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OPTIMALITY CONDITIONS AND DUALITY RESULTS OF THE NONLINEAR PROGRAMMING PROBLEMS UNDER $\rho-(p,r)$-INVEXITY ON DIFFERENTIABLE MANIFOLDS

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Abstract. In this paper, by using the notion of $\rho-(p,r)$-invexity assumptions on the functions involved, optimality conditions and duality results (Mond-Weir, Wolfe and mixed type) are established on differentiable manifolds. Counterexample is constructed to justify that our investigations are more general than the existing work available in the literature.

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1. Introduction

Convexity plays a vital role in the theory of optimization but it is often not enjoyed by real problems. Therefore, several generalizations have been developed for the classical properties of convexity. An important and significant generalization of convexity is invexity which was introduced by Hanson [6], in the year 1981. Later on Zalmai [16] generalized the class of invex functions into $\rho-(\eta,\theta)$-invex functions. In 2001, Antczak [3] introduced (p,r)-invex sets and functions. Mandal and Nahak [9] introduced $(p,r)-\rho-(\eta,\theta)$-invexity which is a generalization of the results of both Zalmai [16] and Antczak [3].
Rapcsak [13] introduced a generalization of convexity called geodesic convexity and extended many results of convex analysis and optimization theory from linear spaces to Riemannian manifolds. Udriste [14] established duality results for a convex programming problem on Riemannian manifolds. Pini [12] introduced the notion of invex functions on a manifold. Motivated by Pini [12], Mititelu [11] generalized invexity by defining (ρ,η)-invex, (ρ,η)-pseudoinvex and

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(ρ, η)-quasiinvex functions. Mititelu [11] also established the necessary and sufficient conditions of Karush-Kuhn-Tucker type for a vector programming problem defined on a differentiable manifold. Mond-Weir type duality for vector programming problems on differentiable manifolds was developed by Ferrara and Mititelu [5]. The concepts of geodesic invex sets, geodesic invex and geodesic preinvex functions were introduced by Barani and Pouryayevali [4] on Riemannian manifolds. Ahmad et al. [2] extended these results by introducing geodesic η-pre-pseudo invex functions and geodesic η-pre-quasi invex functions. Recently, Iqbal et al. [7] defined geodesic E-convex sets and geodesic E-convex functions. Further, Agarwal et al. [1] introduced geodesic α-invex sets, geodesic α-invex and α-preinvex functions.
Motivated to the concept of $(p, r) - \rho - (\eta, \theta)$-invexity which was introduced by Mandal and Nahak [9], in this paper we have defined $\rho - (p, r)$-invex functions on differentiable manifolds. We have studied optimality conditions and duality results of the nonlinear programming problems on differentiable manifolds under this generalized invexity assumptions.

2. Preliminaries

In this section, we recall some definitions and known results about differentiable manifolds which will be used throughout the article. These standard materials can be found in [8, 15].

Definition 2.1. An n-dimensional manifold is a Hausdorff topological space which is connected and has the property that each point has a neighborhood homeomorphic to some open set in Cartesian n-space.
A system S of differentiable coordinates in an n-dimensional manifold M is an indexed family $\{V_j, j \in J\}$ of open sets covering M, and for each j, a homeomorphism $\psi_j : E_j \to V_j$, where E_j is an open set in Cartesian n-space, such that the map

$$\psi_j^{-1}\psi_i : \psi_i^{-1}(V_i \bigcap V_j) \to \psi_j^{-1}(V_i \bigcap V_j),\ i, j \in J$$

is differentiable. If each such map has continuous derivatives of order r, then S is said to be of class r.
Two systems of coordinates $S,\ S'$ in M of class r are said to be r-equivalent if the composite families $\{V_j, V'_k\}$, $\{\psi_j, \psi'_k\}$ form a system of class r.
A differentiable n-manifold M of class r is an n-manifold M, together with an r-equivalence class of systems of coordinates in M.

Definition 2.2. A curve on a differentiable manifold M is a differentiable map α from some interval $J = (-\delta, \delta)$ of the real line into M.

Definition 2.3. A tangent vector on a curve γ at a point p of M is defined as the map

$$\dot{\gamma}_p : C^\infty(M) \to \mathbb{R},\ f \mapsto \dot{\gamma}_p(f) \equiv \frac{\mathrm{d}}{\mathrm{d}t}(f o \gamma)\mid_p . \tag{1}$$

Definition 2.4. The set of all tangent vectors at a point p of M is called the tangent space at p and is denoted by T_pM.

Definition 2.5. A manifold whose tangent spaces are endowed with a smoothly varying inner product with respect to a point $x \in M$ is called a Riemannian manifold. The smoothly varying inner product, denoted by $< \xi_x, \zeta_x >$ for every two elements ξ_x and ζ_x of T_xM, is called a Riemannian metric. If M is a differentiable manifold, then there always exist Riemannian metrices on M. As a result there exists exactly one covariant derivation called Levi-Civita connection denoted by $\nabla_X Y$ for any vector fields X, Y on M.

Let M be an n-dimensional differentiable manifold and T_pM be the tangent space to M at p. Also assume that $TM = \bigcup_{p\in M} T_pM$ is the tangent bundle of M. Let α be a differentiable curve on M with $\alpha(0) = p \in M$. Then the tangent vector to the curve α at p is $v = \alpha'(0) \in T_{\alpha(0)}M = T_pM$. Assume that N is another differentiable manifold and $\phi : M \to N$ is a differentiable map.

Definition 2.6 ([5])**.** The linear map $\mathrm{d}\phi_p : T_pM \to T_{\phi(p)}N$ defined by $\mathrm{d}\phi_p(v) = \phi'(p)v$ is called the differential of ϕ at the point p.

Let $F : M \to \mathbb{R}$ be a differentiable function. The differential of F at p, namely $\mathrm{d}F_p : T_pM \to T_{F(p)}\mathbb{R} \equiv \mathbb{R}$, is introduced by $\mathrm{d}F_p(v) = \mathrm{d}F(p)v$, $v \in T_pM$. The length of a differentiable curve $\gamma : [a, b] \to M$ is defined by

$$L(\gamma) = \int_a^b \|\gamma'(t)\|_{\gamma(t)}\mathrm{d}t.$$

For any two points $p, q \in M$, we define $\mathrm{d}(p, q) = \inf\{L(\gamma) : \gamma$ is a differentiable curve joining p to q $\}$. Then d is a distance which induces the original topology on M.

Definition 2.7 ([4])**.** A geodesic is a smooth curve γ, such that γ satisfies the equation $\nabla_{\frac{d\gamma(t)}{dt}} \frac{d\gamma(t)}{dt} = 0$. The existence theorem for ordinary differential equations implies that for every $v \in TM$ there exist an open interval $J(v)$ containing 0 and exactly one geodesic $\gamma_v : J(v) \to M$ with $\frac{d\gamma(0)}{dt} = v$. This implies that there is an open neighborhood $\tilde{T}M$ of the submanifold M of TM such that for every $v \in \tilde{T}M$ the geodesic $\gamma_v(t)$ is defined for $|t| < 2$. The exponential mapping $\exp : \tilde{T}M \to M$ is then defined as $\exp(v) = J_v(1)$ and the restriction of exp to a fiber T_pM in $\tilde{T}M$ is denoted by $\exp_p$ for every $p \in M$.

We consider now a map $\eta : M \times M \to TM$ such that $\eta(p, q) \in T_qM$ for every $p, q \in M$. For a differentiable function $f : M?rightarrow\mathbb{R}$, Pini [12] defined invexity in the following manner.

Definition 2.8. The differentiable function f is said to be η-invex or invex on a differentiable manifold M if for any $x, y \in M$,

$$f(x) - f(y) \geq \mathrm{d}f_y(\eta(x, y)). \tag{2}$$

Later on Mititelu [11] generalized the above definition as follows.

Definition 2.9. The differentiable function f is said to be (ρ, η)-invex at y if there exist an $\eta : M \times M \to TM$ and $\rho \in \mathbb{R}$ such that

$$\forall x \in M : f(x) - f(y) \geq \mathrm{d}f_y(\eta(x, y)) + \rho \mathrm{d}^2(x, y). \tag{3}$$

Definition 2.10 ([4])**.** A closed η-path joining the points y and $u = \alpha_{x,y}(1)$ is a set of the form $P_{yu} = \{v : v = \alpha(t) : t \in [0, 1]\}$.

3. ρ-(p,r)-Invexity

Definition 3.1 (Mandal and Nahak(2011))**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and p, r be arbitrary real numbers, $\rho \in \mathbb{R}$. The function f is said to be $(p, r) - \rho - (\eta, \theta)$-invex with respect to $\eta, \theta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ at u, if any one of the following conditions holds

$$\frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geq \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - \mathbf{1})] + \rho\|\theta(x, u)\|^2, \ for \ p \neq 0, \ r \neq 0, \tag{4}$$

$$\frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geq \nabla f(u)\eta(x, u) + \rho\|\theta(x, u)\|^2, for \ p = 0, \ r \neq 0, \tag{5}$$

$$f(x) - f(u) \geq \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - \mathbf{1}) + \rho\|\theta(x, u)\|^2, for \ p \neq 0, \ r = 0, \tag{6}$$

$$f(x) - f(u) \geq \nabla f(u)\eta(x, u) + \rho\|\theta(x, u)\|^2, for \ p = 0, \ r = 0. \tag{7}$$

Here the exponentials appearing on the right-hand sides of inequalities above are understood to be taken componentwise and $\mathbf{1} = (1, 1, ..., 1)$.
Motivated by the $(p, r) - \rho - (\eta, \theta)$-invex function, we introduce the $\rho - (p, r)$-invex function and study the sufficient optimality conditions and duality results (weak, strong and converse duality) for optimization problems defined on a differentiable manifold. Throughout the rest of the paper M denotes an n-dimensional differentiable manifold of class r.

Definition 3.2. Let M be an n-dimensional differentiable manifold and $f : M \to \mathbb{R}$ be a differentiable function. Let η be a map $\eta : M \times M \to TM$ such that $\eta(x, u) \in T_uM$ for all $x, u \in M$. The exponential map on M is a map $exp_u : T_uM \to M$ and the differential of the exponential map $(\mathrm{d}exp_u)_a : T_a(T_uM) \cong T_uM \to T_cM$, where $a = t_0\eta(x, u), t_0 \in [0, 1]$, and $c \in P_{xu}$ where P_{xu} is a closed path joining the point x and u. Let p, r and ρ be arbitrary real numbers. If for all $x \in M$, the relations

$$\frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geq \frac{1}{p}\mathrm{d}f_c([(\mathrm{d}exp_u)_a(p\eta(x, u))] - \mathbf{I}) + \rho \mathrm{d}^2(x, u), \tag{8}$$
$$\text{for} \ \ p \neq 0, r \neq 0,$$

$$\frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geq \ \mathrm{d}f_u(\eta(x, u)) + \rho \mathrm{d}^2(x, u), for \ p = 0, \ r \neq 0, \tag{9}$$

$$f(x) - f(u) \geq \frac{1}{p}\mathrm{d}f_c([(\mathrm{d}exp_u)_a(p\eta(x, u))] - \mathbf{I}) + \rho \mathrm{d}^2(x, u), \tag{10}$$
$$\text{for} \ \ p \neq 0, r = 0,$$

$$f(x) - f(u) \geq \mathrm{d}f_u(\eta(x,u)) + \rho \mathrm{d}^2(x,u), \ \text{ for } \ p = 0, \ \ r = 0, \qquad (11)$$

hold, then f is said to be $\rho - (p,r)$-invex function at u on M. Here $\mathbf{I} \in T_c M$ such that for a co-ordinate chart ϕ, $\phi(\mathbf{I}) = \mathbf{1}$, where $\mathbf{1} = (1, 1, ..., 1)$.

Note

1. If $\rho > 0$, then we call the functions as "strongly $\rho - (p,r)$-invex" functions.
2. If $\rho = 0$, then the functions reduce to "(p,r)-invex" functions.
3. If $\rho < 0$, then we call the functions as "weakly $\rho - (p,r)$-invex" functions.

It is clear that every strongly $\rho - (p,r)$-invex function is (p,r)-invex but weakly $\rho - (p,r)$-invex function is not (p,r)-invex in general. We construct the following counter example.

Example 3.3. We consider the circle $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 18^2\}$ of the Euclidean space $\mathbb{R}^2$. In the case of the circle S the possible co-ordinate charts are

$$U_1 = \{(x,y) : x > 0\} \ \phi_1(x,y) = y, \quad U_2 = \{(x,y) : x < 0\} \ \phi_2(x,y) = y$$
$$U_3 = \{(x,y) : y > 0\} \ \phi_3(x,y) = x, \quad U_4 = \{(x,y) : y < 0\} \ \phi_4(x,y) = x.$$

Let $x = (x_1, x_2) \in S$ and we define a differentiable function f on S by $f : S \to \mathbb{R}$, $f(x) = -x_1 + \cos x_2$. Let $u = (u_1, u_2) \in S$ and the angle between x and u is θ°, $(\theta \geq 1)$. Hence $\mathrm{d}(x,u) = \frac{2\pi \times 18\theta}{360} = \frac{11\theta}{35} = .3143\theta$. The tangent space of S at u is the set $T_u S = \{v \in \mathbb{R}^2 : u \cdot v = 0\}$. We choose $\eta : S \times S \to T_u S$ as $\eta(x,u) = (-u_2, u_1) \in T_u S$. Let $a = \eta(x,u) = (-u_2, u_1)$. We now find $\mathrm{d}f_u(a)$. We take a chart $\phi_3(-u_2, u_1) = \phi(-u_2, u_1) = u_2$ at a and the identity mapping as a chart ψ at $f(a)$. Here both S and $\mathbb{R}$ are of dimension 1. We now find the Jacobian matrix $\psi o f o \phi^{-1}$ at $\phi(a)$.

$$\mathrm{d}f_a(\frac{\partial}{\partial \phi})(\psi) = \frac{\partial}{\partial \phi}(\psi o f) = \frac{\partial}{\partial \phi}(f(-u_2, u_1)) = \frac{\partial}{\partial u_2}(u_2 + \cos u_1) = 1.$$

i.e., $\mathrm{d}f_u(\eta(x,u)) = 1$. Now $e^{f(x)-f(u)} - 1 - \mathrm{d}f_u(\eta(x,u)) - \rho \mathrm{d}^2(x,u) = e^{f(x)-f(u)} - 1 - 1 - (.3143)^2 \rho\theta^2 > -2 - .0987\rho\theta^2$ (since $e^{f(x)-f(u)} > 0$). If we take $\rho = -50$, then $e^{f(x)-f(u)} - 1 - \mathrm{d}f_u(\eta(x,u)) - \rho \mathrm{d}^2(x,u) > -2 + 4.935\theta^2 > 0$, $\forall x, u \in S$ (we take $\theta \geq 1$). Hence f is ((-50)-(0,1))-invex on S, i.e., f is weakly $50 - (0,1)$-invex. But if we take $x = (15, 3\sqrt{11}) \in S$, $u = (15, -3\sqrt{11}) \in S$, then $e^{f(x)-f(u)} - 1 - \mathrm{d}f_u(\eta(x,u)) = 1 - 1 - 1 = -1 < 0$, i.e., f is not (0,1)-invex on S.

3.1. Sufficient Optimality Conditions. Recently, many traditional optimization methods have been successfully generalized to minimize objective functions on manifolds. Consider the following primal optimization problem on a differentiable manifold M

(P) $\qquad$ *Minimize* $f(x)$

$\qquad$ subject to $g_i(x) \leq 0, \ i = 1, ..., m,$

where $f : M \to \mathbb{R}, \ g_i : M \to \mathbb{R}, \ i = 1, ..., m$ are differentiable functions. Let D denote the set of all feasible solutions of **(P)**.

Let $\bar{x} \in$ D be an optimal solution of **(P)** and we define the set $J^{\circ} = \{j \in 1, ..., m : g_j(\bar{x}) = 0\}$. Suppose that the domain D satisfies the following constraint qualification [11] at $\bar{x}$:

$$R(\bar{x}) : \exists v \in TM : \mathrm{d}(g_{J^{\circ}})_{\bar{x}}(v) \leq 0.$$

Here $\mathrm{d}(g_{J^{\circ}})_{\bar{x}}(v)$ is the vector components of $\mathrm{d}(g_j)_{\bar{x}}(v)$, $\forall j \in J^{\circ}$, taken in increasing order of j.

Mititelu [11] established necessary and sufficient conditions of Karush-Kuhn-Tucker (KKT) [10] type for a vector programming problem on a differentiable manifold.

Lemma 3.4 ([11])**.** *(Necessary Karush-Kuhn-Tucker (KKT) condition) If a feasible point $\bar{x} \in M$ is an optimal solution of the problem **(P)** and satisfies the constraint qualification $R(\bar{x})$, then there exists multiplier $\xi = (\xi_1, ..., \xi_m)^T \in \mathbb{R}^m$, such that the following conditions hold*

$$\mathrm{d}f_{\bar{x}} + \xi^T \mathrm{d}g_{\bar{x}} = 0, \tag{12}$$

$$\xi^T g(\bar{x}) = 0, \tag{13}$$

$$\xi \geq 0, \ \ i = 1, 2, ..., m, \tag{14}$$

here $g = (g_1, g_2, ..., g_m)^T$.

Theorem 3.5. *(Sufficient Optimality Condition) Assume that a point $\bar{x} \in M$ is feasible for problem **(P)**, and let the KKT conditions (12)-(14) be satisfied at $(\bar{x}, \xi)$. If the objective function f and the function $\xi^T g$ are $\rho_1 - (p, r)$-invex and $\rho_2 - (p, r)$-invex, respectively at $\bar{x}$ on D with respect to the same function η with $(\rho_1 + \rho_2) \geq 0$, then $\bar{x}$ is an optimal solution of the problem **(P)**.*

Proof. Let x be a feasible point for the problem **(P)**. Since f and $\xi^T g$ are $\rho_1 - (p, r)$-invex and $\rho_2 - (p, r)$-invex, respectively at $\bar{x}$ on D with respect to the same function η, $\forall x \in$ D, we have

$$\frac{1}{r}(e^{r(f(x) - f(\bar{x}))} - 1) \geq \frac{1}{p}\mathrm{d}f_{\bar{x}}(\mathrm{d}(exp_{\bar{x}}(p\eta(x, \bar{x}))) - \mathbf{I}) + \rho_1 \mathrm{d}^2(x, \bar{x}), \tag{15}$$

$$\frac{1}{r}(e^{r(\xi^T g(x) - \xi^T g(\bar{x}))} - 1) \geq \frac{\xi^T}{p}\mathrm{d}g_{\bar{x}}(\mathrm{d}(exp_{\bar{x}}(p\eta(x, \bar{x}))) - \mathbf{I}) + \rho_2 \mathrm{d}^2(x, \bar{x}). \tag{16}$$

Adding (15) and (16) we have

$$\begin{aligned} &\frac{1}{r}[(e^{r(f(x) - f(\bar{x}))} - 1 + e^{r(\xi^T g(x) - \xi^T g(\bar{x}))} - 1)] \\ &\geq \frac{1}{p}(\mathrm{d}f_{\bar{x}} + \xi^T \mathrm{d}g_{\bar{x}})(\mathrm{d}(exp_u(p\eta(x, \bar{x}))) - \mathbf{I}) + (\rho_1 + \rho_2)\mathrm{d}^2(x, \bar{x}). \end{aligned}$$

By KKT conditions and as $(\rho_1 + \rho_2) \geq 0$, we have

$$\frac{1}{r}[(e^{r(f(x) - f(\bar{x}))} - 1] \geq \frac{1}{r}(1 - e^{r(\xi^T g(x))}). \tag{17}$$

Without loss of generality, let $r > 0$ (in the case when $r < 0$ the proof is analogous; one should change only the direction of some inequalities below to the opposite one). Since x is a feasible solution of the problem **(P)**, then $g(x) \leq 0$ and $\xi \geq 0$ imply that $(1 - e^{r(\xi^T g(x))}) \geq 0$. From which we get $e^{r(f(x)-f(\bar{x}))} \geq 1$. Hence $f(x) \geq f(\bar{x})$ holds for all feasible $x \in \mathrm{D}$ of the problem **(P)**. Therefore, $\bar{x}$ is an optimal solution of the problem **(P)**. □

3.2. Mond-Weir Type Duality. Duality theory is the central part of optimization. In several optimization problems evaluating the dual maximum is comparatively easier than solving a primal minimization problem. Udriste [14] first introduced the concept of duality for a convex programming problem on a Riemannian manifold. Ferrera and Mititelu [5] developed a duality of Mond-Weir type for a vector mathematical programming problem involving invex functions on a differentiable manifold. In our work, we establish the duality results for the primal problem **(P)** involving $\rho - (p, r)$-invex functions over a differentiable manifold.

For the optimization problem **(P)**, the Mond-Weir type dual problem [5] **(MWD)** is defined in the following form

$$\textbf{(MWD)}\ \mathit{Maximize}\ f(u)$$

$$\textit{subject to}\ \mathrm{d}f_u + y^T \mathrm{d}g_u = 0,$$

$$y^T g(u) \geq 0,\ y \in \mathbb{R}^m_+,$$

where $f, g_i : M \to \mathbb{R},\ i = 1, 2, ..., m$ are differentiable functions. Let W_1 denote the set of all feasible solutions of **(MWD)**.

Remark 3.1. Throughout the remaining sections of this paper, without loss of generality, we assume $r > 0$ (in the case when $r < 0$ the proof is analogous; one should change only the direction of some inequalities to the opposite one but finally will get same results). The theorems will be proved only in the case when $p \neq 0, r \neq 0$ (other cases can be dealt with likewise).

We have established the following duality results between **(P)** and **(MWD)**.

Theorem 3.6. *(Weak Duality) Let x and (u, y) be the feasible solutions of **(P)** and **(MWD)**, respectively. Moreover, assume that f and $y^T g$ are $\rho_1 - (p, r)$-invex and $\rho_2 - (p, r)$-invex, respectively at u on M with respect to the same η and $(\rho_1 + \rho_2) \geq 0$, then* $\inf$ ***(P)*** $\geq \sup$ ***(MWD)***.

Proof. Since f and $y^T g$ are $\rho_1 - (p, r)$-invex and $\rho_2 - (p, r)$-invex, respectively at u with respect to the same η, we have

$$\frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geq \frac{1}{p}\mathrm{d}f_u(\mathrm{d}(exp_u(p\eta(x, u))) - \mathbf{I}) + \rho_1 \mathrm{d}^2(x, u), \tag{18}$$

$$\frac{1}{r}(e^{r(y^T g(x)-y^T g(u))} - 1) \geq \frac{y^T}{p}\mathrm{d}g_u(\mathrm{d}(exp_u(p\eta(x, u))) - \mathbf{I}) + \rho_2 \mathrm{d}^2(x, u). \tag{19}$$

Adding (18) and (19) we get

$$\frac{1}{r}[(e^{r(f(x)-f(u))}-1+e^{r(y^Tg(x)-y^Tg(u))}-1)]$$
$$\geq \frac{1}{p}(\mathrm{d}f_u+y^T\mathrm{d}g_u)(\mathrm{d}(exp_u(p\eta(x,u)))-\mathbf{I})+(\rho_1+\rho_2)\mathrm{d}^2(x,u).$$

Since (u, y) is a feasible solution of **(MWD)** and $(\rho_1+\rho_2)\geq 0$, we get

$$\frac{1}{r}[(e^{r(f(x)-f(u))}-1]\geq \frac{1}{r}(1-e^{r(y^Tg(x))}). \tag{20}$$

Since x is a feasible solution of **(P)** and $y^T\geq 0$, then we have $(1-e^{ry^Tg(x)})\geq 0$
$\Rightarrow e^{r(f(x)-f(u))}\geq 1$,
$\Rightarrow f(x)\geq f(u)$ holds for $\forall x\in$ D and $u\in W_1$.
Therefore, inf **(P)** $\geq$ sup **(MWD)**. □

Theorem 3.7. *(Strong Duality) Let $\bar{x}$ be an optimal solution of the problem **(P)** at which a constraint qualification $R(\bar{x})$ be satisfied. Then there exists $\xi\in\mathbb{R}^m_+$ such that $(\bar{x},\xi)$ is a feasible solution of **(MWD)**. Suppose that the hypotheses of the Weak Duality Theorem 3.6 hold, then $(\bar{x},\xi)$ is an optimal solution of the dual programming problem **(MWD)**, and the objective values of **(P)** and **(MWD)** are equal.*

Proof. Since a constraint qualification $R(\bar{x})$ is satisfied at $\bar{x}$, then from the KKT necessary conditions (12)-(14), $\exists\ \xi$ such that $(\bar{x},\xi)$ is a feasible solution of **(MWD)**. Since the conditions of the Weak Duality Theorem 3.6 hold, then $(\bar{x},\xi)$ is an optimal solution of the dual problem **(MWD)** and the objective values of **(P)** and **(MWD)** are equal. □

Theorem 3.8. *(Converse Duality) Let $(\bar{u}, y)$ be an optimal solution of the dual problem **(MWD)** such that $\bar{u}\in D$. If f and y^Tg are $\rho_1-(p,r)$-invex and $\rho_2-(p,r)$-invex, respectively at $\bar{u}$ on M, with respect to the same η with $(\rho_1+\rho_2)\geq 0$. Then $\bar{u}$ is an optimal solution of **(P)**.*

Proof. Since f and y^Tg are $\rho_1-(p,r)$-invex and $\rho_2-(p,r)$-invex, respectively at $\bar{u}$ with respect to the same η, we have

$$\frac{1}{r}(e^{r(f(x)-f(\bar{u}))}-1)\geq \frac{1}{p}\mathrm{d}f_{\bar{u}}(\mathrm{d}(exp_{\bar{u}}(p\eta(x,\bar{u})))-\mathbf{I})+\rho_1\mathrm{d}^2(x,\bar{u}), \tag{21}$$

$$\frac{1}{r}(e^{r(y^Tg(x)-y^Tg(\bar{u}))}-1)\geq \frac{y^T}{p}\mathrm{d}g_{\bar{u}}(\mathrm{d}(exp_{\bar{u}}(p\eta(x,\bar{u})))-\mathbf{I})+\rho_2\mathrm{d}^2(x,\bar{u}). \tag{22}$$

Adding (21) and (22) we get

$$\frac{1}{r}[(e^{r(f(x)-f(\bar{u}))}-1+e^{r(y^Tg(x)-y^Tg(\bar{u}))}-1)]$$
$$\geq \frac{1}{p}(\mathrm{d}f_{\bar{u}}+y^T\mathrm{d}g_{\bar{u}})(\mathrm{d}(exp_{\bar{u}}(p\eta(x,\bar{u})))-\mathbf{I})+(\rho_1+\rho_2)\mathrm{d}^2(x,\bar{u}).$$

Using feasibility of $(\bar{u}, y)$ and since $(\rho_1 + \rho_2) \geq 0$, we have

$$\frac{1}{r}(e^{r(f(x)-f(\bar{u}))} - 1) \geq \frac{1}{r}(1 - e^{ry^T g(x)}).$$

Since $x \in \mathrm{D}$ and $y \geq 0$ we have, $1 - e^{ry^T g(x)} \geq 0 \Rightarrow e^{r(f(x)-f(\bar{u}))} \geq 1 \Rightarrow f(x) \geq f(\bar{u})$. So $\bar{u}$ is an optimal solution of **(P)**. $\square$

3.3. Wolfe Type Duality. Motivated by the classical Wolfe type duality [10], for the optimization problem **(P)**, we define the Wolfe type dual **(WD)** in the following form

$$\textbf{(WD)}\ \mathit{Maximize}\ f(u) + \sum_{i=1}^{m} \xi_i g_i(u)$$

$$\mathit{subject\ to}\ \mathrm{d}f_u + \sum_{i=1}^{m} \xi_i \mathrm{d}g_{iu} = 0,$$

$$\xi_i \geq 0,\ i = 1, 2, ..., m,$$

where $f, g_i : M \to \mathbb{R},\ i = 1, 2, ..., m$ are differentiable functions. Let W_2 be the set of all feasible solutions of **(WD)**.
We have proved the following duality results between **(P)** and **(WD)**.

Theorem 3.9. *(Weak Duality) Let x and (u, ξ) be feasible solutions for **(P)** and **(WD)**, respectively. Moreover, assume that f and $\sum_{i=1}^{m} \xi_i g_i$ are $\rho_1 - (p, r)$-invex and $\rho_2 - (p, -r)$-invex, respectively at u on M with respect to the same η with $(\rho_1 + \rho_2) \geq 0$, then* $\inf$ ***(P)*** $\geq \sup$ ***(WD)***.

Proof. Since f and $\sum_{i=1}^{m} \xi_i g_i$ are $\rho_1 - (p, r)$-invex and $\rho_2 - (p, -r)$-invex with respect to the same η, we have

$$\frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geq \frac{1}{p}\mathrm{d}f_u(\mathrm{d}(exp_u(p\eta(x, \bar{u}))) - \mathbf{I}) + \rho_1 \mathrm{d}^2(x, u), \tag{23}$$

$$\begin{aligned} &\frac{-1}{r}(e^{-r(\sum_{i=1}^{m} \xi_i g_i(x) - \sum_{i=1}^{m} \xi_i g_i(u))} - 1) \\ &\geq \frac{1}{p}\sum_{i=1}^{m} \xi_i \mathrm{d}g_{iu}(\mathrm{d}(exp_u(p\eta(x, u))) - \mathbf{I}) + \rho_2 \mathrm{d}^2(x, u). \end{aligned} \tag{24}$$

Adding (23) and (24), we get

$$\begin{aligned} &\frac{1}{r}[(e^{r(f(x)-f(u))} - 1 + e^{-r(\sum_{i=1}^{m} \xi_i g_i(x) - \sum_{i=1}^{m} \xi_i g_i(u))} + 1)] \\ &\geq \frac{1}{p}(df_u + \sum_{i=1}^{m} \xi_i dg_{iu})(d(exp_u(p\eta(x, u))) - \mathbf{I}) + (\rho_1 + \rho_2)\mathrm{d}^2(x, u). \end{aligned} \tag{25}$$

Since (u, ξ) is a feasible solution of **(WD)**, we have

$$df_u + \sum_{i=1}^{m} \xi_i dg_{iu} = 0.$$

Again $(\rho_1 + \rho_2) \geq 0$, hence

$$e^{r(f(x)-f(u))} \geq e^{r(-\sum_{i=1}^{m}\xi_i g_i(x)+\sum_{i=1}^{m}\xi_i g_i(u))}. \tag{26}$$

Since x is a feasible solution of **(P)** and $\xi_i \geq 0$, we have $\sum_{i=1}^{m}\xi_i g_i(x) \leq 0$. Hence

$$e^{r(f(x)-f(u)} \geq e^{r(\sum_{i=1}^{m}\xi_i g_i(u))},$$

$$\Rightarrow f(x) - f(u) \geq \sum_{i=1}^{m}\xi_i g_i(u),$$

$$\Rightarrow f(x) \geq f(u) + \sum_{i=1}^{m}\xi_i g_i(u)$$

holds for $\forall x \in$ D and $u \in W_2$. Therefore, inf **(P)** $\geq$ sup **(WD)**. □

Theorem 3.10. *(Strong Duality) Let $\bar{x}$ be an optimal solution of the problem **(P)** at which a constraint qualification $R(\bar{x})$ be satisfied. Then there exists $\xi \in \mathbb{R}_+^m$ such that $(\bar{x}, \xi)$ is a feasible solution of **(WD)**. Suppose that the hypotheses of the Weak Duality Theorem 3.9 hold, then $(\bar{x}, \xi)$ is an optimal solution of the dual programming problem **(WD)**, and the objective values of **(P)** and **(WD)** are equal.*

Proof. Since a constraint qualification $R(\bar{x})$ is satisfied at $\bar{x}$, then from the KKT necessary conditions (12)-(14), $\exists\, \xi$ such that $(\bar{x}, \xi)$ is a feasible solution of **(WD)**. Since the conditions of the Weak Duality Theorem 3.9 hold, then $(\bar{x}, \xi)$ is an optimal solution of the dual problem **(WD)** and the objective values of **(P)** and **(WD)** are equal. □

Theorem 3.11. *(Converse Duality) Let (u, ξ) be an optimal solution of the dual problem **(WD)** such that $u \in D$. If f and $\sum_{i=1}^{m}\xi_i g_i$ are $\rho_1 - (p, r)$-invex and $\rho_2 - (p, -r)$-invex, respectively at u on M with respect to the same η with $(\rho_1 + \rho_2) \geq 0$. Then u is an optimal solution of **(P)**.*

Proof. We prove it by contradiction. Let u is not an optimal solution of **(P)**. Hence $\exists x \in D \ni f(x) < f(u)$. Since (u, ξ) is an optimal solution of **(WD)**, we have

$$f(u) + \sum_{i=1}^{m}\xi_i g_i(u) > f(x) + \sum_{i=1}^{m}\xi_i g_i(x), \tag{27}$$

$$or, f(x) - f(u) < -\sum_{i=1}^{m}\xi_i g_i(x) + \sum_{i=1}^{m}\xi_i g_i(u). \tag{28}$$

Since f is $\rho_1 - (p, r)$-invex and $\sum_{i=1}^{m}\xi_i g_i$ is $\rho_2 - (p, -r)$- invex we have from (26)

$$e^{r(f(x)-f(u))} \geq e^{r(-\sum_{i=1}^{m}\xi_i g_i(x)+\sum_{i=1}^{m}\xi_i g_i(u))},$$

$$\Rightarrow f(x) - f(u) \geq -\sum_{i=1}^{m}\xi_i g_i(x) + \sum_{i=1}^{m}\xi_i g_i(u),$$

which is a contradiction to (28). Hence u is an optimal solution of **(P)**. □

3.4. Mixed Type Duality. For the problem **(P)**, we consider the mixed type dual problem **(MDP)** in the following form

$$\textbf{(MDP)}\ Maximize\ f(u)+\sum_{i=1}^{m}\xi_i g_i(u)$$

$$subject\ to\ \mathrm{d}f_u+\sum_{i=1}^{m}\xi_i \mathrm{d}g_{iu}=0,$$

$$\sum_{i=1}^{m}\xi_i g_i(u)\geq 0,\ \xi_i\geq 0,\ i=1,2,...,m.$$

Let W_3 be the set of all feasible solutions of **(MDP)**.
We have established the following duality results between **(P)** and **(MDP)**, whose proofs are omitted as they are very similar to Theorem 3.9 to Theorem 3.11.

Theorem 3.12. *(Weak Duality) Let x and (u,ξ) be feasible solutions for **(P)** and **(MDP)** respectively. Moreover, we assume that f and $\sum_{i=1}^{m}\xi_i g_i$ are $\rho_1-(p,r)$-invex and $\rho_2-(p,-r)$-invex, respectively at u on M with respect to the same η with $(\rho_1+\rho_2)\geq 0$, then* inf ***(P)*** $\geq$ sup ***(MDP)***.

Theorem 3.13. *(Strong Duality) Let $\bar{x}$ be an optimal solution of the problem **(P)** at which a constraint qualification $R(\bar{x})$ be satisfied. Then there exists $\xi\in\mathbb{R}_+^m$, such that $(\bar{x},\xi)$ is a feasible solution of **(MDP)**. Suppose that the hypotheses of the Weak Duality Theorem 3.12 hold, then $(\bar{x},\xi)$ is an optimal solution of the dual programming problem **(MDP)**, and the objective values of **(P)** and **(MDP)** are equal.*

Theorem 3.14. *(Converse Duality) Let (u,ξ) be an optimal solution of the dual problem **(MDP)** such that $u\in D$. If f and $\sum_{i=1}^{m}\xi_i g_i$ are $\rho_1-(p,r)$-invex and $\rho_2-(p,-r)$-invex, respectively at u on M with respect to the same η with $(\rho_1+\rho_2)\geq 0$. Then u is an optimal solution of **(P)**.*

4. Conclusions

The notion of $\rho-(p,r)$-invex functions on differentiable manifolds is introduced in this paper which generalizes invex functions. We establish optimality conditions and duality results under $\rho-(p,r)$-invexity assumptions for a general nonlinear programming problem that is built upon on differentiable manifolds. In future we aim to study variational problems and control problems on differentiable manifolds under generalized invexity assumptions.
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