) < h^+(G) \leq$ $g^+(G)$ holds.

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1. Introduction

The distance d(u, v) between two vertices u and v in a connected graph G is the length of a shortest u - v path in G. An u - v path of length d(u, v) is called an u - v geodesic. It is well known that this distance is a metric on the vertex set V(G). For more information, refer to [6]. We define I[u, v] to be the set of all vertices lying on some u - v geodesic of G, and for a nonempty subset S of V(G), $I[S] = \bigcup_{u,v \in S} I[u, v]$.

The set S is convex if I[S] = S. Clearly, if $S = \{v\}$ or S = V, then S is convex. The convexity number c(G) of G is the cardinality of a maximum proper convex subset of V. The convex hull [S] is the smallest convex set containing S. Since the intersection of two convex sets is convex, the convex hull is well defined. Note that $S \subseteq I[S] \subseteq [S] \subseteq V$. A subset $S \subseteq V$ is called a geodetic set if I[S] = V and the geodetic number g(G) of G is the minimum cardinality of its geodetic sets. A subset $S \subseteq V$ is called a hull set if [S] = V and the hull number h(G) of G is the minimum cardinality of its hull sets. As a historical background, convexity in graphs was discussed in the book by Buckly and Harary

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[1] and studied by Harary and Niemenen [8]. The geodetic number of a graph was introduced by Chartrand *et al.* [3], and studied by Chartrand *et al.* [4] and Kim [10]. Furthermore the hull number of a graph was introduced by Everett and Seidman [5], and studied by Chartrand *et al.* [2].

For an oriented graph D, the directed distance d(u, v) from a vertex u to a vertex v is the length of a shortest directed u-v path. Although this distance is not a metric, as it clearly lacks the symmetry property, the above three numbers of D can be defined in a natural manner. A connected graph has orientations with different geodetic numbers and orientations with different hull numbers. Also a connected graph has orientations with different convexity numbers if there are no end-vertices. The aim of this paper is to survey characterizations of the three orientable numbers and their relations.

In the following we introduce some definitions used in directed graph. If the direction of the edge e in a directed graph is from vertex u to vertex v, then we write e = (u, v). The outdegree od(v) of a vertex v is the number of edges incident from v and indegree id(v) of a vertex v is the number of edges incident to v. A vertex u of a directed graph D is a transitive vertex if od(u) > 0, id(u) > 0, and for every $(x, u), (u, y) \in E(D), (x, y) \in E(D)$. A vertex v of D is an extreme vertex if od(v) = 0, id(v) = 0, or a transitive vertex. A source(sink) of an oriented graph D is a vertex with $id_D(v) = 0(od_D(v) = 0)$. Note that a vertex v is extreme iff V - v is a convex set, iff v is contained in every hull set and every geodetic set.

For an undirected graph G, $con^{-}(G)$ and $con^{+}(G)$ are the minimum and maximum convexity numbers over all orientations of G. We can make an arbitrary vertex v extreme by orienting all incident edges away from v for any graph G, so we always have $con^{+}(G) = n - 1$, where n is the order of G. Moreover, if G contains an end-vertex x, then in every orientation x has indegree 0 or outdegree 0. So in this case, $con^{-}(G) = n - 1$ too. If G has no end-vextices, it is straightforward to find an orientation with no vertices indegree 0 or outdegree 0. A. Farrugia [7] was interested in whether $con^{-}(G) < con^{+}(G)$. We state his theorem.

Theorem 1.1. A graph with minimum degree 2 can be oriented so that all its vertices are non-extreme. Thus, for a connected graph G with at least 3 vertices, $con^{-}(G) < con^{+}(G)$ iff G has no end-vertices.

2. The relation $h^{-}(G) \leq g^{-}(G) < h^{+}(G) \leq g^{+}(G)$

In general, for a connected graph G, there are orientations of G that have distinct orientable numbers. This suggests the following definitions. For a connected graph G of order $n \ge 2$, the lower orientable geodetic number $g^-(G)$ is defined as the minimum geodetic number of an orientation of G and the upper orientable geodetic number $g^+(G)$ of G as the maximum such geodetic number, that is,

$$g^{-}(G) = \min\{g(D) : D \text{ is an orientation of } G\},\$$

$$q^+(G) = \max\{g(D) : D \text{ is an orientation of } G\}.$$

Similarly, the lower orientable hull number $h^-(G)$ and the upper orientable hull number $h^+(G)$ are defined. So we have

$$h^{-}(G) = \min\{h(D) : D \text{ is an orientation of } G\},\$$

 $h^+(G) = \max\{h(D) : D \text{ is an orientation of } G\}.$

Hence, for every connected graph G of order $n \ge 2$, $2 \le g^-(G) \le g^+(G) \le n$ and $2 \le h^-(G) \le h^+(G) \le n$.

Lemma 2.1. Let G be a complete graph of order $n \ge 2$. then $g^-(G) = 2 = h^-(G)$ and $g^+(G) = n = h^+(G)$.

Proof. Let G be a complete graph with vertices v_1, \dots, v_n . We first orient G transitively (that is, $(v_i, v_j) \in E(G)$ iff i < j). Since every vertex is extreme, this orientation shows that $g^+(G) = n = h^+(G)$. Reversing the orientation of $v_1v_2 \cdots v_{n-1}v_n$ makes $\{v_1, v_2\}$ geodetic set. Thus $g^-(G) = 2 = h^-(G)$. \Box

G. Chartrand and P. Zhang [4] showed that if G is any connected graph of order $n \geq 3$, then $g^{-}(G) < g^{+}(G)$.

Theorem 2.2. For any connected graph G of order $n \ge 3$, then $g^-(G) < g^+(G)$.

Proof. We have already seen by Lemma 2.1 that the result is true if G is complete, so we can assume that G contains two vertices v_0 and v_2 with $d(v_0, v_2) = 2$. Let v_0, v_1, v_2 be a path of length 2 in G. Clearly, $v_0v_2 \notin E(G)$. Let $U = V(G) - \{v_0, v_1, v_2\}$. Since $g^-(P_3) \neq g^+(P_3)$, we may assume that $U \neq \emptyset$

To verify the result, it suffices to show that there exist two orientations of G having distinct geodetic numbers. An orientation D_1 of G is obtained by directing v_0v_1 and v_1v_2 so that $(v_0, v_1), (v_1, v_2) \in E(D_1)$. Every edge joining a vertex of U and v_0 is directed away from v_0 , while every edge joining a vertex of U and v_1 (or v_2) is directed toward v_1 (or v_2). The edges joining two vertices of U are directed arbitrarily. This completes the construction of D_1 . Since $id_{D_1}(v_0) = od_{D_1}(v_2) = 0$, the vertices v_0 and v_2 belong to every geodetic set of D_1 .

We now describe an orientation D_2 of G. First, we direct v_0v_1 and v_1v_2 so that $(v_0, v_1), (v_2, v_1) \in E(D_2)$. Every edge joining a vertex of U and v_1 is directed toward v_1 , while every edge joining a vertex of U and v_0 (or v_2) is directed away v_0 (or v_2). The edges joining two vertices of U are directed exactly as in the orientation D_1 . Since $id_{D_2}(v_0) = od_{D_2}(v_1) = od_{D_2}(v_2) = 0$, all of v_0, v_1, v_2 belong to every geodetic set of D_2 . Also, since $I_{D_2}[\{v_0, v_1, v_2\}] = \{v_0, v_1, v_2\}$, every geodetic set of D_2 must contain some vertices of U. Let $g(D_2) = t + 3$, where $T = \{u_1, u_2, \dots, u_t, v_0, v_1, v_2\}$ is a minimum geodetic set of D_2 .

We now show that $S = \{u_1, u_2, \dots, u_t, v_0, v_2\}$ is a geodetic set of D_1 . Let $w \in V(D_1)$. We show that w lies on some x - y geodesic for $x, y \in S$. Since this is clearly true if $w \in S$, we may assume that $w \in V(D_1) - S$. Since v_1 lies on the geodesic v_0, v_1, v_2 in D_1 , we need only consider the case where $w \in U - S$.

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Now w lies on some a - b geodesic in D_2 , where $a, b \in \{u_1, u_2, \dots, u_t\}$, that is, w lies on a $u_i - u_j$ geodesic in D_2 for some i and j with $1 \le i \ne j \le t$, where $V(P) \subseteq U$. Since there is no directed $u_r - u_s$ path in D_1 containing any of v_0, v_1, v_2 for all pairs r, s with $1 \le r \ne s \le t$, it follows that P is $u_i - u_j$ geodesic in D_1 as well. Hence S is a geodetic set for D_1 and $g(D_1) \le t + 2$, implying that $g(D_1) < g(D_2)$ and so $g^-(G) < g^+(G)$.

A. Farrugia [7] also showed that if G is any connected graph of order $n \ge 3$, then $h^-(G) < h^+(G)$.

Theorem 2.3. For any connected graph G of order $n \ge 3$, $h^-(G) < h^+(G)$.

Since every geodetic set is a hull set, we have $h(D) \leq g(D)$ for every directed graph D. Therefore we have $h^-(G) \leq g^-(G)$ and $h^+(G) \leq g^+(G)$. There are many classes for which $h^-(G) = g^-(G) < h^+(G) = g^+(G)$.

Theorem 2.4. (1) If T is a tree of order $n \ge 2$ with k end-vertices, then $g^{-}(T) = h^{-}(T) = k$ and $g^{+}(T) = h^{+}(T) = n$.

(2) For $n \ge 3$, $g^{-}(C_n) = h^{-}(C_n) = 2$ and $g^{+}(C_n) = h^{+}(C_n) = n$ if n = 3 or n is even; otherwise $g^{+}(C_n) = h^{+}(C_n) = n - 1$.

(3) For integers r and s with $2 \leq r \leq s$, $g^{-}(K_{r,s}) = h^{-}(K_{r,s}) = 2$ and $g^{+}(K_{r,s}) = h^{+}(K_{r,s}) = r + s$.

Proof. (1) Since a tree T is bipartite, $g^+(T) = h^+(T) = n$. And since every geodetic set S of T contains all the end-vertices, we have $|S| \ge k$. For every vertex $u \in V(T) - S'$, u lies in some v - w geodesic, where S' is a set of all the end-vertices and $v, w \in S'$ with |S'| = k. Therefore, S' is a minimum geodetic set and $g^-(T) = h^-(T) = k$.

(2) Every pair of distinct vertices of the directed cycle $\overrightarrow{C_n}$ is a geodetic set. So $g^-(C_n) = h^-(C_n) = 2$.

If n = 3, then there is an transitive orientation of C_3 and so $g^+(C_n) = h^+(C_n) = n$.

If n is even, we direct the two edges incident with v_i towards v_i for all even i. Then every vertex of C_n has indegree or outdegree 0. So any geodetic set contains all the vertices of C_n . Therefore, $g^+(C_n) = h^+(C_n) = n$.

If n is odd $(n \ge 5)$, let $V(C_n) = \{v_1, v_2, \dots, v_{2k+1}\}$. Now we direct the two edges incident with v_i towards v_i for all *i* even and have an arc (v_1, v_{2k+1}) . Then each vertex $v_i(1 \le i \le 2k)$ has indegree or outdegree 0 and vertex v_{2k+1} lies in $v_1 - v_{2k}$ geodesic. So $\{v_1, v_2, \dots, v_{2k}\}$ is a minimum geodetic set and $g^+(C_n) = h^+(C_n) = 2k = n - 1$.

(3) There exists an orientation of $K_{r,s}$ having directed Hamiltonian path. So $g^{-}(K_{r,s}) = h^{-}(K_{r,s}) = 2$. And since $K_{r,s}$ is bipartite, $g^{+}(K_{r,s}) = h^{+}(K_{r,s}) = r + s$.

Now we focus our study on the relation $g^-(G) < h^+(G)$. J-T. Hung *et al.* [9] proved that this relation holds for every connected graph of order $n \ge 3$.

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Lemma 2.5. Suppose that G is a connected graph of order $n \ge 3$ and T is a spanning tree of G with k leaves. Then $g^-(G) \le k$.

Proof. Let S be the set of all leaves of T and r be a leaf of T. Define an orientation D of G by (i) if $uv \in E(T)$ and $d_T(r, u) > d_T(r, v)$, then $(v, u) \in E(D)$, and (ii) if $uv \in E(G) \setminus E(T)$ and $d_T(r, u) > d_T(r, v)$, then $(u, v) \in E(D)$, and (iii) assign any arbitrary directions to the other edges. In D, we have that the paths from r to the other leaves in T are geodesics of D. That is, S is a geodetic set of D. By the definition, $g^-(G) \leq g(D) \leq |S| = k$.

Theorem 2.6. If a graph G of order $n \ge 3$ has a Hamiltonian path, then $2 = h^-(G) = g^-(G) < h^+(G)$. Furthermore, if $n \ge 4$, then $2 = h^-(G) = g^-(G) < h^+(G) - 1$.

Proof. Since G has a Hamiltonian path, by Lemma 2.5, $g^-(G) \leq 2$. Thus, $2 \leq h^-(G) \leq g^-(G) \leq g(D) = 2$. Therefore $2 = h^-(G) = g^-(G)$.

If n = 3 and G is connected, then G is either a complete graph or a path. It is easy to see that $h^+(G) = 3$; that is, the statement of the theorem is true for n = 3. For G being a complete graph, the hull number of an acyclic tournament of order n is n. So $2 = g^-(G) < h^+(G) = n$ for $n \ge 3$. We assume that G is not a complete graph and the order of G is at least 4, then there exist $u, v \in V(G)$ such that $uv \notin E(G)$.

<u>Case 1.</u> $N_G(u) \cap N_G(v) = \emptyset$.

Take $x \in N_G(u)$ and $y \in N_G(v)$. Construct an orientation D_1 of G by the following steps:

(1) if $us, tx \in E(G)$, then $(u, s), (t, x) \in E(D_1)$,

(2) if $ys, tv \in E(G)$, then $(y, s), (t, v) \in E(D_1)$,

and (3) assign any arbitrary directions to the other edges.

Since $N_G(u) \cap N_G(v) = \emptyset$, the vertices u and y are sources and the vertices x and v are sinks in D_1 . Therefore the vertices u, v, x, y must be in every hull set of D_1 . Hence $h^+(G) \ge h(D_1) \ge 4 = h^-(G) + 2$.

<u>Case 2.</u> $N_G(u) \cap N_G(v) \neq \emptyset$.

Take $x \in N_G(u) \cap N_G(v)$. Construct an orientation D_2 of G by the following steps:

(1) if $ut, vs \in E(G)$, then $(u, t), (v, s) \in E(D_2)$,

(2) if $tx \in E(G)$, then $(t, x) \in E(D_2)$,

and (3) assign any arbitrary directions to the other edges.

In D_2 , the vertices u and v are sources and the vertex x is a sink. Then the vertices u, v and x must be in every hull set of D_2 . Since $|V(G)| \ge 4$ and $[\{u, v, x\}]_{D_2} = \{u, v, x\} \neq V(G)$. We have that $h^+(G) \ge h(D_2) \ge 4 = h^-(G) + 2$.

Theorem 2.7. If G is a connected graph with at least 3 vertices, then $g^-(G) < h^+(G)$.

Proof. Suppose that G is a connected graph with order $n \ge 3$. If G has a Hamiltonian path, then by Theorem 2.6, the statement of the theorem is true.

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So we assume that G has no Hamiltonian path. Let T be a spanning tree of G obtained by the depth-first search algorithm where $V(T) = \{u_1, u_2, \dots, u_n\}$ (the order is given by the search with u_1 as the root vertex of T) and the leaves of T be v_1, v_2, \dots, v_k . By G having no Hamiltonian path, $k \ge 3$, and if $u_i u_j \in E(G)$ with i < j, then u_i is a vertex in the path from u_1 to u_j in T.

If u_1 is not a leaf in T, then by the depth-first search algorithm, v_1, v_2, \dots, v_k are independent in G. (If $u_i = v_a, u_j = v_b, i < j$, and $v_a v_b \in E(G)$, then v_a is not a leaf of T.) Construct an orientation D_1 of G by $(u_i, u_j) \in E(D_1)$ if and only if $u_i u_j \in E(G)$ and i < j. Then u_1 is a source and v_1, v_2, \dots, v_k are sinks in D_1 ; that is, every hull set of D_1 contains $u_1, v_1, v_2, \dots, v_k$. Thus, $h^+(G) \ge h(D_1) \ge k + 1$. By Lemma 2.5, $g^-(G) \le k < k + 1 \le h^+(G)$. If u_1 is a leaf in T, then $u_1 = v_1$. There exits:

(a) If $u_1v_i \notin E(G)$ for all $i = 2, 3, \dots, k$, then v_1, v_2, \dots, v_k are independent. We can construct an orientation D_2 of G in which v_1, v_2, \dots, v_k are sinks in D_2 . Then every hull set of D_2 contains v_1, v_2, \dots, v_k . Since $[\{v_1, v_2, \dots, v_k\}]_{D_2} = \{v_1, v_2, \dots, v_k\}, h^+(G) \ge h(D_2) \ge k + 1$. By Lemma 2.5, $g^-(G) \le k < h^+(G)$.

(b) If there exists $j \in \{2, 3, \dots, k\}$ such that $u_1v_j \in E(G)$, then let i be a smallest number such that $deg_T(u_i) \geq 3$, and T' be a spanning tree of G with $E(T') = E(T) \setminus \{u_{i-1}u_i\} \cup \{u_1v_j\}$. Then the number of leaves in T' is k-1. By Lemma 2.5, $g^-(G) \leq k-1$. Let D_3 be an orientation of G defined by $(u_i, u_j) \in E(D_3)$ if and only if $u_iu_j \in E(G)$ and i < j. In D_3 , u_1 is a source and v_2, v_3, \dots, v_k are sinks; that is, $h^+(G) \geq h(D_3) \geq k$.

Hence $g^-(G) < h^+(G)$ for G of order $n \ge 3$. The theorem then follows. \Box

The following is a direct consequence of Theorem 2.7.

Corollary 2.8. If G is a connected graph of order $n \ge 3$, then $h^-(G) \le g^-(G) < h^+(G) \le g^+(G)$.

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