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# COMPACT INTERPOLATION ON AX = Y IN $ALG\mathcal{L}^{\dagger}$

## JOO HO KANG

ABSTRACT. In this paper the following is proved: Let  $\mathcal{L}$  be a subspace lattice on a Hilbert space  $\mathcal{H}$  and X and Y be operators acting on  $\mathcal{H}$ . Then there exists a compact operator A in Alg $\mathcal{L}$  such that AX = Y if and only if  $\sup \left\{ \frac{\|E^{\perp} Yf\|}{\|E^{\perp} Xf\|} : f \in \mathcal{H}, \ E \in \mathcal{L} \right\} = K < \infty$  and Y is compact. Moreover, if the necessary condition holds, then we may choose an operator A such that AX = Y and  $\|A\| = K$ .

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## 1. Introduction

Let  $\mathcal{C}$  be a collection of operators acting on a Hilbert space  $\mathcal{H}$ . An interpolation question for  $\mathcal{C}$  asks for which operators X and Y on  $\mathcal{H}$  when is there a bounded linear operator A (usually satisfying some other conditions) such that AX = Y? The "other conditions" that have been of interest to us involve restricting A to lie in the algebra associated with a subspace lattice. The simplest case of the operator interpolation problem relaxes all restrictions on A, requiring it simply to be a bounded operator. In this case, the existence of A is nicely characterized by Douglas [2]. Another interpolation question for a given subalgebra  $\rfloor$  of  $B(\mathcal{H})$  asks for which vectors x and y in  $\mathcal{H}$  is there a bounded operator  $A \in \mathcal{C}$  that maps x to y. Lance [6] initiated the discussion by considering a nest  $\mathcal{N}$  and asking what conditions on x and y will guarantee the existence of an operator A in Alg $\mathcal{N}$  such that Ax = y. Hopenwasser [3] extended Lance's result to the case where the nest  $\mathcal{N}$  is replaced by an arbitrary commutative subspace lattice  $\mathcal{L}$ . Munch [7] considered the problem of finding a Hilbert-Schmidt operator A in Alg $\mathcal{N}$  that maps x to y, whereupon Hopenwasser [4] again extended

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to Alg $\mathcal{L}$ . In [1], authors studied the problem of finding A so that Ax = y and A is required to lie in certain ideals contained in Alg $\mathcal{L}$  (for a nest  $\mathcal{L}$ ).

Roughly speaking, when an operator maps one thing to another, we think of the operator as the interpolating operator and the equation representing the mapping as the interpolation equation. The equations Ax = y and AX = Y are indistinguishable if spoken aloud, but we mean the change to capital letters to indicate that we intend to look at fixed operators X and Y, and ask under what conditions there will exist an operator A satisfying the equation AX = Y.

Let x and y be vectors in a Hilbert space. Then  $\langle x, y \rangle$  means the inner product of vectors x and y. Note that the "vector interpolation" problem is a special case of the "operator interpolation" problem. Indeed, if we denote by  $x \otimes u$  the rank-one operator defined by the equation  $x \otimes u(w) = \langle w, u \rangle x$ , and if we set  $X = x \otimes u$ , and  $Y = y \otimes u$ , then the equations AX = Y and Ax = yrepresent the same restriction on A.

Let  $\mathcal{H}$  be a Hilbert space and  $\mathbb{N} = \{1, 2, ...\}$ . A bounded operator A on  $\mathcal{H}$  has *finite-rank* if rangeA is finite dimensional. A bounded operator A on  $\mathcal{H}$  is called *compact* if  $A(\text{ball}\mathcal{H})$  has compact closure in  $\mathcal{H}$ , where  $\text{ball}\mathcal{H} = \{h \in \mathcal{H} : \|h\| \leq 1\}$ . We denote  $B_0(\mathcal{H})$  the set of all compact operators on  $\mathcal{H}$ .

We will study finite-rank operator interpolation problems on  $\mathcal{L}$  and find a compact operator A in Alg $\mathcal{L}$  such that AX = Y for given X and Y in  $B(\mathcal{H})$  as convergence of finite-rank operators. Also, we will study this problem for given countable operators  $X_1, X_2, \ldots$  and  $Y_1, Y_2, \ldots$ 

**Theorem 1.1** ([2]). Let X and Y be bounded operators acting on a Hilbert space  $\mathcal{H}$ . Then the following statements are equivalent:

(1) range  $Y^* \subseteq$  range  $X^*$ 

(2)  $Y^*Y \leq \lambda^2 X^*X$  for some  $\lambda \geq 0$ 

(3) there exists a bounded operator A on  $\mathcal{H}$  so that AX = Y.

Moreover, if (1), (2), and (3) are valid, then there exists a unique operator A so that

 $(a) ||A||^2 = \inf\{\mu : Y^*Y \le \mu X^*X\}$ 

(b) 
$$kerY^* = kerA^*$$
 and

(c) 
$$rangeA^* \subseteq rangeX$$
.

**Lemma 1.2.** Let A and X be bounded operators acting on a Hilbert space  $\mathcal{H}$ .

(a) If  $X = y \otimes x$  is a rank-one operator and  $Ay \neq 0$ , then AX is a rank-one operator.

(b) If  $X = (y1 \otimes x1) + (y2 \otimes x2)$  is a rank-two operator and  $Ay_1$  and  $Ay_2$  are linearly independent, then AX is a rank-two operator.

(c) If  $X = \sum_{i=1}^{n} y_i \otimes x_i$  is a rank-*n* operator and  $Ay_1, \ldots, Ay_n$  are linearly

independent, then AX is a rank-n operator.

(d) If X is a rank-n operator, then AX is a rank-m operator for  $m \leq n$ .

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*Proof.* Let  $f \in \mathcal{H}$ . Then (a)  $AXf = A(y \otimes x)f = A\langle f, x \rangle y = \langle f, x \rangle Ay$ . So AX is a rank-one operator.

(b) 
$$AXf = A((y_1 \otimes x_1) + (y_2 \otimes x_2))f = A\langle f, x_1 \rangle y_1 + A\langle f, x_2 \rangle y_2 = \langle f, x_1 \rangle Ay_1 + \langle f, x_2 \rangle Ay_2$$
 for all  $f \in \mathcal{H}$ .

(c) 
$$AXf = A(\sum_{i=1}^{n} y_i \otimes x_i)f = A\sum_{i=1}^{n} \langle f, x_i \rangle y_i = \sum_{i=1}^{n} \langle f, x_i \rangle Ay_i \text{ for all } f \in \mathcal{H}.$$
  
(d) can be easily obtained by using (a), (b) and (c).

**Theorem 1.3.** Let X and Y be bounded operators acting on a Hilbert space  $\mathcal{H}$ . Then the following are equivalent:

(1) There is a rank-one operator  $A = y \otimes x$  such that AX = Y and range  $X \not\subset \{x\}^{\perp}$ .

(2)  $rangeY^* \subseteq rangeX^*$  and Y is a rank-one operator.

*Proof.* If there exists a rank-one operator  $A = y \otimes x$  such that AX = Y. Then  $rangeY^* \subseteq rangeX^*$ . Since  $rangeX \not\subset \{x\}^{\perp}$ , there exists a vector h in  $\mathcal{H}$  such that  $\langle Xh, x \rangle \neq 0$ . For any f in  $\mathcal{H}$ ,  $Yf = (y \otimes x)Xf = \langle Xf, x \rangle y$ . So Y is a rank-one operator.

Conversely, suppose  $rangeY^* \subseteq rangeX^*$  and  $Y = y_1 \otimes x_1$ . Then there exists an operator B in  $B(\mathcal{H})$  such that BX = Y. Then for each  $h \in \mathcal{H}$ ,

$$(y_1 \otimes x_1)h = \langle h, x_1 \rangle y_1 = BXh.$$

So  $B(rangeX) = rangeY = sp\{y_1\}$ . Define  $A : \overline{rangeX} \to \mathcal{H}$  by Ah = Bhif  $h \in rangeX$  and Ah = 0 if  $h \in B(H)$ . Since  $rangeA = B(\overline{rangeX}) \subset \overline{B(rangeX)} = sp\{y_1\}$ ,  $rangeA \subset sp\{y_1\}$ . Since rangeA is a linear subspace containing  $y_1, sp\{y_1\} \subset \overline{rangeA}$ . Hence A is a rank-one operator and AX = Y.

**Theorem 1.4.** Let X and Y be bounded operators acting on a Hilbert space  $\mathcal{H}$ . If range  $Y^* \subseteq \operatorname{range} X$ . and Y is a finite-rank operator, then there exists a finite-rank operator A such that AX = Y.

*Proof.* If Y is a finite-rank operator, then  $Y = \sum_{i=1}^{n} y_i \otimes x_i$ , where  $y_1, \ldots, y_n$  are linearly independent. Since  $rangeY^* \subseteq rangeX^*$ , there exists an operator B in  $B(\mathcal{H})$  such that BX = Y. For each  $h \in \mathcal{H}$ ,

$$BXh = Yh = \sum_{i=1}^{n} (y_i \otimes x_i)h = \sum_{i=1}^{n} \langle h, x_i \rangle y_i.$$

So  $B(rangeX) = sp\{y_1, \ldots, y_n\}$ . Define  $A : rangeX \to \mathcal{H}$  by Ah = Bh if  $h \in rangeX$  and Ah = 0 if  $h \in rangeX^{\perp}$ . Then  $A \in B(\mathcal{H})$ . And  $rangeA = B(rangeX) \subset \overline{B(rangeX)} = sp\{y_1, \ldots, y_n\}$ . Therefore dim(rangeA) = n and AX = Y.

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## 2. Results

**Theorem 2.1.** Let  $\mathcal{L}$  be a subspace lattice on  $\mathcal{H}$  and let X and Y be operators in  $B(\mathcal{H})$ . Assume that the range of X is dense in  $\mathcal{H}$ . Then the following are equivalent:

(1) There exists a compact operator A in Alg $\mathcal{L}$  such that AX = Y.

(2)  $\sup\left\{\frac{\|E^{\perp}Yf\|}{\|E^{\perp}Xf\|}: f \in \mathcal{H}, \ E \in \mathcal{L}\right\} = K < \infty \text{ and } Y \text{ is compact.}$ 

Moreover, if condition (2) holds, we may choose an operator A such that ||A|| =K.

*Proof.* Assume that  $\sup \left\{ \frac{\|E^{\perp}Yf\|}{\|E^{\perp}Xf\|} : f \in \mathcal{H}, E \in \mathcal{L} \right\} = K < \infty$ . Then there exists an operator A in Alg $\mathcal{L}$  such that AX = Y by Theorem 3.1 [5]. Since Y is compact, there is a sequence  $\{Y_n\}$  of finite-rank operators that converges to Y in the norm topology on  $B(\mathcal{H})$ . From the construction of  $Y_n$ , since  $rangeY_n^* \subseteq$  $rangeY^*$  for each  $n \in \mathbb{N}$ ,  $rangeY^*_n \subseteq rangeX^*$  for each  $n \in \mathbb{N}$ . By Theorems 1.4, there is a finite-rank operator  $A_n$  such that  $A_n X = Y_n$  for each  $n \in \mathbb{N}$ . Since  $Y_n \to Y$  in the norm topology on  $B(\mathcal{H}), ||An - A|| \to 0$ . Hence A is compact. The proof of the converse is obvious. 

**Theorem 2.2.** Let  $\mathcal{L}$  be a subspace lattice on  $\mathcal{H}$  and let  $X_1, \ldots, X_n$  and  $Y_1, \ldots, Y_n$ be bounded operators acting on  $\mathcal{H}$ . Let k be a fixed natural number in  $\{1, 2, \ldots, n\}$ and assume that  $X_k$  has dense range. Then the following are equivalent:

(1) There exists a compact operator A in Alg $\mathcal{L}$  such that  $AX_i = Y_i$  for each

 $i = 1, 2, \dots, n.$   $(2) \sup \left\{ \frac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H}, \ E \in \mathcal{L} \right\} = K < \infty \text{ and } Y_k \text{ is compact.}$   $(2) \sup \left\{ \frac{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H}, \ E \in \mathcal{L} \right\} = K < \infty \text{ and } Y_k \text{ is compact.}$ Moreover, if condition (2) holds, we may choose an operator A such that ||A|| =K.

*Proof.* If  $\sup\left\{\frac{\|E^{\perp}(\sum_{i=1}^{n}Y_{i}f_{i})\|}{\|E^{\perp}(\sum_{i=1}^{n}X_{i}f_{i})\|}: f_{i} \in \mathcal{H}, E \in \mathcal{L}\right\} = K < \infty$ , and for given k in  $\{1, 2, \ldots, n\}, Y_k$  is compact, then by Theorem 3.2 [5], there exists an operator A in Alg $\mathcal{L}$  such that  $AX_i = Y_i$  for  $i = 1, 2, \dots, n$ . Since  $Y_k$  is compact, there is a sequence  $\{Y_{k_m}\}$  of finite-rank operators that converges to  $Y_k$  in the norm topology on  $B(\mathcal{H})$ . From the construction of  $Y_{k_m}$ , we know that  $rangeY_{k_m}^* \subseteq rangeY_k^*$  for each  $m \in \mathbb{N}$ . Therefore  $rangeY_{k_m}^* \subseteq rangeX^*$  for each  $m \in \mathbb{N}$ . By Theorem 2.1, for each  $m \in \mathbb{N}$ , there is a finite-rank operator  $A_{k_m}$  such that  $A_{k_m}X_k = Y_{k_m}$ . Since  $Y_{k_m} \to Y_k$  in the norm topology on B(H),  $||A_{k_m} - A|| \to 0$ . Hence A is compact. We omit the proof of the converse since it can be proved easily. 

**Theorem 2.3** ([5]). Let  $\mathcal{L}$  be a subspace lattice on  $\mathcal{H}$  and let  $X_1, \ldots, X_n$  and  $Y_1, \ldots, Y_n$  be bounded operators acting on  $\mathcal{H}$ . Let k be a fixed natural number in  $\{1, 2, \ldots, n\}$  and assume that  $X_k$  has dense range and  $Re\langle E^{\perp}X_i f, E^{\perp}X_j g\rangle \geq 0$ for each E in  $\mathcal{L}$ , i < j and all f, g in  $\mathcal{H}$ . Then the following are equivalent:

(1)  $range E^{\perp} Y_i^* \subseteq range E^{\perp} X_i^*$  for each E in  $\mathcal{L}$  and i = 1, 2, ..., n.

(2) There exists an operator A in AlgL such that  $AX_i = Y_i$  for i = 1, 2, ..., n.

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$$(3) \sup\left\{\frac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H}, \ E \in \mathcal{L}\right\} < \infty.$$

By the Theorem 2.3, we can get the following Theorem.

**Theorem 2.4.** Let  $\mathcal{L}$  be a subspace lattice on  $\mathcal{H}$  and let  $X_1, \ldots, X_n$  and  $Y_1, \ldots, Y_n$ be bounded operators acting on  $\mathcal{H}$ . Let k be a fixed natural number in  $\{1, 2, \ldots, n\}$ and assume that  $X_k$  has dense range. If for i < j in  $\{1, 2, \ldots, n\}$  and all f, g in  $\mathcal{H}, \operatorname{Re}\langle E^{\perp}X_i f, E^{\perp}X_j g \rangle \geq 0$ , then the following are equivalent:

(1)  $range E^{\perp} Y_i^* \subseteq range E^{\perp} X_i^*$  for each E in  $\mathcal{L}$  and i = 1, 2, ..., n, and  $Y_k$  is compact.

(2) There exists a compact operator A in Alg $\mathcal{L}$  such that  $AX_i = Y_i$  for i = 1, 2, ..., n.

If we observe the proof of the above theorems, we can generalize Theorem 2.2 to the countable case easily.

**Theorem 2.5.** Let  $\mathcal{L}$  be a subspace lattice on  $\mathcal{H}$  and let  $X_i$  and  $Y_i$  be bounded operators acting on  $\mathcal{H}$  for all i = 1, 2, ..., n. Let k be a fixed natural number in  $\{1, 2, ..., n\}$  and assume that  $X_k$  has dense range. Then the following are equivalent:

(1) There exists a compact operator A in Alg $\mathcal{L}$  such that  $AX_i = Y_i$  for i = 1, 2, ..., n.

 $(2) \sup\left\{\frac{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\|}{\|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\|} : f_i \in \mathcal{H}, \ E \in \mathcal{L}, \ m \in \mathbb{N}\right\} = K < \infty \ and \ Y_k \ is \ compact.$ 

Moreover, if condition (2) holds, we may choose an operator an operator A such that ||A|| = K.

**Theorem 2.6.** Let  $\mathcal{L}$  be a subspace lattice on  $\mathcal{H}$  and let  $X_i$  and  $Y_i$  be bounded operators acting on  $\mathcal{H}$  for all i = 1, 2, ..., n. Let k be a fixed natural number in  $\{1, 2, ..., n\}$  and assume that  $X_k$  has dense range and  $\operatorname{Re}\langle E^{\perp}Y_i f, E^{\perp}Y_j g \rangle \leq$  $\operatorname{Re}\langle E^{\perp}X_i f, E^{\perp}X_j g \rangle$  for each E in  $\mathcal{L}$ , i < j and all f, g in  $\mathcal{H}$ , then the following are equivalent:

(1) There exists  $M \ge 0$  such that  $\sup\left\{\frac{\|E^{\perp}Y_if\|}{\|E^{\perp}X_if\|} : f \in \mathcal{H}, \ E \in \mathcal{L}\right\} < M$  for each  $i \in \mathbb{N}$  and  $Y_k$  is compact.

(2) There is a compact operator A in AlgL such that  $AX_i = Y_i$  for i = 1, 2, ..., n.

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**Joo Ho Kang** received her Ph.D. at the University of Alabama under the direction of Tavan T. Trent. She has been a professor of Daegu University since 1977. Her research interest is an operator theory.

Department of Mathematics, Daegu University, Kyungpook, Korea. e-mail: jhkang@daegu.ac.kr

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