# OSCILLATION OF HIGHER ORDER STRONGLY SUPERLINEAR AND STRONGLY SUBLINEAR DIFFERENCE EQUATIONS ${ }^{\dagger}$ 

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#### Abstract

We establish some new criteria for the oscillation of $m$ th order nonlinear difference equations. We study the case of strongly superlinear and the case of strongly sublinear equations subject to various conditions. We also present a sufficient condition for every solution to be asymptotic at $\infty$ to a factorial expression $(t)^{(m-1)}$.

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## 1. Introduction

In what follows, we shall denote $\mathbb{N}=\{0,1, \ldots\}, \mathbb{N}(a)=\{a, a+1, \ldots\}$ where $a \in \mathbb{N}$ and $\mathbb{N}(a, b)=\{a, a+1, \ldots, b\}, b \in \mathbb{N}(a)$.

Consider the $m$ th order nonlinear difference equation

$$
\begin{equation*}
\triangle^{m} x(t)+f(t, x(t))=0 \tag{1}
\end{equation*}
$$

where $\triangle$ is the forward difference operator defined by $\triangle x(t)=x(t+1)-x(t)$, $m$ is a positive even integer. We shall assume that

$$
\left\{\begin{array}{l}
f: \mathbb{N}\left(t_{0}\right) \times \mathbb{R} \rightarrow \mathbb{R} \text { is continuous, } \operatorname{sgn} f(t, x)=\operatorname{sgn} x \text { and }  \tag{2}\\
f(t, x) \leq f(t, y) \text { for } x \leq y \text { and } t \geq t_{0} \in \mathbb{N}\left(t_{0}\right)
\end{array}\right.
$$

By a solution of equation (1), we mean a nontrivial sequence $\{x(t)\}$ satisfying equation (1) for all $t \in \mathbb{N}\left(t_{0}\right)$, where $t_{0} \in \mathbb{N}$. A solution $\{x(t)\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative and it is

[^0]nonoscillatory otherwise. An equation is said to be oscillatory if all its solutions are oscillatory.

Equation (1) (or the function $f$ ) is said to be strongly superlinear if there exists a constant $\beta>1$ such that

$$
\begin{equation*}
\frac{|f(t, x)|}{|x|^{\beta}} \leq \frac{|f(t, y)|}{|y|^{\beta}} \text { for }|x| \leq|y|, x y>0, t>t_{0} \tag{3}
\end{equation*}
$$

and it is said to be strongly sublinear if there exists a constant $\gamma \in(0,1)$ such that

$$
\begin{equation*}
\frac{|f(t, x)|}{|x|^{\gamma}} \geq \frac{|f(t, y)|}{|y|^{\gamma}} \text { for }|x| \leq|y|, x y>0, t \geq t_{0} \tag{4}
\end{equation*}
$$

(3) holds with $\beta=1$, then equation (1) is called superlinear and (4) holds with $\gamma=1$ is called sublinear.

The literature on oscillation of solutions of difference equantions is almost devoted to study of equation (1) when $m=1$ and 2 and for recent contribution we refer to Agarwal et.al. [1, 2, 3]. Only few results are available for the oscillation of equation (1) when $m>2$, see Agarwal et.al.[2, 4, 5].

Therefore, the purpose of this paper is to establish some new results for the oscillation of strongly superlinear and strongly sublinear difference equations. We also provide conditions, which guarantee that every solution defined for all large $t \in \mathbb{N}\left(t_{0}\right)$ is asymptotic at $\infty$ to $(t)^{(m-1)}$.

The obtained results improve and unify these which have appeared in the recent literature.

## 2. Preliminaries

We shall need the following lemmas given in [2].
Lemma 2.1. (Discrete Toylors Formula) Let $x(t)$ be defined on $\mathbb{N}\left(t_{0}\right)$. Then for all $t \in \mathbb{N}\left(t_{0}\right)$ and $0 \leq n \leq j-1$

$$
\begin{equation*}
\Delta^{n} x(t)=\sum_{i=n}^{j-1} \frac{\left(t-t_{0}\right)^{(i-n)}}{(i-n)!} \Delta^{i} x\left(t_{0}\right)+\sum_{l=t_{0}}^{t-j+n} \frac{(t-l-1)^{(j-n-1)}}{(j-n-1)!} \Delta^{j} x(l) \tag{5}
\end{equation*}
$$

Further, for all $t \in \mathbb{N}\left(t_{0}, z\right)$, where $z \in \mathbb{N}\left(t_{0}\right)$ and $0 \leq n \leq j-1$

$$
\begin{align*}
\Delta^{n} x(t)= & \sum_{i=n}^{j-1}(-1)^{i-n} \frac{(z+i-n-1-t)^{(i-n)}}{(i-n)!} \Delta^{i} x(z) \\
& -(-1)^{j-n-1} \sum_{l=t}^{z-1} \frac{(l+j-n-1-t)^{(j-n-1)}}{(j-n-1)!} \Delta^{j} x(l) \tag{6}
\end{align*}
$$

Lemma 2.2. (Discrete Kneser's Theorem) Let $x(t)$ be defined on $\mathbb{N}\left(t_{0}\right), x(t)>0$ and $\Delta^{m} x(t)$ be eventually of one sign on $\mathbb{N}\left(t_{0}\right)$. Then there exists an integer $k$,
$0 \leq k \leq m$ with $m+k$ odd for $\Delta^{m} x(t) \leq 0$ and $(m+k)$ even for $\Delta^{m} x(t) \geq 0$ such that

$$
\left\{\begin{array}{l}
k \leq m-1 \text { implies }(-1)^{i+k} \Delta^{i} x(t)>0 \text { for all } t \in \mathbb{N}\left(t_{0}\right), k \leq i \leq m-1,  \tag{7}\\
k \geq 1 \text { implies } \Delta^{i} x(t)>0 \text { for all } t \in \mathbb{N}\left(t_{0}\right), 1 \leq i \leq k-1
\end{array}\right.
$$

Lemma 2.3. Let $x(t)$ be defined on $\mathbb{N}\left(t_{0}\right)$ and $x(t)>0$ with $\Delta^{m} x(t) \leq 0$ for $t \in \mathbb{N}\left(t_{0}\right)$ and not identically zero. Then there exists large $t_{1} \in \mathbb{N}\left(t_{0}\right)$ such that

$$
\begin{equation*}
x(t) \geq \frac{\left(t-t_{1}\right)^{m-1}}{(m-1)!} \Delta^{m-1} x\left(2^{m-k-1} t\right) \text { for } t \geq t_{1} \tag{8}
\end{equation*}
$$

where $k$ is as in Lemma 2.2. Furthermore, if $x(t)$ is increasing, then

$$
\begin{equation*}
x(t) \geq \frac{1}{(m-1)!}\left(\frac{t}{2^{m-1}}\right)^{(m-1)} \Delta^{m-1} x(t), t \geq 2^{m-1} t_{1} \tag{9}
\end{equation*}
$$

Lemma 2.4. (Gronwal Inequality) Let for all $t \in \mathbb{N}\left(t_{0}\right)$ the following inequality be satisfied

$$
\begin{equation*}
x(t) \leq p(t)+q(t) \sum_{l=t_{0}}^{t-1} f(l) x(l) \tag{10}
\end{equation*}
$$

where $\{p(t)\},\{q(t)\},\{f(t)\}$ and $\{x(t)\}$ are non-negative real-valued sequence defined on $\mathbb{N}\left(t_{0}\right)$. Then for all $t \in \mathbb{N}\left(t_{0}\right)$,

$$
\begin{equation*}
x(t) \leq p(t)+q(t) \sum_{l=t_{0}}^{t-1} p(l) f(l) \prod_{r=l+1}^{t-1}(1+q(r) f(r)) . \tag{11}
\end{equation*}
$$

## 3. Oscillation Criteria

We shall study the oscillatory behavior of all solutions of equation (1) when it is either strongly superlinear or strongly sublinear.

We begin with strongly superlinear case of equation(1).
Theorem 3.1. Suppose that equation (1) is strongly superlinear. If

$$
\begin{equation*}
\sum_{t=t_{0}}^{\infty} \sum_{s=t}^{\infty}\left(\prod_{i=0}^{k-1}\left(1-\frac{t}{s-i}\right)\right)^{\beta}(s-t)^{(m-k-1)}\left|f\left(s, c(s)^{(k-1)}\right)\right|=\infty, t \geq t_{0} \tag{12}
\end{equation*}
$$

for some $c \neq 0$ and $k \in\{1,3, \ldots, m-1\}$, then equation (1) is oscillatory.
Proof. Let $\{x(t)\}$ be a non-oscillatory solution of equation (1), say $x(t)>0$ for $t \geq t_{0} \in \mathbb{N}\left(t_{0}\right)$. By Lemma 2.2, there exists an integer $k \in\{1,3, \ldots, m-1\}$ such that (7) holds for $t \geq t_{1} \geq t_{0}$.

Clearly, $\Delta^{k-1} x(t)$ is positive and increasing for $t \geq t_{1}$. Thus from (5), we find for $s \geq t \geq t_{0}$

$$
x(s)=\sum_{i=0}^{k-2} \frac{(s-t)^{(i)}}{i!} \Delta^{i} x(t)+\sum_{l=t}^{s-k+1} \frac{(s-l-1)^{(k-2)}}{(k-2)!} \Delta^{k-1} x(l)
$$

$$
\begin{equation*}
\geq \frac{(s-t)^{(k-1)}}{(k-1)!} \Delta^{k-1} x(t), \quad \text { for } s \geq t \geq t_{1} \tag{13}
\end{equation*}
$$

On the other hand, there exists a constant $c>0$ such that

$$
\begin{equation*}
x(s) \geq c(s)^{(k-1)} \text { for } s \geq t_{1} \tag{14}
\end{equation*}
$$

From (6) with $n=k$ and $j=m$, and equation (1), we have

$$
\begin{equation*}
\Delta^{k} x(t) \geq \sum_{s=t}^{\infty} \frac{(s-t+m-k-1)^{(m-k-1)}}{(m-k-1)!} f(s, x(s)) \tag{15}
\end{equation*}
$$

Using the strong superlinearity of $f$ we obtain

$$
\begin{equation*}
\Delta^{k} x(t) \geq \sum_{s=t}^{\infty} \frac{(s-t+m-k-1)^{(m-k-1)}}{(m-k-1)!} f\left(s, c(s)^{(k-1)}\right)\left(\frac{x(s)}{c(s)^{(k-1)}}\right)^{\beta}, t \geq t_{1} \tag{16}
\end{equation*}
$$

Using (13) in (16) and the fact that $m-1 \geq k$, we have

$$
\begin{aligned}
\Delta^{k} x(t) & \geq \sum_{s=t}^{\infty} \frac{(s-t)^{(m-k-1)}}{(m-k-1)!} f\left(s, c(s)^{(k-1)}\right)\left(\frac{(s-t)^{(k-1)}}{c(k-1)!(s)^{(k-1)}}\right)^{\beta}\left(\Delta^{k-1} x(t)\right)^{\beta} \\
& =\sum_{s=t}^{\infty}\left(\frac{(s-t)^{(m-k-1)}}{(m-k-1)!}\left(\frac{1}{c(k-1)!} \prod_{i=1}^{k-1}\left(1-\frac{t}{s-i}\right)\right)^{\beta} f\left(s, c(s)^{(k-1)}\right)\left(\Delta^{k-1} x(t)\right)^{\beta}\right.
\end{aligned}
$$

for $t \geq t_{1}$. Or

$$
\begin{equation*}
\frac{\Delta^{k} x(t)}{\left(\Delta^{k-1} x(t)\right)^{\beta}} \geq C \sum_{s=t}^{\infty}\left[\prod_{i=0}^{k-1}\left(1-\frac{t}{s-i}\right)\right]^{\beta}(s-t)^{(m-k-1)} f\left(s, c(s)^{(k-1)}\right) \tag{17}
\end{equation*}
$$

where $C=\left(\frac{1}{c(k-1)!}\right)^{\beta} \frac{1}{(m-k-1)!}$.
Now, since $\Delta^{k} x(t)$ is positive and decreasing for $t \geq t_{1}+1$, we find

$$
\frac{\Delta^{k} x(t-1)}{\left(\Delta^{k-1} x(t)\right)^{\beta}} \geq C \sum_{s=t}^{\infty}\left[\prod_{i=0}^{k-1}\left(1-\frac{t}{s-i}\right)\right]^{\beta}(s-t)^{(m-k-1)} f\left(s, c(s)^{(k-1)}\right)
$$

Summing this inequality from $t_{1}+1$ to $T \geq t_{1}+1$ we get

$$
\begin{aligned}
& C \sum_{t=t_{1}+1}^{T}\left[\prod_{i=0}^{k-1}\left(1-\frac{t}{s-i}\right)\right]^{\beta}(s-t)^{(m-k-1)} f\left(s, c(s)^{(k-1)}\right) \\
\leq & \sum_{t=t_{1}+1}^{T} \frac{\Delta^{k} x(t-1)}{\left(\Delta^{k-1} x(t)\right)^{\beta}} \\
\leq & \int_{x\left(t_{1}+1\right)}^{x(T+1)} u^{-\beta} d u=\left.\frac{u^{1-\beta}}{1-\beta}\right|_{u=x(t+1)} ^{x(T+1)} \\
< & \frac{1}{\beta-1}\left(x\left(t_{1}+1\right)\right)^{1-\beta}<\infty \text { as } T \rightarrow \infty,
\end{aligned}
$$

which contradicts condition (12). This completes the proof.

When $k=1$, condition (12) is reduced to

$$
\begin{equation*}
\sum_{t=t_{0}}^{\infty} \sum_{s=t}^{\infty}\left(1-\frac{t}{s}\right)^{\beta}(s-t)^{(m-2)}|f(s, c)|=\infty \tag{18}
\end{equation*}
$$

For the case when $m=2$, we obtain
Corollary 3.2. Suppose that equation (1) with $m=2$ is strongly superlinear. If

$$
\begin{equation*}
\sum_{t=t_{0}}^{\infty} \sum_{s=t}^{\infty}|f(s, c)|=\infty, \text { for some constand } c \neq 0 \tag{19}
\end{equation*}
$$

then equation (1) with $m=2$ is oscillatory.
For strongly sublinear equation (1), we have
Theorem 3.3. Let equation (1) be strongly sublinear. If

$$
\begin{equation*}
\sum_{s=t_{0}}^{\infty}\left|f\left(s, c(s)^{(m-1)}\right)\right|=\infty \tag{20}
\end{equation*}
$$

for some constant $c \neq 0$, then equation (1) is oscillatory.
Proof. Let $\{x(t)\}$ be a non-oscillatory solution of equation (1), say $x(t)>0$ for $t \in \mathbb{N}\left(t_{0}\right)$. By Lemma 2.2, these exists a $t_{1} \geq t_{0}$ and constants $c_{1}$ and $c_{2}$ such that

$$
\begin{align*}
\Delta^{m-1} x(t) & >0 \text { and } \Delta x(t)>0 \text { for } t \geq t_{1},  \tag{21}\\
x(t) & \leq c_{1}(t)^{(m-1)} \text { for } t \geq t_{1}, \tag{22}
\end{align*}
$$

and by Lemma 2.3, we find

$$
\begin{equation*}
x(t) \geq c_{2}(t)^{(m-1)} \Delta^{m-1} x(t), \text { for } t \geq t_{2}=2^{m-1} t_{1} \tag{23}
\end{equation*}
$$

Summing equation (1) from $t \geq t_{2}$ to $u \geq t$ and letting $u \rightarrow \infty$, we get

$$
\Delta^{m-1} x(t) \geq \sum_{s=t}^{\infty} f(s, x(s))
$$

Using the strong sublinearity of $f$ in the above inequality, we obtain

$$
\begin{equation*}
\Delta^{m-1} x(t) \geq \sum_{s=t}^{\infty} f\left(s, c_{1}(s)^{(m-1)}\right)\left(\frac{x(s)}{c_{1}(s)^{(m-1)}}\right)^{\gamma}, \quad \text { for } t \geq t_{2} \tag{24}
\end{equation*}
$$

By applying (23) in the (24), we find

$$
\begin{equation*}
\frac{x(t)}{(t)^{(m-1)}} \geq c_{2} \Delta^{m-1} x(t) \geq \frac{c_{2}}{c_{1}^{\gamma}} \sum_{s=t}^{\infty} f\left(s, c_{1}(s)^{(m-1)}\right)\left(\frac{x(s)}{(s)^{(m-1)}}\right)^{\gamma} . \tag{25}
\end{equation*}
$$

Denoting the right-hand side of (25) by $z(t)$, we find

$$
-\Delta z(t)=\frac{c_{2}}{c_{1}^{\gamma}} f\left(t, c_{1}(t)^{(m-1)}\right)\left(\frac{x(t)}{(t)^{(m-1)}}\right)^{\gamma}
$$

$$
\geq \frac{c_{2}}{c_{1}^{\gamma}} f\left(t, c_{1}(t)^{(m-1)}\right) z^{\gamma}(t) \text { for } t \geq t_{2}
$$

or

$$
\frac{c_{2}}{c_{1}^{\gamma}} f\left(t, c_{1}(t)^{(m-1)}\right) \leq-z^{-\gamma}(t) \Delta z(t) \text { for } t \geq t_{2}
$$

Summing this inequality from $t_{2}+1$ to $T \geq t_{2}+1$, we get

$$
\begin{aligned}
\frac{c_{2}}{c_{1}^{\gamma}} \sum_{t=t_{2}+k}^{T} f\left(t, c_{1}(t)^{(m-1)}\right) & \leq-\sum_{t=t_{2}+k}^{T} z^{-\gamma}(t) \Delta z(t) \\
& \leq \int_{0}^{x\left(t_{2}+1\right)} z^{-\gamma}(t) \mathrm{d} z(t)=\frac{z^{1-\gamma}\left(t_{2}\right)}{1-\gamma}<\infty \text { as } T \rightarrow \infty
\end{aligned}
$$

which contradicts condition (20). This completes the proof.
Remark 3.1. One can easily see that equation (1) is oscillatory if

$$
\begin{equation*}
\sum_{t=t_{0}}^{\infty}\left|f(s, c)^{(m-1)}\right|=\infty \tag{26}
\end{equation*}
$$

for some constants $c \neq 0$.
Remark 3.2. The results of this section are presented in a form which is essentially new. It extend and improve many of the existing results appeared in the literature, see $[1,2,3,4,5]$.
Remark 3.3. When $m=2$, the results obtained include many of the known oscillation results for related second order nonlinear difference equations, see [1, 2, 3].
Remark 3.4. The results of this section can be extended to $m$ th order nonlinear difference equation with deviating arguments of the form

$$
\begin{equation*}
\Delta^{m} x(t)+f(t, x(g(t)))=0 \tag{27}
\end{equation*}
$$

where $f$ is as in equation (1) and $g \in \mathbb{G}\left\{g: \mathbb{N}\left(t^{*}\right) \rightarrow \mathbb{N}\right.$ for some $t^{*} \in \mathbb{N}: g(t) \leq$ $\left.t, \lim _{t \rightarrow \infty} g(t)=0\right\},\{\mathrm{g}(\mathrm{t})\}$ is a nondecreasing sequence.

In fact, we may replace $s$ in conditions (12) and (20) by $g(s)$. The details are left to the reader.

## 4. Asymptotic Behavior

In this section we give a sufficient condition for every solution $x$ defined for all large $t \in \mathbb{N}\left(t_{0}\right)$ of equation (1) to satisfy

$$
\Delta^{m-1} x(t) \rightarrow c \text { for } t \rightarrow \infty \text { and so } \frac{x(t)}{(t)^{(m-1)}} \rightarrow \frac{c}{(m-1)!} \text { for } t \rightarrow \infty
$$

where $c$ is some real number (depending on solution $\{x(t)\}$.
We assume that

$$
\begin{equation*}
|f(t, x)| \leq a(t)+b(t)|x|^{\gamma} \text { for all }(t, x) \in \mathbb{N}\left(t_{0}\right) \times \mathbb{R} \tag{28}
\end{equation*}
$$

where $\gamma \in(0,1],\{a(t)\}$ and $\{b(t)\}$ are nonnegative real-valued sequences.

Theorem 4.1. If

$$
\begin{equation*}
\sum_{s=t_{0}}^{\infty} a(s)<\infty \quad \text { and } \sum_{s=t_{0}}^{\infty}\left((s)^{(m-1)}\right)^{\gamma} b(s)<\infty \tag{29}
\end{equation*}
$$

then every solution $\{x(t)\}, t \in \mathbb{N}\left(t_{0}\right)$ of equation (1) satisfies

$$
\begin{equation*}
\Delta^{m-1} x(t)=c+o(1), \quad \text { for } t \rightarrow \infty, \tag{30}
\end{equation*}
$$

and

$$
\begin{equation*}
x(t)=\frac{c}{(m-1)!}(t)^{(m-1)}+o\left((t)^{(m-1)}\right) \text { for } t \rightarrow \infty, \tag{31}
\end{equation*}
$$

where $c$ is some constant (depending on solution $\{x(t)\}$.
Proof. Let $\{x(t)\}$ be a solution for $t \geq t_{0} \in \mathbb{N}\left(t_{0}\right)$ of equation (1). Then (1) gives

$$
x(t)=\sum_{1=0}^{m-1} \frac{\left(t-t_{0}\right)^{(i)}}{i!} \Delta^{i} x\left(t_{0}\right)+\sum_{s=t_{0}}^{t-m} \frac{(t-s-1)^{(m-1)}}{(m-1)!}(-f(s, x(s))) \text { for } t \geq t_{0} .
$$

Thus, by (29), we obtain for $t \geq t_{0}$,

$$
\begin{aligned}
|x(t)| \leq & \sum_{1=0}^{m-1} \frac{\left(t-t_{0}\right)^{(i)}}{i!}\left|\Delta^{i} x\left(t_{0}\right)\right|+\sum_{s=t_{0}}^{t-m} \frac{(t-s-1)^{(m-1)}}{(m-1)!}\left[a(s)+b(s)\left|x^{\gamma}(s)\right|\right] \\
\leq & \sum_{i=0}^{m-1} \frac{(t)^{(i)}}{i!}\left|\Delta^{i} x\left(t_{0}\right)\right|+\frac{(t)^{(m-1)}}{(m-1)!} \sum_{s=t_{0}}^{t-m} a(s) \\
& +\frac{(t)^{(m-1)}}{(m-1)!} \sum_{s=t_{0}}^{t-m} b(s)\left((s)^{(m-1)}\right)^{\gamma}\left(\frac{x(s)}{(s)^{(m-1)}}\right)^{\gamma},
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{|x(t)|}{(t)^{(m-1)}} \leq & \sum_{i=0}^{m-1} \frac{1}{i!} \frac{(t)^{(i)}}{(t)^{(m-1)}}\left|\Delta^{i} x\left(t_{0}\right)\right|+\frac{1}{(m-1)!} \sum_{s=t_{0}}^{t-m} a(s) \\
& +\frac{1}{(m-1)!} \sum_{s=t_{0}}^{t-m}\left((s)^{(m-1)}\right)^{\gamma} b(s)\left(\frac{x(s)}{(s)^{(m-1)}}\right)^{\gamma} .
\end{aligned}
$$

Using the elementary inequality

$$
|x|^{\gamma} \leq 1+|x|, \text { for } x \in \mathbb{R} \text { and } \gamma \in[0,1],
$$

we find

$$
\begin{aligned}
\frac{|x(t)|}{(t)^{(m-1)}} \leq & \sum_{i=0}^{m-1} \frac{1}{i!} \frac{(t)^{(i)}}{(t)^{(m-1)}}\left|\Delta^{i} x\left(t_{0}\right)\right|+\frac{1}{(m-1)!} \sum_{s=t_{0}}^{t-m}\left[a(s)+\left((s)^{(m-1)}\right)^{\gamma} b(s)\right] \\
& +\frac{1}{(m-1)!} \sum_{s=t_{0}}^{t-m}\left((s)^{(m-1)}\right)^{\gamma} b(s)\left(\frac{|x(s)|}{(s)^{(m-1)}}\right) .
\end{aligned}
$$

By the assumption (29), these exists constat $C>0$ such that

$$
\frac{|x(t)|}{(t)^{(m-1)}} \leq C+\sum_{s=t_{0}}^{t-m} \frac{1}{(m-1)!}\left((s)^{(m-1)}\right)^{\gamma} b(s)\left(\frac{|x(s)|}{(s)^{(m-1)}}\right) .
$$

Applying Lemma 2.4 and using condition (29) we can conclude that there exists a positive constant $M$ such that

$$
\begin{equation*}
\frac{|x(t)|}{(t)^{(m-1)}} \leq M \text { for } t \geq t_{0} \in \mathbb{N}\left(t_{0}\right) \tag{32}
\end{equation*}
$$

Now, by using (28) and (32), we derive

$$
\begin{aligned}
|f(t, x(t))| & \leq a(t)+b(t)|x(t)|^{\gamma} \\
& =a(t)+b(t)\left((t)^{(m-1)}\right)^{\gamma}\left(\frac{|x(t)|}{(t)^{(m-1)}}\right)^{\gamma} \\
& \leq a(t)+M^{\gamma} b(t)\left((t)^{(m-1)}\right)^{\gamma} \text { for } t \geq t_{0} .
\end{aligned}
$$

Thus, because of (29), it follows that

$$
\sum_{s=t_{0}}^{\infty} f(s, x(s)) \text { exists (as a real number). }
$$

But, (1) gives

$$
\Delta^{m-1} x(t)=\Delta^{m-1} x\left(t_{0}\right)+\sum_{s=t_{0}}^{t-1} f(s, x(s))
$$

Therefore,

$$
\lim _{t \rightarrow \infty} \Delta^{m-1} x(t)=\Delta^{m-1} x\left(t_{0}\right)+\sum_{s=t_{0}}^{\infty} f(s, x(s)):=C \in \mathbb{R}
$$

i.e. (30) holds. Finally, by L'Hospital rule, we obtain

$$
\lim _{t \rightarrow \infty} \frac{x(t)}{(t)^{(m-1)}}=\frac{1}{(m-1)!} \lim _{t \rightarrow \infty} \Delta^{m-1} x(t)=\frac{c}{(m-1)!}
$$

and consequently the solution $\{x(t)\}$ satisfies (31). This completes the proof.
Of course, Theorem 4.1 remains valid for the equations of the form

$$
\begin{equation*}
\Delta^{m} x(t)= \pm f(t, x(t)) \tag{33}
\end{equation*}
$$

where $f$ satisfies conditions (28) and (29).
We may note that the second part of the condition (29) can be replaced by

$$
\begin{equation*}
\sum^{\infty} s^{(m-1) \gamma} b(s)<\infty \tag{34}
\end{equation*}
$$

and also the conclusion (31) can be replace by

$$
\begin{equation*}
x(t)=\frac{c t^{(m-1)}}{(m-1)!}+o\left(t^{(m-1)}\right) \text { for } t \rightarrow \infty \tag{35}
\end{equation*}
$$

As illustrative example, we consider the equation

$$
\begin{equation*}
\Delta^{4} x(t)=p(t)+q(t)|x(t)|^{\gamma} \operatorname{sgn} x(t) \tag{36}
\end{equation*}
$$

where the $\{p(t)\}$ and $\{q(t)\}$ are non negative real sequence and $\gamma \in(0,1]$ is a constant. Now, if

$$
\sum^{\infty} p(s)<\infty \text { and } \sum^{\infty} s^{(3) \gamma} q(s)<\infty
$$

then by Theorem 4.1, we conclude that every solution of equation (36) satisfies

$$
\Delta^{3} x(t)=C+o(1) \text { for } t \rightarrow \infty \text { and } x(t)=\frac{C}{3!} t^{(3)}+o\left((t)^{(3)}\right) \text { for } t \rightarrow \infty
$$

where $C$ is some real number depending on the solution $\{x(t)\}$.

## 5. General Remarks

1. We many note that conditions (12) and (18) can be replaced by the stronger condition

$$
\begin{equation*}
\sum_{t=t_{0}}^{\infty} s^{(m-1)}|f(s, c)|=\infty \text { for some constant } c \neq 0 \tag{37}
\end{equation*}
$$

while condition (20) takes the form

$$
\begin{equation*}
\sum_{t=t_{0}}^{\infty}\left|f\left(s, c(s)^{(m-1)}\right)\right|=\infty \text { for some constant } c \neq 0 \tag{38}
\end{equation*}
$$

2. Theorem 4.1 when $m=2$ is a discrete analog of the results in $[6,7,8,9]$. Moreover, it improve and unify some of them.

## 6. Example

Consider the following $m$ th order nonlinear difference equation

$$
\begin{equation*}
\Delta^{m} x(t)+x(t)^{\frac{1}{3}}=0 \tag{39}
\end{equation*}
$$

Let $m>4$ is a positive even integer, $f(t, x(t))=x(t)^{\frac{1}{3}}$, when $|x| \leq|y|, \gamma=\frac{1}{2}$, we have

$$
\frac{|f(t, x)|}{|x|^{\frac{1}{2}}} \geq \frac{|f(t, y)|}{|y|^{\frac{1}{2}}}
$$

so $f(t, x(t))$ is strong sublinear.
Since $c \neq 0$,

$$
f\left(s, c(s)^{(m-1)}\right)=c^{\frac{1}{3}}\left((s)^{(m-1)}\right)=c^{\frac{1}{3}}(s(s-1)(s-2) \ldots(s-m+2))^{\frac{1}{3}} .
$$

Then,

$$
\begin{equation*}
\sum_{s=t_{0}}^{\infty}\left|f\left(s, c(s)^{(m-1)}\right)\right|=\sum_{s=t_{0}}^{\infty}\left|c^{\frac{1}{3}}(s(s-1)(s-2) \ldots(s-m+2))^{\frac{1}{3}}\right| . \tag{40}
\end{equation*}
$$

Because $m>4$ is a positive even integer, so $\frac{1}{3}(m-1)>1$, let $t_{0}=m-2$, so

$$
(s(s-1)(s-2) \ldots(s-m+2))^{\frac{1}{3}}>0
$$

and $\lim _{s \rightarrow \infty}(s(s-1)(s-2) \ldots(s-m+2))^{\frac{1}{3}}=+\infty$, then (40) is divergence, so $\sum_{s=t_{0}}^{\infty}\left|f\left(s, c(s)^{(m-1)}\right)\right|=\infty$. By Theorem 3.3, the solution of function (39) is oscillatory.

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