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BOUNDED OSCILLATION FOR SECOND-ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS†

XIA SONG* AND QUANXIN ZHANG

Abstract. Two necessary and sufficient conditions for the oscillation of the bounded solutions of the second-order nonlinear delay differential equation

$$\big(a(t)x'(t)\big)' + q(t)f(x[\tau(t)]) = 0$$

are obtained by constructing the sequence of functions and using inequality technique.

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1. Introduction

Consider the second-order nonlinear delay differential equation

$$(a(t)x'(t))' + q(t)f(x[\tau(t)]) = 0, \quad t \geq t_0. \tag{1.1}$$

The paper assumes the following conditions hold:

(H_1) $a(t) \in C^1([t_0,\infty),(0,\infty))$, $q(t) \in C([t_0,\infty),[0,\infty))$, and with $t \to \infty$,

$$A(t) = \int_{t_0}^{t} \frac{1}{a(s)} \mathrm{d}s \to \infty;$$

(H_2) $f(x) \in C(R,R)$ is non-decreasing function , and $\frac{f(x)}{x} \geq \delta > 0$ with $x \neq 0$;

(H_3) $\tau(t) \in C([t_0,\infty),R)$, $\tau(t) \leq t$, and $\lim\limits_{t\to\infty} \tau(t) = \infty$.

We call that $x(t) \in C^1([T_x,\infty),\mathrm{R})$ $(T_x \geq t_0)$ is the solution of equation (1.1) if $a(t)x'(t) \in C^1([T_x,\infty),\mathrm{R})$ and $x(t)$ satisfy (1.1) for $t \in [T_x,\infty)$. We suppose that every solution of (1.1) can be extended in $[t_0,+\infty)$. In any infinite

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*Corresponding author.
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interval $[T, +\infty)$, we call $x(t)$ is a regular solution of (1.1) if $x(t)$ is not the eventually identically zero. The regular solution of (1.1) is said to be oscillatory in case it has arbitrarily large zero point; otherwise, the solution is said to be nonoscillatory.

For the equation (1.1), if $a(t) \equiv 1$, the equation (1.1) becomes

$$x''(t) + q(t)f(x[\tau(t)]) = 0, \quad t \geq t_0. \tag{1.2}$$

For the equation (1.2), if $f(x) = x,\ \tau(t) = t$, and $q(t) = c(t)$, the equation (1.2) is simplified to be the second-order linear differential equation

$$x''(t) + c(t)x(t) = 0, \quad t \geq t_0. \tag{1.3}$$

There are some oscillation criteria for the equation (1.3), and one of the most important criteria is given by Wintner [1] as follows: If

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} c(x)\mathrm{d}x\mathrm{d}s = \infty, \tag{1.4}$$

the equation (1.3) is oscillatory. In 1978, Kamenev [2] improved the result of Wintner. He proved that if

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda}} \int_{t_0}^{t} (t-s)^{\lambda} c(s)\mathrm{d}s = \infty \tag{1.5}$$

whereλ is a constant and $\lambda > 1$, the equation (1.3) is oscillatory.

In recent years, the oscillation theory and its application of differential equations have been greatly concerned. For example, you can see the recent monographs [3-5]. In particular, the result on oscillation criteria of second-order differential equation is very rich (see [6-16]), but the most results obtained establish the sufficient condition of the oscillation for differential equations. Generally, the necessary and sufficient condition is difficult to obtain. This article discusses the oscillation for the bounded solution of the equation (1.1), and establishes two necessary and sufficient conditions of oscillation for the bounded solution of (1.1) by constructing a sequence of functions and using inequality technique.

2. Main results

Theorem 2.1. *Suppose that $(H_1)-(H_3)$ hold. Then the necessary and sufficient condition of the oscillation for every bounded solution of the equation (1.1) is*

$$\int_{t_0}^{\infty} A(s)q(s)\mathrm{d}s = \infty. \tag{2.1}$$

Proof. Sufficiency. Suppose that there is nonoscillatory bounded solution $x(t)$ of the equation (1.1). Without loss of generality we assume that $x(t)$ is eventually positive, then there exists $t_1 (t_1 \geq t_0)$ such that as $t \geq t_1$,

$$x(t) > 0,\ x(\tau(t)) > 0.$$

From the equation (1.1), we get

$$(a(t)x'(t))' = -q(t)f(x[\tau(t)]) \le 0,\ t \ge t_1.$$

Thus, we can determine that

$$a(t)x'(t) \ge 0,\ t \ge t_1. \tag{2.2}$$

Actually, if there exists $t_2(t_2 \ge t_1)$ such that

$$a(t_2)x'(t_2) = c < 0,$$

noticing $a(t)x'(t)$ is nonincreasing, we can obtain

$$a(t)x'(t) \le c$$

as $t \ge t_2$, i.e.

$$x'(t) \le \frac{c}{a(t)},\ t \ge t_2.$$

Integrating the above formula from t_2 to t, by (H_1), we get that

$$x(t) \le x(t_2) + c\int_{t_2}^{t} \frac{1}{a(s)}\mathrm{d}s \to -\infty\ (t \to \infty).$$

This contradicts with that $x(t)$ is eventually positive, so (2.2) holds. Thus, for $t \ge t_1$, we have

$$x(t) > 0,\ x'(t) \ge 0,\ (a(t)x'(t))' \le 0.$$

Then there exists $t_3(t_3 \ge t_2)$ and $l(l > 0)$ such that when $t \ge t_3$,

$$f(x(\tau(t))) \ge \delta x(\tau(t)) \ge l. \tag{2.3}$$

Substituting (2.3) into equation (1.1), we get

$$(a(t)x'(t))' + lq(t) \le 0,\ t \ge t_3. \tag{2.4}$$

Multiplying the both ends of the above formula by $A(t)$, we can get

$$A(t)\left(a(t)x'(t)\right)' + lA(t)q(t) \le 0,\ t \ge t_3. \tag{2.5}$$

Because

$$(A(t)a(t)x'(t))' = A(t)\left(a(t)x'(t)\right)' + x'(t),$$

we can obtain

$$A(t)\left(a(t)x'(t)\right)' = (A(t)a(t)x'(t))' - x'(t).$$

Substituting the above into (2.5), we get

$$(A(t)a(t)x'(t))' - x'(t) + lA(t)q(t) \le 0,\ t \ge t_3.$$

Integrating the above formula from t_3 to t, we obtain

$$\begin{aligned} l\int_{t_3}^{t} A(s)q(s)\mathrm{d}s &\le x(t) - x(t_3) - A(t)a(t)x'(t) + A(t_3)a(t_3)x'(t_3) \\ &\le x(t) - x(t_3) + A(t_3)a(t_3)x'(t_3). \end{aligned}$$

Because $x(t)$ is increasing and bounded, there exists $C(C>0)$ such that

$$\int_{t_3}^{\infty} A(s)q(s)\mathrm{d}s \le C.$$

This contradicts with the condition (2.1). The proof of sufficient section is completed.

Necessity. If

$$\int_{t_0}^{\infty} A(s)q(s)\mathrm{d}s < \infty,$$

then there is $T \ge t_0$ such that for $t \ge T$, we have

$$\int_{t}^{\infty} A(s)q(s)\mathrm{d}s \le \frac{1}{f(2)}.$$

Constructing the sequence of functions such that

$$x_0(t) \equiv 2,$$

$$x_{k+1}(t) = \begin{cases} 1 + \int_T^t A(s)q(s)f(x_k[\tau(s)])\mathrm{d}s + A(t)\int_t^{\infty} q(s)f(x_k[\tau(s)])\mathrm{d}s, t \ge T; \\ x_{k+1}(T), \tau(T) \le t < T, \end{cases} \tag{2.6}$$

for $k = 1, 2, \cdots$. Then

$$x_1(t) = \begin{cases} 1 + f(2)\left(\int_T^t A(s)q(s)\mathrm{d}s + A(t)\int_t^{\infty} q(s)\mathrm{d}s\right) \le x_0(t) = 2, t \ge T; \\ x_1(T) \le x_0(T) = 2, \tau(T) \le t < T. \end{cases}$$

Suppose that

$$1 \le x_k(t) \le x_{k-1}(t) \le \cdots \le x_1(t) \le x_0(t) = 2,\ t \ge \tau(T).$$

Noticing $f(u)$ is non-decreasing, then

$$1 \le x_{k+1}(t)$$
$$= \begin{cases} 1 + \int_T^t A(s)q(s)f(x_k[\tau(s)])\mathrm{d}s + A(t)\int_t^{\infty} q(s)f(x_k[\tau(s)])\mathrm{d}s \le x_k(t), t \ge T; \\ x_{k+1}(T) \le x_k(T), \tau(T) \le t < T. \end{cases}$$

Thus, by mathematical induction, for any positive integer k, we get

$$1 \le x_k(t) \le x_{k-1}(t) \le 2,\ t \ge \tau(T).$$

Therefore, the limit of the sequence of functions $\{x_k(t)\}$ exists, i.e.

$$\lim_{k\to\infty} x_k(t) = x(t)$$

and $1 \le x(t) \le 2,\ t \ge \tau(T)$. Applying Lebesgue control convergence theorem to (2.6), we can get

$$x(t) = \begin{cases} 1 + \int_T^t A(s)q(s)f(x[\tau(s)])\mathrm{d}s + A(t)\int_t^{\infty} q(s)f(x[\tau(s)])\mathrm{d}s, t \ge T; \\ x(T), \tau(T) \le t < T. \end{cases}$$

Derivation of the both sides of the above formula and multiplying them by $a(t)$, we get that for $t > T$

$$\begin{aligned} & a(t)x'(t) \\ &= a(t)A(t)q(t)f(x[\tau(t)]) + a(t)\left[A'(t)\int_t^\infty q(s)f(x[\tau(s)])\mathrm{d}s - A(t)q(t)f(x[\tau(t)])\right] \\ &= \int_t^\infty q(s)f(x[\tau(s)])\mathrm{d}s. \end{aligned}$$

Continually after derivation of the both sides of the above formula, we get

$$(a(t)x'(t))' = -q(t)f(x[\tau(t)]),\ t > T.$$

It is easy to see that $x(t)$ is an eventually positive bounded solution of (1.1), which is contradictory to that every bounded solution of the equation (1.1) is oscillatory. The proof is completed. □

Remark 2.1. For linear case of (1.1) (i.e. $f(x) = x$), oscillation criterion relative to Theorem 2.1 has been obtained in Corollary 2 in [17].

By Theorem 2.1, we can obtain for equation (1.2)

Corollary 2.2. *Suppose that $(H_1) - (H_3)$ hold. Then every bounded solution of equation (1.2) is oscillatory if and only if*

$$\int_{t_0}^\infty (s - t_0)q(s)\mathrm{d}s = \infty. \tag{2.7}$$

Theorem 2.3. *Assume that $(H_1) - (H_3)$ hold. Then every bounded solution of (1.1) is oscillatory if and only if*

$$\int_{t_0}^\infty \frac{1}{a(s)}\left(\int_s^\infty q(u)\mathrm{d}u\right)\mathrm{d}s = \infty. \tag{2.8}$$

Proof. Sufficiency. Suppose that there is nonoscillatory bounded solution $x(t)$ of the equation (1.1). Without loss of generality, we assume that $x(t)$ is eventually positive, then there exists $t_1 (t_1 \geq t_0)$ such that

$$x(t) > 0,\ x(\tau(t)) > 0$$

for $t \geq t_1$. Using arguments similar to ones in the proof of Theorem 2.1, we can get (2.4), i.e.

$$(a(t)x'(t))' + lq(t) \leq 0,\ t \geq t_2 \geq t_1.$$

Integrating the above from t to $t + v$, we get

$$a(t+v)x'(t+v) - a(t)x'(t) + l\int_t^{t+v} q(s)\mathrm{d}s \leq 0.$$

Noticing (2.2) and letting $v \to \infty$, we get that

$$l\frac{1}{a(t)}\int_t^\infty q(s)\mathrm{d}s \leq x'(t).$$

Integrating the above from $T(T \geq t_2)$ to $t(t \geq T)$, it follows

$$l \int_T^t \frac{1}{a(s)} \left(\int_s^\infty q(u)\mathrm{d}u \right) \mathrm{d}s \leq x(t) - x(T) \leq x(t).$$

Let $t \to \infty$ to acquire the limits of both sides of the above. Because $x(t)$ is bounded and increasing, it is easy to get

$$\int_T^t \frac{1}{a(s)} \left(\int_s^\infty q(u)\mathrm{d}u \right) \mathrm{d}s \leq \infty.$$

This is contradictory to the condition (2.8). The proof of sufficiency is completed.

Necessity. Suppose that

$$\int_{t_0}^\infty \frac{1}{a(s)} \left(\int_s^\infty q(u)\mathrm{d}u \right) \mathrm{d}s < \infty,$$

and there exists $T(T \geq t_0)$ such that

$$\int_t^\infty \frac{1}{a(s)} \left(\int_s^\infty q(u)\mathrm{d}u \right) \mathrm{d}s \leq \frac{1}{f(2)}$$

for $t \geq T$. Construct the sequence of functions and let

$$x_0(t) \equiv 2,$$

$$x_{k+1}(t) = \begin{cases} 1 + \displaystyle\int_T^t \frac{1}{a(s)} \left(\int_s^\infty q(u) f(x_k[\tau(u)])\mathrm{d}u \right) \mathrm{d}s, & t \geq T; \\ x_{k+1}(T), & \tau(T) \leq t < T, \end{cases} \tag{2.9}$$

$k = 1, 2, \cdots$. Similarly to the proof of Theorem 2.1, by the mathematical induction for any positive integer k, we have

$$1 \leq x_k(t) \leq x_{k-1}(t) \leq 2, \ t \geq \tau(T).$$

So the limit of $\{x_k(t)\}$ exists, i.e.

$$\lim_{k \to \infty} x_k(t) = x(t)$$

and $1 \leq x(t) \leq 2, \ t \geq \tau(T)$. By Lebesgue control convergence theorem to (2.9), it follows that

$$x(t) = \begin{cases} 1 + \displaystyle\int_T^t \frac{1}{a(s)} \left(\int_s^\infty q(u) f(x[\tau(u)])\mathrm{d}u \right) \mathrm{d}s, & t \geq T; \\ x(T), & \tau(T) \leq t < T. \end{cases}$$

Derivation of the both sides of the above and multiplying them by $a(t)$, we get that for $t > T$

$$a(t)x'(t) = \int_t^\infty q(u) f(x[\tau(u)])\mathrm{d}u.$$

Continuing to take the derivatives of the both sides of the above, we can get

$$(a(t)x'(t))' = -q(t) f(x[\tau(t)]), \ t > T.$$

Thus $x(t)$ is an eventually positive bounded solution of (1.1), which is contradictory to that every bounded solution of the equation (1.1) is oscillatory. The proof is completed. □

By Theorem 2.3, we can get for equation (1.2).

Corollary 2.4. *Suppose that* $(H_1)-(H_3)$ *hold. Then every bounded solution of (1.2) is oscillatory if and only if*

$$\int_{t_0}^{\infty}\int_{s}^{\infty} q(u)\mathrm{d}u\mathrm{d}s = \infty. \tag{2.10}$$

Example 2.5. Consider second-order linear differential equation

$$(t \cdot x'(t))' + \frac{1}{t}x(t) = 0,\ t \geq 1. \tag{2.11}$$

Here

$$a(t) = t,\ q(t) = \frac{1}{t},\ f(x) = x,\ \tau(t) = t.$$

The conditions (H_1)-(H_3), (2.1) and (2.8) are clearly satisfied. Altogether, by Theorems 2.1 and 2.3, every bounded solution of the equation (2.11) is oscillatory. In fact, $x(t) = \cos \ln t$ is a bounded oscillatory solution of the equation (2.11).

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Xia Song received M.Sc. from Shandong University, China. She is now a lecturer of Department of Mathematics in Binzhou University, Shandong, China. Her current research interests include differential equations and dynamical systems.

Department of Mathematics, Binzhou University, Shandong 256603, P.R. China.
e-mail: songxia119@163.com

Quanxin Zhang was made professor in 1997. He is the incumbent Dean of the Department of Mathematics in Binzhou University, he was selected as the "Excellent Well-known Teacher of Shandong Province". His current research interests include differential equations and dynamical systems.

Department of Mathematics, Binzhou University, Shandong 256603, P.R. China.
e-mail: 3314744@163.com