J. Appl. Math. & Informatics Vol. **32***(2014), No. 3 - 4, pp. 465 - 474*
http://dx.doi.org/10.14317/jami.2014.465

AN EXTENSION OF GENERALIZED EULER POLYNOMIALS OF THE SECOND KIND

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Abstract. Many mathematicians have studied various relations beween Euler number E_n, Bernoulli number B_n and Genocchi number G_n (see [1-18]). They have found numerous important applications in number theory. Howard, T.Agoh, S.-H.Rim have studied Genocchi numbers, Bernoulli numbers, Euler numbers and polynomials of these numbers [1,5,9,15]. T.Kim, M.Cenkci, C.S.Ryoo, L. Jang have studied the q-extension of Euler and Genocchi numbers and polynomials [6,8,10,11,14,17]. In this paper, our aim is introducing and investigating an extension term of generalized Euler polynomials. We also obtain some identities and relations involving the Euler numbers and the Euler polynomials, the Genocchi numbers and Genocchi polynomials.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.
Key words and phrases : the generalized Euler polynomials of the second kind, Euler numbers, Genocchi numbers, Bernoulli numbers, Stirling numbers of the first kind, Stirling numbers of the second kind.

1. Introduction

The Genocchi number G_n, the Bernoulli number $B_n (n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\})$ and the Euler number E_n are defined by the following generating function.

$$\begin{aligned} \frac{2}{e^t+1} &= \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \qquad (\mid t \mid < \pi), \\ \frac{t}{e^t-1} &= \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \qquad (\mid t \mid < 2\pi), \\ \frac{2t}{e^t+1} &= \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} \qquad (\mid t \mid < \pi), \end{aligned} \tag{1.1}$$

Received October 30, 2013. Revised January 17, 2014. Accepted January 20, 2014.
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For a real or complex parameter α, the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{Z}$, and the generalized Euler polynomials $E_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{Z}$ are defined by the following generating functions (see, for details, [4, p.253 et seq.], [14, Section 2.8] and [18, Section 1.6]).

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} \quad (|\, t \,| \quad < 2\pi \quad) \tag{1.2}$$

and

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!} \quad (|\, t \,| \quad < \pi \quad). \tag{1.3}$$

The Genocchi polynomials $G_n(x)$ of order $k \in \mathbb{N}$ are defined by

$$\left(\frac{2t}{e^t+1}\right)^{k} e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!} \quad (|\, t \,| \quad < \pi \quad). \tag{1.3}$$

The Euler numbers E_n and Euler polynomials $E_n(x)$ are defined by

$$\begin{aligned} E_0 = 1, \quad E_n &= -\sum_{k=1}^{[\frac{n}{2}]} \binom{n}{2k} E_{n-2k} \quad (n \geq 1), \\ E_n(x) &= \frac{1}{2^n}\sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} (2x-1)^{n-2k} E_{2k} \quad (n \geq 0), \end{aligned} \tag{1.4}$$

where $[x]$ is the greatest integer not exceeding x (see [6,8,9,10,11,13,15,16]).

By(1.1), we have

$$G_{2n+1} = B_{2n+1} = 0 \quad (n \in \mathbb{N}), \quad G_n = 2(1-2^n)B_n, \tag{1.5}$$

where $\mathbb{N}$ is the set of positive integers. The Genocchi number G_n satisfy the recurrence relation

$$G = 0, \quad G_1 = 1, \quad G_n = -\frac{1}{2}\sum_{k=1}^{n-1} \binom{n}{k} G_k \quad (n \geq 2). \tag{1.6}$$

Therefore, we find out that $G_2 = -1$, $G_4 = 1$, $G_6 = -3$, $G_8 = 17$, $G_{10} = -155$, $G_{12} = 2073$, $G_{14} = -38227, \cdots$. That is, $G_{2n+1} = 0 (n \geq 1)$.

The Stirling number of the first kind $s(n,k)$ can be defined by means of

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k. \tag{1.7}$$

or by the generating function

$$(\log(1+x))^k = k!\sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!}. \tag{1.8}$$

We get (1.9) from (1.7) and (1.8)

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k), \tag{1.9}$$

with $s(n,0) = 0(n > 0)$, $s(n,n) = 1(n \geq 0)$, $s(n,1) = (-1)^{n-1}(n-1)!(n > 0)$, $s(n,k) = 0(k > n$ or $k < 0)$. Stirling number of the second kind $S(n,k)$ can be defined by

$$x^n = \sum_{k=0}^{n} S(n,k)\frac{x^n}{n!}, \tag{1.10}$$

or by the generating function

$$(e^x - 1)^k = k!\sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!}. \tag{1.11}$$

We get (1.12) from (1.10) and (1.11)

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \tag{1.12}$$

with $S(n,0) = (n > 0)$, $S(n,n) = 1(n \geq 0)$, $S(n,1) = 1(n > 0)$, $S(n,k) = 0$ $(k > n$ or $k < 0)$.

We begin with discussing Euler numbers, Genocchi numbers, Bernoulli numbers, Stirling numbers of the first kind, Stirling numbers of the second kind. In the paper, we organized the entire contents as follows. In Section 2, we define the extension term of generalized Euler polynomials of the second kind and prove them. We also study some interesting relations about $\widetilde{\mathcal{E}}_n(x)$ and $\widetilde{\mathcal{E}}_n^{(\alpha)}(x)$, a polynomial of x and α with integers coefficients. In Section3, the extension term of generalized Euler polynomials of the second kind will be used to induce the main results of this paper. We also obtain some identities involving the Genocchi numbers, Genocchi polynomials, the Euler numbers, Euler polynomials and prove them.

2. Some relations within the an extension terms of the generalized Euler polynomials of the second kind

In this section, we study some relations of the extension term of generalized Euler polynomials of the second kind and research for properties between $\widetilde{\mathcal{E}}_n(x)$ and $\widetilde{\mathcal{E}}_n^{(\alpha)}(x)$. First of all, we define the generalized Euler polynomials $\widetilde{\mathcal{E}}_n(x)$ of the second kind as follows. In [7], we introduced the generalized Euler polynomials $\widetilde{\mathcal{E}}_n(x)$ of the second kind and investigate their properties. First of all, we introduce the generalized Euler polynomials $\widetilde{\mathcal{E}}_n(x)$ of the second kind as follows. This completes with the usual convention of replacing $\widetilde{\mathcal{E}}_n^{(x)}$ by $\widetilde{\mathcal{E}}_n(x)$(see, for details, [7]).

Definition 2.1. Let x be a real or complex parameter, $n \geq k(n,k \in \mathbb{N})$. Then we define

$$\left(\frac{2e^t}{e^{2t}+1}\right)^x = \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_n(x)\frac{t^n}{n!} \quad (\mid t \mid < \frac{\pi}{2}), \tag{2.1}$$

We derive that

$$\widetilde{\mathcal{E}}_n(x) = \sum_{k=0}^{n} c(n,k)x^k, \tag{2.2}$$

where

$$c(n,k) = \sum_{j=0}^{k} \binom{n}{j} (-1)^{k-j} \sum_{l=k-j}^{n-j} s(l,k-j)S(n-j,l)2^{n-j-l}. \tag{2.3}$$

For a real and complex parameter α, the generalized Euler polynomials $\widetilde{\mathcal{E}}_n^{(\alpha)}(x)$, each of degree n in x as well as in α, are defined by means of the generating function.

Definition 2.2. Let α be a real or complex parameter. Then we define

$$\left(\frac{2e^t}{e^{2t}+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_n^{(\alpha)}(x)\frac{t^n}{n!} \quad (| \, t \, |< \frac{\pi}{2}). \tag{2.4}$$

By using Definition 2.2, we have the addition theorem of polynomials $\widetilde{\mathcal{E}}_n^{(\alpha)}(x)$ and the relation of polynomials $\widetilde{\mathcal{E}}_n^{(\alpha)}(x)$ and numbers $\widetilde{\mathcal{E}}_n^{(\alpha)}$.

Theorem 2.3. *(Addition theorem) Let $\alpha, x, y \in \mathbb{C}$ and n be non-negative integers. Then we get*

$$\widetilde{\mathcal{E}}_n^{(\alpha)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} \widetilde{\mathcal{E}}_{n-k}^{(\alpha)}(x)y^k.$$

Proof. For n be non-negative integers, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_n^{(\alpha)}(x+y)\frac{t^n}{n!} &= \left(\frac{2e^t}{e^{2t}+1}\right)^{\alpha} e^{(x+y)t} = \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_n^{(\alpha)}(x)\frac{t^n}{n!} \sum_{n=0}^{\infty} y^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} \widetilde{\mathcal{E}}_{n-k}^{(\alpha)}(x)y^k \frac{t^n}{n!}.\end{aligned}$$

By comparing the coefficients of both side, we complete the proof of the Theorem 2.3. ☐

By using the Definition 2.2, we have the following Theorem 2.4.

Theorem 2.4. *Let $n \geq k (n,k,l \in \mathbb{N})$. Then we derive that*

$$\widetilde{\mathcal{E}}_n^{(\alpha)}(x) = \sum_{l=0}^{n} \rho^{(\alpha)}(n,l)x^l, \tag{2.5}$$

where

$$\rho^{(\alpha)}(n,l) = \binom{n}{l} \sum_{k=0}^{n-l} c(n-l,k)\alpha^k. \tag{2.6}$$

Proof. By (2.1) and (2.2), we easily have

$$\left(\frac{2e^t}{e^{2t}+1}\right)^{\alpha}=\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_n(\alpha)\frac{t^n}{n!}(|\,t\,|<\frac{\pi}{2}),\quad \widetilde{\mathcal{E}}_n(\alpha)=\sum_{k=0}^{n}c(n,k)\alpha^k.$$

By Definition 2.2, (1.4) and (1.8) we have

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_n^{(\alpha)}(x)\frac{t^n}{n!}&=\left(\frac{2e^t}{e^{2t}+1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}c(n,k)\alpha^k\frac{t^n}{n!}\sum_{n=0}^{\infty}x^k\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\sum_{k=0}^{n-l}c(n-l,k)\alpha^k x^l\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\rho^{(\alpha)}(n,l)x^l\frac{t^n}{n!}\end{aligned}$$

which readily yields

$$\widetilde{\mathcal{E}}_n^{(\alpha)}(x)=\sum_{l=0}^{n}\rho^{(\alpha)}(n,l)x^l.$$

Therefore, we complete the proof of Theorem 2.4 □

Remark 2.1. From (2.3) and Theorem 2.4, we find out that

$\rho^{(\alpha)}(0,0)=1,$

$\rho^{(\alpha)}(1.0)=0,\rho^{(\alpha)}(1,1)=1,$

$\rho^{(\alpha)}(2,0)=-\alpha,\rho^{(\alpha)}(2,1)=0,\rho^{(\alpha)}(2,2)=1,$

$\rho^{(\alpha)}(3,0)=0,\rho^{(\alpha)}(3,1)=-3\alpha,\rho^{(\alpha)}(3,2)=0,\rho^{(\alpha)}(3,3)=1,$

$\rho^{(\alpha)}(4,0)=2\alpha+3\alpha^2,\rho^{(\alpha)}(4,1)=0,\rho^{(\alpha)}(4,2)=-6\alpha,\rho^{(\alpha)}(4,3)=0,\rho^{(\alpha)}(4,4)=1,$

$\cdots$

Thus, we know that $\widetilde{\mathcal{E}}_n^{(\alpha)}(x)$ is a polynomial of x. Setting $n=1,2,3,4,5$ in Theorem 2.4, we get to

$$\begin{aligned}&\widetilde{\mathcal{E}}_0^{(\alpha)}(x)=1,\quad \widetilde{\mathcal{E}}_1^{(\alpha)}(x)=x,\\&\widetilde{\mathcal{E}}_2^{(\alpha)}(x)=x^2-\alpha,\quad \widetilde{\mathcal{E}}_3^{(\alpha)}(x)=x^3-3\alpha x,\\&\widetilde{\mathcal{E}}_4^{(\alpha)}(x)=x^4-6\alpha x^2+3\alpha^2+2\alpha,\quad \widetilde{\mathcal{E}}_5^{(\alpha)}(x)=x^5-10\alpha x^3+(15\alpha^2+10\alpha)x.\end{aligned}$$

We also find out an extension terms of the generalized Euler polynomials that can be represented by $c(n,k)$ with Stirling numbers of the first kind, Stirling numbers of the second kind.

Since

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_n^{(\alpha)}(-x)\frac{t^n}{n!}&=\left(\frac{2e^t}{e^{2t}+1}\right)^{\alpha}e^{-xt}=\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_n^{(\alpha)}\frac{t^n}{n!}\sum_{n=0}^{\infty}(-1)^n x^n\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\sum_{k=0}^{n-l}c(n-l,k)\alpha^k(-1)^l x^l\frac{t^n}{n!},\end{aligned}$$

we have the following theorem.

Theorem 2.5. *Let $n, k \in \mathbb{N}$, then by (1.1) and Definition 2.2, we have*

$$\widetilde{\mathcal{E}}_n^{(\alpha)}(x) = (-1)^n \widetilde{\mathcal{E}}_n^{(\alpha)}(-x).$$

3. Some relations between an extension of the generalized Euler polynomials and Euler numbers, Genocchi numbers and themselves

In this section, we access some relations between an extension terms of generalized Euler polynomials and Euler numbers, Euler polynomials, Genocchi numbers, Genocchi polynomials of order k. We construct relations among an extension terms of generalized Euler polynomials themselves.

Theorem 3.1. *Let $n \geq k(n, k \in \mathbb{N})$. Relation between $\widetilde{\mathcal{E}}_n^{(\alpha)}(x)$ and Genocchi numbers G_n, we have*

$$\widetilde{\mathcal{E}}_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} (n-k)! \sum_{\substack{v_1+\cdots+v_\alpha=n-k \\ 0\leq v_i}} \frac{G^*(v_1)G^*(v_2)\cdots G^*(v_\alpha)}{(v_1!v_2!\cdots v_\alpha!)} x^k \tag{3.1}$$

where

$$G^*(n) = \sum_{k=0}^{n} \binom{n}{k} G_{k+1} \frac{2^k}{k+1}. \tag{3.2}$$

Proof. By (1.1), we have

$$\frac{2}{e^{2t}+1} = \sum_{n=0}^{\infty} G_{n+1} \frac{2^n}{n+1} \frac{t^n}{n!} \qquad (|\, t \,| < \pi),$$

$$\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} G_{k+1} \frac{2^k}{k+1} \frac{t^n}{n!}.$$

Let us that

$$G^*(n) = \sum_{k=0}^{n} \binom{n}{k} G_{k+1} \frac{2^k}{k+1}.$$

Therefore, we have

$$\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} G^*(n) \frac{t^n}{n!}. \tag{3.3}$$

And we expand to degree of α, we obtain

$$\left(\frac{2e^t}{e^{2t}+1}\right)^\alpha = \sum_{n=0}^{\infty} \sum_{\substack{v_1+\cdots+v_\alpha=n \\ 0\leq v_i}} \frac{G^*(v_1)G^*(v_2)\cdots G^*(v_\alpha)}{(v_1!v_2!\cdots v_\alpha!)} t^n \tag{3.4}$$

We deduce that by the generalized Euler polynomials

$$\left(\frac{2e^t}{e^{2t}+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(n-k)! \sum_{\substack{v_1+\cdots+v_\alpha=n-k\\ 0\le v_i}} \frac{G^*(v_1)G^*(v_2)\cdots G^*(v_\alpha)}{(v_1!v_2!\cdots v_\alpha!)} x^k \frac{t^n}{n!}$$

By (2.4), we may immediately obtain Theorem 3.1. This completes the proof of Theorem 3.1. □

We find out that $G^*(0)=1,\ G^*(1)=0,\ G^*(2)=-1,\ G^*(3)=0,\ G^*(4)=5,\ G^*(5)=0,\ G^*(6)=-61,\cdots$.

Remark 3.1. Let $n \geq k(n,k \in \mathbb{N})$. Then we have

$$\rho^{(\alpha)}(n,k) = \binom{n}{k}(n-k)! \sum_{\substack{v_1+\cdots+v_\alpha=n-k\\ 0\le v_i}} \frac{G^*(v_1)G^*(v_2)\cdots G^*(v_\alpha)}{(v_1!v_2!\cdots v_\alpha!)}, \tag{3.5}$$

where

$$G^*(n) = \sum_{k=0}^{n}\binom{n}{k}G_{k+1}\frac{2^k}{k+1}, \quad \rho^{(\alpha)}(n,l) = \binom{n}{l}\sum_{k=0}^{n-l} c(n-l,k)\alpha^k.$$

Theorem 3.2. *Let $n \geq k(n,k \in \mathbb{N})$. Then we obtain*

$$k!\rho^{(\alpha)}(n,k) = \widetilde{\mathcal{E}}_{n-k}^{(\alpha)}\frac{n!}{(n-k)!}. \tag{3.6}$$

Proof. By applying Theorem 2.4, we have

$$k!\rho^{(\alpha)}(n,k) = \frac{d^k}{dx^k}\widetilde{\mathcal{E}}_{n}^{(\alpha)}(x) \mid_{x=0} \tag{3.7}$$

It follows from Definition 2.2 that

$$\left(\frac{2e^t}{e^{2t}+1}\right)^{\alpha} t^k = \sum_{n=0}^{\infty}\frac{d^k}{dx^k}\left(\widetilde{\mathcal{E}}_{n}^{(\alpha)}(x)\mid_{x=0}\right)\frac{t^n}{n!} = k!\sum_{n=k}^{\infty}\rho^{(\alpha)}(n,k)\frac{t^n}{n!} \tag{3.8}$$

On the other hand, we have from (2.1)

$$\left(\frac{2e^t}{e^{2t}+1}\right)^{\alpha} t^k = \sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n}^{(\alpha)}\frac{t^n}{n!}t^k \tag{3.9}$$

Substituting (3.9) in (3.8) we get

$$k!\sum_{n=k}^{\infty}\rho^{(\alpha)}(n,k)\frac{t^n}{n!} = \sum_{n=k}^{\infty}\widetilde{\mathcal{E}}_{n-k}^{(\alpha)}\frac{n!}{(n-k)!}\frac{t^n}{n!} \tag{3.10}$$

By (3.10), we may immediately obtain Theorem 3.2. This completes the proof of Theorem 3.2. □

Theorem 3.3. *Let $n \geq k (n, k \in \mathbb{N})$. Relation between $\widetilde{\mathcal{E}}_n^{(\alpha)}(x)$ and Euler numbers E_n, we have*

$$\widetilde{\mathcal{E}}_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} (n-k)! \sum_{\substack{v_1+\cdots+v_\alpha=n-k \\ 0 \leq v_i}} \frac{E^*(v_1)E^*(v_2)\cdots E^*(v_\alpha)}{(v_1!v_2!\cdots v_\alpha!)} x^k \tag{3.11}$$

where

$$E^*(n) = \sum_{k=0}^{n} \binom{n}{k} E_k 2^k. \tag{3.12}$$

Proof. By definition (1.1)

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} E_n \frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} nE_{n-1} \frac{t^n}{n!} \qquad (|\, t \,| < \pi),$$

Therefore, according to (3.1), (3.2), we have

$$G^*(n) = \sum_{k=0}^{n} \binom{n}{k} G_{k+1} \frac{2^k}{k+1} \quad = \sum_{k=0}^{n} \binom{n}{k} E_k 2^k = E^*(n).$$

By Theorem 3.1, we may immediately obtain Theorem 3.3. □

We find out that $E^*(0) = 1,\ E^*(1) = 0,\ E^*(2) = -1,\ E^*(3) = 0,\ E^*(4) = 5,\ E^*(5) = 0,\ E^*(6) = -61, \cdots$. Thus, we easily see that $G^*(n) = E^*(n)$.

By Definition 2.2, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_n^{(\alpha)}(x) \frac{t^n}{n!} &= \left(\frac{2e^t}{e^{2t}+1}\right)^{\alpha} e^{xt} = \left(\frac{2}{e^{2t}+1}\right)^{\alpha} e^{(\alpha+x)t} \\ &= \sum_{n=0}^{\infty} 2^n E_n^{(\alpha)}\left(\frac{x}{2} + \frac{\alpha}{2}\right) \frac{t^n}{n!}.\end{aligned}$$

Therefore, we have the following theorem.

Theorem 3.4. *Let α be a real or complex parameter. Relation between $\widetilde{\mathcal{E}}_n^{(\alpha)}(x)$ and Euler polynomials $E_n^{(\alpha)}(x)$, we have*

$$\widetilde{\mathcal{E}}_n^{(\alpha)}(x) = 2^n E_n^{(\alpha)}\left(\frac{x}{2} + \frac{\alpha}{2}\right) \tag{3.13}$$

For $\alpha = 1$ in (3.13), we have the following corollary.

Corollary 3.5. *For $\alpha = 1$, we have*

$$\widetilde{\mathcal{E}}_n(x) = 2^n E_n\left(\frac{x}{2} + \frac{1}{2}\right)$$

Theorem 3.6. *Let $n, \alpha \in \mathbb{N}$. Relation between $\widetilde{\mathcal{E}}_n^{(\alpha)}(x)$ and Genocchi polynomials $G_n^{(\alpha)}(x)$, we have*

$$\frac{1}{2^n} \binom{\alpha+n}{\alpha} \alpha! \widetilde{\mathcal{E}}_n^{(\alpha)}(x) = G_{\alpha+n}^{(\alpha)}\left(\frac{x}{2} + \frac{\alpha}{2}\right) \tag{3.14}$$

Proof. By Definition 2.2,

$$\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_n^{(\alpha)}(x)\frac{t^n}{n!}=\left(\frac{2e^t}{e^{2t}+1}\right)^{\alpha}e^{xt}=\frac{1}{(2t)^{\alpha}}\left(\frac{4t}{e^{2t}+1}\right)^{\alpha}e^{(\alpha+x)t}$$
$$=\frac{1}{2^{\alpha}}\sum_{n=0}^{\infty}\frac{2^{\alpha+n}G_{\alpha+n}^{(\alpha)}(\frac{x}{2}+\frac{\alpha}{2})}{\binom{\alpha+n}{\alpha}\alpha!}\frac{t^n}{n!} \tag{3.15}$$
$$=\sum_{n=0}^{\infty}\frac{2^{n}G_{\alpha+n}^{(\alpha)}(\frac{x}{2}+\frac{\alpha}{2})}{\binom{\alpha+n}{\alpha}\alpha!}\frac{t^n}{n!}.$$

Therefore, we may immediately obtain Theorem 3.6. This completes the proof of Theorem 3.6. □

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