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VARIOUS TYPES OF WELL-POSEDNESS FOR MIXED VECTOR QUASIVARIATIONAL-LIKE INEQUALITY USING BIFUNCTIONS

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ABSTRACT. In this paper, we investigate the α-well-posedness and α-L-well-posedness for a mixed vector quasivariational-like inequality using bifunctions. Some characterizations are derived for the above mentioned well-posedness concepts. The concepts of α-well-posedness and α-L-well-posedness in the generalized sense are also given and similar characterizations are derived.

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1. Introduction

The notion of well-posedness is significant as it plays a crucial role in the stability theory for optimization problems and has been studied in different areas of optimization such as mathematical programming, calculus of variations and optimal control. Such a study becomes important for problems wherein, we may not be able to find the exact solution of the problem. Under these circumstances, the well-posedness of an optimization problem is pivotal in the sense that it ensures the convergence of the sequence of approximate solutions obtained through iterative techniques to the exact solution of the problem.

Well-posedness of a minimization problem was first considered by Tykhonov [23] according to which every minimizing sequence converges towards the unique minimum solution. Practically, a problem may have more than one solution. Hence, the notion of well-posedness in the generalized sense was introduced. The nonemptiness of the set of minimizers and the convergence of subsequence of the minimizing sequence towards a member of this set guarantees well-posedness in the generalized sense. Zolezzi [26, 27] introduced and studied the extended

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well-posedness for an optimization problem by embedding the original problem into a parametric optimization problem. For further details, one may refer to the text by Lucchetti [17].

Variational inequality provides suitable mathematical models for a wide range of practical problems and have been intensively studied in [7, 14]. Since, a minimization problem is closely related to a variational inequality, hence it is important to study the well-posedness of variational inequality. Lucchetti and Patrone [15] introduced the notion of well-posedness for a variational inequality by means of Ekeland's Variational Principle. Lignola and Morgan [16] introduced parametric well-posedness for variational inequalities whereas in [18], Lignola introduced the notions of well-posedness and L-well-posedness for quasivariational inequalities and derived some metric characterizations. The corresponding results of Lignola and Morgan [16] were extended to the vector case by Fang and Huang [8]. Prete et al. [21] introduced the concept of α-well-posedness for the classical variational inequality. Fang, Huang and Yao [10] introduced the notion of well-posedness for a mixed variational inequality and studied its relationship with the well-posedness of corresponding inclusion and fixed point problems which was further generalized by Ceng and Yao [2] for generalized mixed variational inequality. Parametric variational inequalities are problems where a parameter is allowed to vary in a certain subset of a metric space. It has been shown that the parametric variational inequality is a central ingredient in the class of Mathematical Programs with Equilibrium Constraints which appear in many applied contexts and have been studied by many authors [16, 21].

A quasivariational inequality is an extension of the classical variational inequality in which the defining set of the problem varies with a variable. The interest in quasivariational inequalities lies in the fact that many economic or engineering problems are modeled through them. Very recently, Ceng et al. [3] studied the concepts of well-posedness and L-well-posedness for mixed quasivariational-like inequality problems (MQVLI). Fang and Hu [9] and Hu, Fang and Huang [12] studied well-posedness for parametric variational inequality and quasivariational inequality respectively using bifunctions.

Motivated by the above mentioned research work,in this paper, we generalize the concepts of α-well-posedness and α-L-well-posedness for parametric mixed vector quasivariational-like inequality (MVQVLI$_p$) having a unique solution and in the generalized sense if (MVQVLI$_p$) has more than one solution. Necessary and sufficient conditions for α-well-posedness and α-L-well-posedness are formulated in terms of the diameters of the approximate solution sets. In a similar way, α-well-posedness and α-L-well-posedness in the generalized sense is shown to be equivalent to a condition involving a regular measure of non compactness of the approximate solution sets.

The paper is organized as follows: In Section 2, necessary notations, definitions and lemmas have been recalled. Section 3 establishes necessary and sufficient conditions for α-well-posedness and α-L-well-posedness for (MVQVLI$_p$) using bifunctions, while in Section 4, necessary and sufficient conditions are obtained for α-well-posedness and α-L-well-posedness in the generalized sense. Finally in Section 5, α-well-posedness and α-L-well-posedness of (MVQVLI$_p$) are shown to be equivalent to the existence and uniqueness of their respective solutions.

2. Preliminaries

Throughout this paper, we suppose that $\alpha \geq 0$, K is a nonempty closed subset of a real Banach space X. Let $\eta : K \times K \to X$ be a map. Let P be a parametric norm space, $S : P \times K \to 2^K$ be a set-valued map. Let Y be a real Banach space endowed with a partial order induced by a pointed, closed and convex cone C with $intC$ nonempty;

$$x \geq_C y \Leftrightarrow x - y \in C, \qquad x \geq_{intC} y \Leftrightarrow x - y \in intC,$$
$$x \ngeq_C y \Leftrightarrow x - y \notin C, \qquad x \ngeq_{intC} y \Leftrightarrow x - y \notin intC.$$

Let $h : P \times K \times X \to Y$ be a function. Let $\phi : K \times K \to Y$ be a bifunction. We consider the following parametric mixed vector quasivariational-like inequality using bifunctions;

$$\text{MVQVLI}_p(h, S) \quad \text{Find } x \in K \text{ such that } x \in S(p, x),$$
$$h(p, x, \eta(x, y)) + \phi(x, y) \ngeq_{intC} 0, \quad \forall\, y \in S(p, x).$$

It is observed that MVQVLI$_p(h, S)$ provides very general formulations of variational inequalities which include the classical Stampacchia variational inequality as a special case (see [14]), mixed quasivariational-like inequalities (see [3]), variational inequalities defined using bifunctions (see [9]), parametric quasivariational inequality (see [19]) and parametric quasivariational inequality defined using bifunctions (see [12]).

In particular, we observe that, if $\phi(x, y) = \phi(x) - \phi(y)$ and $Y = \bar{\mathbb{R}}$, then MVQVLI$_p(h, S)$ reduces to mixed quasivariational-like inequality studied in [3]. If $\phi(x, y) = 0$, $\eta(x, y) = x - y$, $\forall\, x, y \in K$ and $Y = \bar{\mathbb{R}}$, then MVQVLI$_p(h, S)$ reduces to the parametric Stampacchia quasivariational inequality using bifunctions SQVI$_p(h, S)$ which has been dealt in [12]. If further, $S(p, x) = K$, $\forall\, x \in K$, then it reduces to the parametric Stampacchia variational inequality using bifunctions studied in [9]. The solution set of MVQVLI$_p(h, S)$ is denoted by T_p. In the sequel, we introduce some notions of well-posedness for MVQVLI$_p(h, S)$.

Definition 2.1. Let $\alpha \geq 0$. Let $p \in P$ and $\{p_n\} \subset P$ be a sequence converging to p. A sequence $\{x_n\} \subset X$ is said to be an **α-approximating sequence** [respectively an **α-L-approximating sequence**] for MVQVLI$_p(h, S)$ corresponding to $\{p_n\}$ if and only if:

(i) $x_n \in K$, $\forall\, n \in \mathbb{N}$.
(ii) there exists a sequence of positive numbers $\{\epsilon_n\}$ with $\epsilon_n \downarrow 0$ such that:

$$d(x_n, S(p_n, x_n)) \leq \epsilon_n, \;\; h(p_n, x_n, \eta(x_n, y)) + \phi(x_n, y) \ngeq_{intC} \left(\frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n\right)e,$$
$$y \in S(p_n, x_n), \;\; \forall\, n \in \mathbb{N},$$

[respectively if:

(i) $x_n \in K, \forall\, n \in \mathbb{N}$.

(ii) there exists a sequence of positive numbers $\{\epsilon_n\}$ with $\epsilon_n \downarrow 0$ such that:

$$d(x_n, S(p_n, x_n)) \leq \epsilon_n; h(p_n, y, \eta(x_n, y)) + \phi(x_n, y) \not\geq_{intC} \left(\frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n\right)e,$$
$$y \in S(p_n, x_n),\ \forall\, n \in \mathbb{N},]$$

where e is any fixed point in $intC$.

Remark 2.2. Definition 2.1 generalizes Definition 2.3 of Lignola [18], Definition 2.2 of Ceng et al. [3] and Definition 1 of Hu et al. [12].

Definition 2.3. The family $\{\text{MVQVLI}_p(h, S) : p \in P\}$ is said to be **α-well-posed** [respectively **α-L-well-posed**] if $\forall\, p \in P$, $\text{MVQVLI}_p(h, S)$ has a unique solution x_p and for all sequences $\{p_n\} \to p$, every α-approximating sequence [respectively α-L-approximating sequence] corresponding to $\{p_n\}$ converges to x_p.

Remark 2.4. Definition 2.3 generalizes Definition 2.4 of [18], Definition 2.3 of [3] and Definition 2 of [12].

Definition 2.5. The family $\{\text{MVQVLI}_p(h, S) : p \in P\}$ is said to be **α-well-posed in the generalized sense** [respectively **α-L-well-posed in the generalized sense**] if $\forall\, p \in P$, $\text{MVQVLI}_p(h, S)$ has a nonempty solution set and for all sequences $\{p_n\} \to p$, every α-approximating sequence [respectively α-L-approximating sequence] corresponding to $\{p_n\}$ has a subsequence which converges to some point of the solution set.

In order to characterize the well-posedness of the quasivariational inequality, Lignola [18] defined some concepts of approximate solutions for quasivariational inequalities. Motivated by these concepts, for every $\alpha \geq 0$, $\epsilon \geq 0$, $\delta \geq 0$, we consider the following α-approximate and α-L-approximate solution sets;

$$\text{Q}_p(\delta, \epsilon) = \bigcup_{\bar{p} \in B(p,\delta)} \Big\{ x \in K : d(x, S(\bar{p}, x)) \leq \epsilon \text{ and}$$
$$h(\bar{p}, x, \eta(x, y)) + \phi(x, y) \not\geq_{intC} \left(\frac{\alpha}{2}\|x - y\|^2 + \epsilon\right)e,\ \forall\, y \in K \Big\}.$$

$$\text{L}_p(\delta, \epsilon) = \bigcup_{\bar{p} \in B(p,\delta)} \Big\{ x \in K : d(x, S(\bar{p}, x)) \leq \epsilon \text{ and}$$
$$h(\bar{p}, y, \eta(x, y)) + \phi(x, y) \not\geq_{intC} \left(\frac{\alpha}{2}\|x - y\|^2 + \epsilon\right)e,\ \forall\, y \in K \Big\}.$$

To investigate the α-well-posedness and α-L-well-posedness of $\text{MVQVLI}_p(h, S)$, we need the following concepts and results.

Definition 2.6 ([13])**.** Let H be a non empty subset of X. The **measure of noncompactness** μ of the set H is defined by

$$\mu(H) = \inf\left\{\epsilon > 0 : H \subseteq \bigcup_{i=1}^{n} H_i, \operatorname{diam} H_i < \epsilon, i = 1, 2, \ldots, n\right\},$$

where $\operatorname{diam} H_i = \sup\{d(a_1, a_2) : a_1, a_2 \in H_i\}$.

Definition 2.7 ([13])**.** The **Hausdorff Distance** between two nonempty bounded subsets A and B of a metric space (X, d) is

$$\mathcal{H}(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup_{a \in A} d(a, B) = \sup_{a \in A} \inf_{b \in B} d(a, b)$.

Definition 2.8 ([3, 8])**.** Let $h : P \times K \times X \to Y$ be a function and let $\phi : K \times K \to Y$ be a bifunction. Let $\eta : K \times K \to X$ be a map. Then h is said to be

(i) **C-η-monotone** if for any $x, y \in K$,

$$h(p, x, \eta(x, y)) - h(p, y, \eta(x, y)) \geq_{intC} 0.$$

(ii) **C-η-pseudomonotone** with respect to ϕ if for any $x, y \in K$,

$$\begin{aligned} & h(p, x, \eta(x, y)) + \phi(x, y) \ngeq_{intC} 0 \\ \Rightarrow \quad & h(p, y, \eta(x, y)) + \phi(x, y) \ngeq_{intC} 0. \end{aligned}$$

Definition 2.9 ([13])**.** Let (E, τ) and (F, σ) be two 1st countable topological spaces. A set valued map $G : E \to 2^F$ is said to be,

(i) (τ, σ)**-closed** if for all $x \in E$, for all sequences $\{x_n\}$ τ-converging to x and for all sequences $\{y_n\}$ σ-converging to y such that $y_n \in G(x_n)$, $\forall\ n \in \mathbb{N}$, one has $y \in G(x)$, that is, $G(x) \supset \limsup_n G(x_n)$.

(ii) (τ, σ)**-lower semicontinuous** if for all $x \in E$, for all sequences $\{x_n\}$ τ-converging to x and for all $y \in G(x)$, there exists a sequence $\{y_n\}$ σ-converging to y such that $y_n \in G(x_n)$ for sufficiently large n, that is, $G(x) \subset \liminf_n G(x_n)$.

(iii) (τ, σ)**-subcontinuous** if for all $x \in E$, for all sequences $\{x_n\}$ τ-converging to x and for all sequences $\{y_n\}$ with $y_n \in G(x_n)$, y_n has a σ-convergent subsequence.

Definition 2.10. A function $g : X \to \mathbb{R}$ is said to be **positively homogeneous** if $g(\lambda x) = \lambda g(x)$, $\forall\ x \in X$, $\forall\ \lambda > 0$.

Lemma 2.11. *Let K be convex and $x \in K$ be a given point. Let $h : P \times K \times X \to Y$ be a positively homogeneous function in 3rd variable, $y \mapsto h(p, x, \eta(x, y))$ be concave and $\eta(x, x) = 0$, ϕ be a bifunction with $\phi(x, x) = 0$ for fixed x and $y \mapsto \phi(x, y)$ concave. Then,*

$$\begin{aligned} & h(p, x, \eta(x, y)) + \phi(x, y) \ngeq_{intC} 0, \quad \forall\ y \in K \\ \Leftrightarrow \quad & h(p, x, \eta(x, y)) + \phi(x, y) \ngeq_{intC} \frac{\alpha}{2}\|x - y\|^2 e, \quad \forall\ y \in K. \end{aligned}$$

Proof. Obviously, necessary condition holds true.

For sufficient condition, let $h(p,x,\eta(x,y))+\phi(x,y) \not\geq_{intC} \frac{\alpha}{2}\|x-y\|^2 e, \forall\, y \in K$. For any $v \in K$, let $y(t) = x + t(v-x) \in K, \forall\, t \in [0,1]$ with $y(t) \neq x$. Hence,

$$h(p,x,\eta(x,y(t))) + \phi(x,y(t)) \not\geq_{intC} \frac{\alpha}{2}\|x-y(t)\|^2 e.$$

Now, $y \mapsto h(p,x,\eta(x,y))$ is concave and $\eta(x,x) = 0$. So, $h(p,x,\eta(x,y(t))) \geq h(p,x,t\eta(x,v))$. Since, h is positively homogeneous in the 3rd variable, we obtain,

$$h(p,x,\eta(x,y(t))) + \phi(x,y(t)) \geq th(p,x,\eta(x,v)) + \phi(x,y(t)).$$

As $\phi(x,.)$ is concave and $\phi(x,x) = 0$, we get that

$$\phi(x,y(t)) = \phi(x,x+t(v-x)) = t\phi(x,v).$$

Therefore, using the fact that $t \in [0,1]$, we have,

$$h(p,x,\eta(x,y(t))) + \phi(x,y(t)) \geq th(p,x,\eta(x,v)) + t\phi(x,v).$$

Thus, $th(p,x,\eta(x,v)) + t\phi(x,v) \not\geq_{intC} \frac{\alpha}{2}t^2\|x-v\|^2 e$.

Dividing by $t > 0$ and taking limit as $t \to 0$, we get the required sufficient condition. $\square$

Remark 2.12. Lemma 2.11 is a generalization of Lemma 2 of [12].

3. Mixed Vector Quasivariational-like Inequality Having a Unique Solution

In this section, we give some metric characterizations of α-well-posedness and α-L-well-posedness for $\mathrm{MVQVLI}_p(h,S)$.

Theorem 3.1. *Let K be a closed and convex subset of a real Banach space X. Let $\phi : K \times K \to Y$ be a continuous bifunction with $\phi(x,x) = 0$ for fixed x and $\phi(x,.)$ concave. Let $\eta : K \times K \to X$ be a continuous mapping with $y \mapsto h(p,x,\eta(x,y))$ concave and $\eta(x,x) = 0$. Let $S : P \times K \to 2^K$ be a nonempty set-valued map which is convex valued, (s,w)-closed, (s,w)-subcontinuous and (s,s)-lower semicontinuous. Let $h : P \times K \times X \to Y$ be a continuous function which is positively homogeneous in the 3rd variable. Then, $MVQVLI_p$ is α-well-posed if and only if, $\forall\, p \in P$,*

$$Q_p(\delta,\epsilon) \neq \emptyset,\ \forall\, \delta,\epsilon > 0 \text{ and } \operatorname{diam} Q_p(\delta,\epsilon) \to 0 \text{ as } (\delta,\epsilon) \to (0,0). \tag{3.1}$$

Proof. Let MVQVLI_p be α-well-posed. Then, $T_p \neq \emptyset$ and $T_p \subset \mathrm{Q}_p(\delta,\epsilon)$. Thus, $\mathrm{Q}_p(\delta,\epsilon) \neq \emptyset, \forall\, \delta,\epsilon > 0$. Assume on the contrary, $\operatorname{diam} \mathrm{Q}_p(\delta,\epsilon) \nrightarrow 0$ as $(\delta,\epsilon) \to (0,0)$. Then, there exist sequences $\{\epsilon_n\}$, $\{\delta_n\}$, $\{u_n\}$, $\{v_n\}$ with $\epsilon_n \to 0$, $\delta_n \to 0$, $u_n, v_n \in \mathrm{Q}_p(\delta_n,\epsilon_n)$ and a positive number l such that

$$\|u_n - v_n\| > l, \quad \forall\, n\,. \tag{3.2}$$

As $u_n, v_n \in \mathrm{Q}_p(\delta_n, \epsilon_n)$, there exist $p_n, \bar{p}_n \in B(p, \delta_n)$ such that

$$d(u_n, S(p_n, u_n)) \leq \epsilon_n \text{ and } h(p_n, u_n, \eta(u_n, y)) + \phi(u_n, y) \ngeq_{intC} \left(\frac{\alpha}{2}\|u_n - y\|^2 + \epsilon_n\right)e, \quad \forall\, y \in K,\ \forall\, n \in \mathbb{N}$$

$$d(v_n, S(\bar{p}_n, v_n)) \leq \epsilon_n \text{ and} h(\bar{p}_n, v_n, \eta(v_n, y)) + \phi(v_n, y) \ngeq_{intC} \left(\frac{\alpha}{2}\|v_n - y\|^2 + \epsilon_n\right)e, \quad \forall\, y \in K,\ \forall\, n \in \mathbb{N}.$$

Thus, $\{u_n\}$ and $\{v_n\}$ are α-approximating sequences for MVQVLI_p corresponding to $\{p_n\}$ and $\{\bar{p}_n\}$ respectively. As MVQVLI_p is α-well-posed, both the α-approximating sequences converge to the unique solution of MVQVLI_p , which is a contradiction to (3.2).

Conversely, suppose $\forall\, p \in P$, (3.1) holds. We will first show that MVQVLI_p cannot have more than one solution. Assume z_1 and z_2 are its solutions with $z_1 \neq z_2$. Then, $z_1, z_2 \in \mathrm{Q}_p(\delta, \epsilon)$, $\forall\, \delta, \epsilon \geq 0$. Taking (3.1) into account, we get $z_1 = z_2$, which is a contradiction.

Now, let $\{p_n\} \to p \in P$ and $\{x_n\}$ be an α-approximating sequence for MVQVLI_p. Then, there exists sequence $\{\epsilon_n\}$ with $\epsilon_n \downarrow 0$ such that $x_n \in K$,

$$d(x_n, S(p_n, x_n)) \leq \epsilon_n \text{ and} h(p_n, x_n, \eta(x_n, y)) + \phi(x_n, y) \ngeq_{intC} \left(\frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n\right)e, \quad \forall\, y \in S(p_n, x_n),\ \forall\, n \in \mathbb{N}.$$

Take $\delta_n = \|p_n - p\|$ then $x_n \in \mathrm{Q}_p(\delta_n, \epsilon_n)$. By the given condition, $\{x_n\}$ is a Cauchy sequence and it strongly converges to a point say $x_0 \in K$. We will now prove that x_0 is the unique solution of MVQVLI_p by two steps.

(i) We show that $x_0 \in S(p, x_0)$. Since, $d(x_n, S(p_n, x_n)) \leq \epsilon_n < \epsilon_n + \frac{1}{n}$. Therefore, there exists $y_n \in S(p_n, x_n)$ such that

$$\|x_n - y_n\| < \epsilon_n + \frac{1}{n}.$$

As, S is (s, w)-subcontinuous and (s, w)-closed, the sequence $\{y_n\}$ has a subsequence $\{y_{n_k}\}$ weakly converging to $y \in S(p, x_0)$. Hence,

$$d(x_0, S(p, x_0)) \leq \|x_0 - y\| \leq \liminf \|x_{n_k} - y_{n_k}\| \leq \lim \left(\epsilon_{n_k} + \frac{1}{k}\right) = 0.$$

Thus, $x_0 \in S(p, x_0)$.

(ii) Let $z \in S(p, x_0)$ be an arbitrary element. Since, S is (s, s)-lower semicontinuous, there exists $z_n \in S(p_n, x_n) : z_n \to z$. Thus, there exists sequence $\{\epsilon_n\} \downarrow 0$ such that

$$h(p_n, x_n, \eta(x_n, z_n)) + \phi(x_n, z_n) \ngeq_{intC} \left(\frac{\alpha}{2}\|x_n - z_n\|^2 + \epsilon_n\right)e.$$

h, η and ϕ being continuous, we get

$$h(p, x_0, \eta(x_0, z)) + \phi(x_0, z) \ngeq_{intC} \frac{\alpha}{2}\|x_0 - z\|^2 e.$$

Thus, by Lemma 2.11, we have

$$h(p, x_0, \eta(x_0, z)) + \phi(x_0, z) \ngeq_{intC} 0, \quad \forall\ z \in S(p, x_0),$$

which shows that x_0 is a solution of MVQVLI$_p$.
Hence, MVQVLI$_p$ is α-well-posed. □

Theorem 3.2. *Let $S : P \times K \to 2^K$ be convex valued. Then $MVQVLI_p$ is α-well-posed if and only if its solution set $T_p \neq \emptyset$, $\forall\ p \in P$ and* diam $Q_p(\delta, \epsilon) \to 0$ *as $(\delta, \epsilon) \to (0, 0)$.*

Proof. The necessary condition has been proved in Theorem 3.1.

For sufficiency, let $\{p_n\}$ be a sequence such that $p_n \to p \in P$ and $\{x_n\}$ be an α-approximating sequence for MVQVLI$_p$ corresponding to $\{p_n\}$. Then, there exists sequence $\{\epsilon_n\}$ with $\epsilon_n \downarrow 0$ such that

$$d(x_n, S(p_n, x_n)) \leq \epsilon_n \ \text{ and } h(p_n, x_n, \eta(x_n, y)) + \phi(x_n, y) \ngeq_{intC} \left(\frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n\right)e,$$
$$\forall\ y \in S(p_n, x_n),\ \forall n \in \mathbb{N}.$$

Thus, $x_n \in \mathrm{Q}_p(\delta_n, \epsilon_n)$ with $\delta_n = \|p_n - p\|$. Let x_0 be the unique solution of MVQVLI$_p$. Then, $x_0 \in \mathrm{Q}_p(\delta_n, \epsilon_n)\ \forall\ n$. Thus, $\|x_n - x_0\| \leq \mathrm{diam}\, \mathrm{Q}_p(\delta_n, \epsilon_n) \to 0$, that is, $x_n \to x_0$ and hence, MVQVLI$_p$ is α-well-posed. □

We now have analogous results for α-L-well-posedness.

Theorem 3.3. *Suppose that the hypothesis of Theorem 3.1 hold and let h be C-η-pseudomonotone with respect to ϕ. Then $MVQVLI_p$ is α-L-well-posed if and only if $L_p(\delta, \epsilon) \neq \emptyset$, $\forall\ \delta, \epsilon > 0$ and* diam $L_p(\delta, \epsilon) \to 0$ *as $(\delta, \epsilon) \to (0, 0)$.*

Proof. Let MVQVLI$_p$ be α-L-well-posed. As h is C-η-pseudomonotone with respect to ϕ, $\mathrm{L}_p(\delta, \epsilon) \neq \emptyset$. On the same lines of Theorem 3.1, we get diam $\mathrm{L}_p(\delta, \epsilon) \to 0$ as $(\delta, \epsilon) \to (0, 0)$.

Conversely, let the given condition hold. As h is C-η-pseudomonotone with respect to ϕ, every solution of MVQVLI$_p$ belongs to $\mathrm{L}_p(\delta, \epsilon)$, $\forall\ \delta, \epsilon > 0$ which would also be unique. Also, an α-L-approximating sequence exists. Let $\{x_n\}$ be an α-L-approximating sequence which converges to x_0, as in Theorem 3.1. x_0 would then be the solution of MVQVLI$_p$. Thus, MVQVLI$_p$ is α-L-well-posed. □

Theorem 3.4. *Let $S : P \times K \to 2^K$ be convex valued and h be C-η-pseudomonotone with respect to ϕ. Then, $MVQVLI_p$ is α-L-well-posed if and only if its solution set $T_p \neq \emptyset$ and* diam $L_p(\delta, \epsilon) \to 0$ *as $(\delta, \epsilon) \to (0, 0)$.*

Proof. The necessary condition holds as in Theorem 3.3.

For sufficient condition, let $\{p_n\}$ be a sequence converging to $p \in P$ and $\{x_n\}$ be an α-L-approximating sequence for MVQVLI$_p$ corresponding to $\{p_n\}$. Then, there exists sequence $\{\epsilon_n\}$ with $\epsilon_n \downarrow 0$ such that,

$$d(x_n, S(p_n, x_n)) \leq \epsilon_n \ \text{and} h(p_n, y, \eta(x_n, y)) + \phi(x_n, y) \ngeq_{intC} \left(\frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n\right)e,$$

$$\forall\, y\in S(p_n,x_n),\ \forall\, n\in\mathbb{N}.$$

Thus, $x_n \in \mathrm{L}_p(\delta_n,\epsilon_n)$ with $\delta_n = \|p_n - p\|$. Let x_0 be the unique solution of MVQVLI$_p$. Then, $x_0 \in \mathrm{L}_p(\delta_n,\epsilon_n)$, $\forall\, n$. Thus, $\|x_n - x_0\| \le \operatorname{diam} \mathrm{L}_p(\delta_n,\epsilon_n) \to 0$, that is, $x_n \to x_0$ and the problem is α-L-well-posed. □

4. Mixed Vector Quasivariational-like Inequality having more than One Solution

In this section, we give some metric characterizations of α-well-posedness and α-L-well-posedness in the generalized sense for MVQVLI$_p(h,S)$.

Theorem 4.1. *Let all the assumptions of Theorem 3.1 be true and let P be finite dimensional. Then, $MVQVLI_p$ is α-well-posed in the generalized sense if and only if,* $\forall\, p \in P$,

$$Q_p(\delta,\epsilon) \ne \emptyset\ \forall\ \delta,\epsilon > 0 \text{ and } \lim_{\delta\to 0,\epsilon\to 0} \mu(Q_p(\delta,\epsilon)) = 0.$$

Proof. Let MVQVLI$_p$ be α-well-posed in the generalized sense. Then, $T_p \ne \emptyset$ and $T_p \subset \mathrm{Q}_p(\delta,\epsilon)$. Thus, $\mathrm{Q}_p(\delta,\epsilon) \ne \emptyset\ \forall\ \delta,\epsilon > 0$. Also, T_p is compact as when $\{x_n\}$ is any sequence in T_p, then $\{x_n\}$ would be an α-approximating sequence for MVQVLI$_p$ which is α-well-posed in the generalized sense, therefore $\{x_n\}$ would have a subsequence converging strongly to some point of T_p. Now,

$$\mathcal{H}(\mathrm{Q}_p(\delta,\epsilon),T_p) = \max\{e(\mathrm{Q}_p(\delta,\epsilon),T_p), e(T_p,\mathrm{Q}_p(\delta,\epsilon))\} = e(\mathrm{Q}_p(\delta,\epsilon),T_p).$$

Also, $\mu(\mathrm{Q}_p(\delta,\epsilon)) \le 2\mathcal{H}(\mathrm{Q}_p(\delta,\epsilon),T_p) + \mu(T_p) = 2e(\mathrm{Q}_p(\delta,\epsilon),T_p)$. It is now sufficient to show that $e(\mathrm{Q}_p(\delta,\epsilon),T_p) \to 0$ as $(\delta,\epsilon) \to (0,0)$. If $e(\mathrm{Q}_p(\delta,\epsilon),T_p) \nrightarrow 0$ as $(\delta,\epsilon) \to (0,0)$, there exists $\tau > 0$ and sequences $\{\delta_n\},\{\epsilon_n\}$ with $\delta_n \downarrow 0$, $\epsilon_n \downarrow 0$, $x_n \in K$ with $x_n \in \mathrm{Q}_p(\delta_n,\epsilon_n)$ such that

$$x_n \notin T_p + B(0,\tau). \tag{4.1}$$

Since, $x_n \in \mathrm{Q}_p(\delta_n,\epsilon_n)$, $\{x_n\}$ is an α-approximating sequence of MVQVLI$_p$ which is α-well-posed in the generalized sense. Hence, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some point of T_p which is a contradiction to (4.1).

Conversely, let $\mathrm{Q}_p(\delta,\epsilon) \ne \emptyset$ and $\lim_{\delta\to 0,\epsilon\to 0} \mu(\mathrm{Q}_p(\delta,\epsilon)) = 0$. We first show that $\mathrm{Q}_p(\delta,\epsilon)$ is closed, $\forall\ \delta,\epsilon > 0$. Let $x_n \in \mathrm{Q}_p(\delta,\epsilon)$ such that $x_n \to x$. Then, there exists $p_n \in B(p,\delta)$ such that $d(x_n, S(p_n,x_n)) \le \epsilon$ and

$$h(p_n,x_n,\eta(x_n,y)) + \phi(x_n,y) \not\geq_{intC} \left(\frac{\alpha}{2}\|x_n - y\|^2 + \epsilon\right)e, \quad \forall\ y \in S(p_n,x_n).$$

P being finite dimensional, $p_n \to \bar{p} \in B(p,\delta)$. As, $d(x_n,S(p_n,x_n)) \le \epsilon < \epsilon + \frac{1}{n}$, there exists $y_n \in S(p_n,x_n)$ such that $\|x_n - y_n\| < \epsilon + \frac{1}{n}$. S being (s,w)-closed and (s,w)-subcontinuous, $\{y_n\}$ has a subsequence $\{y_{n_k}\}$ which converges weakly to $y \in S(\bar{p},x)$. Thus,

$$d(x,S(\bar{p},x)) \le \|x-y\| \le \liminf \|x_{n_k} - y_{n_k}\| \le \liminf \left(\epsilon + \frac{1}{k}\right) = \epsilon,$$

that is, $d(x, S(\bar{p}, x)) \leq \epsilon$. Let $z \in S(\bar{p}, x)$. S being (s, s)-lower semicontinuous, there exists $z_n \in S(p_n, x_n)$ such that $z_n \to z$. Thus, $h(p_n, x_n, \eta(x_n, z_n)) + \phi(x_n, z_n) \not\geq_{intC} \left(\frac{\alpha}{2}\|x_n - z_n\|^2 + \epsilon\right)e$. By continuity of h, η and ϕ, we have,

$$h(\bar{p}, x, \eta(x, z)) + \phi(x, z) \not\geq_{intC} \left(\frac{\alpha}{2}\|x - z\|^2 + \epsilon\right)e, \quad \forall\, z \in S(\bar{p}, x).$$

Hence, $x \in \mathrm{Q}_p(\delta, \epsilon)$ which shows that $\mathrm{Q}_p(\delta, \epsilon)$ is nonempty and closed. Also, $T_p = \bigcap_{\delta>0,\epsilon>0} \mathrm{Q}_p(\delta, \epsilon)$. Since, $\mu(\mathrm{Q}_p(\delta, \epsilon)) \to 0$ as $(\delta, \epsilon) \to (0, 0)$, by the Theorem on Page 412 of [13], we conclude that T_p is nonempty, compact and

$$e(\mathrm{Q}_p(\delta, \epsilon), T_p) = \mathcal{H}(\mathrm{Q}_p(\delta, \epsilon), T_p) \to 0 \text{ as} (\delta, \epsilon) \to (0, 0).$$

Let $p_n \to p$ and $\{x_n\}$ be an α-approximating sequence for MVQVLI_p. There exists $\epsilon_n \downarrow 0$ such that $d(x_n, S(p_n, x_n)) \leq \epsilon_n$ and

$$h(p_n, x_n, \eta(x_n, y)) + \phi(x_n, y) \not\geq_{intC} \left(\frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n\right)e,$$
$$\forall\, y \in S(p_n, x_n),\ \forall\, n \in \mathbb{N}.$$

Take $\delta_n = \|p_n - p\|$, $x_n \in \mathrm{Q}_p(\delta_n, \epsilon_n)$. There exists a sequence $\{\bar{x}_n\} \in T_p$ such that

$$\|x_n - \bar{x}_n\| = d(x_n, T_p) \leq e(\mathrm{Q}_p(\delta_n, \epsilon_n), T_p) \to 0.$$

Since, T_p is compact, $\{\bar{x}_n\}$ has a subsequence $\{\bar{x}_{n_k}\}$ converging to $\{\bar{x}\} \in T_p$. Hence, the corresponding sequence $\{x_{n_k}\}$ of $\{x_n\}$ converges strongly to $\{\bar{x}\}$ proving that MVQVLI_p is α-well-posed in the generalized sense. □

Theorem 4.2. *Let the assumptions be as in Theorem 3.3 and let P be finite dimensional. Then, $MVQVLI_p$ is α-L-well-posed in the generalized sense if and only if, $\forall\, p \in P$, $L_p(\delta, \epsilon) \neq \emptyset$, $\forall\, \delta, \epsilon > 0$ and $\lim_{\delta\to0,\epsilon\to0} \mu(L_p(\delta, \epsilon)) = 0$.*

Proof. Let MVQVLI_p be α-L-well-posed in the generalized sense. As h is C-η-pseudomonotone with respect to ϕ, $\mathrm{L}_p(\delta, \epsilon) \neq \emptyset$, $\forall\, \delta, \epsilon > 0$. To show that $\lim_{\delta\to0,\epsilon\to0} \mu(\mathrm{L}_p(\delta, \epsilon)) = 0$, the proof is similar as in Theorem 4.1.

Conversely, let

$$\mathrm{L}_p(\delta, \epsilon) \neq \emptyset, \forall \delta, \epsilon > 0 \text{and} \lim_{\delta\to0,\epsilon\to0} \mu(\mathrm{L}_p(\delta, \epsilon)) = 0.$$

As done in the previous theorem, we get that every α-L-approximating sequence has a convergent subsequence and this limit is a solution of MVQVLI_p, proving MVQVLI_p is α-L-well-posed in the generalized sense. □

Corollary 4.3. *$MVQVLI_p$ is α-well-posed in the generalized sense (respectively α-L-well-posed in the generalized sense) if and only if, $\forall\, p \in P$, the solution set of $MVQVLI_p$, that is, T_p is nonempty compact and $e(Q_p(\delta, \epsilon), T_p) \to 0$ as $(\delta, \epsilon) \to (0, 0)$ (respectively if and only if T_p is nonempty compact and $e(L_p(\delta, \epsilon), T_p) \to 0$ as $(\delta, \epsilon) \to (0, 0)$.)*

Proof. Let MVQVLI$_p$ be α-well-posed in the generalized sense. Thus, $T_p \neq \emptyset$ and compact. If $e(\mathrm{Q}_p(\delta,\epsilon),T_p) \nrightarrow 0$ as $(\delta,\epsilon) \to (0,0)$ then there exists $\tau > 0$, sequences $\{\delta_n\},\{\epsilon_n\}$ with $\delta_n \to 0$, $\epsilon_n \to 0$, $x_n \in K$ with $x_n \in \mathrm{Q}_p(\delta_n,\epsilon_n)$ such that $x_n \notin T_p + B(0,\tau)$. Since, $x_n \in \mathrm{Q}_p(\delta_n,\epsilon_n)$, $\{x_n\}$ is an α-approximating sequence of MVQVLI$_p$ which is α-well-posed in the generalized sense. Hence, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some point of T_p, which is a contradiction.

Conversely, let T_p be nonempty compact and $e(\mathrm{Q}_p(\delta,\epsilon),T_p) \to 0$ as $(\delta,\epsilon) \to (0,0)$. Let $p_n \to p$ and $\{x_n\}$ be an α-approximating sequence for MVQVLI$_p$. There exists $\epsilon_n \downarrow 0$ such that $d(x_n,S(p_n,x_n)) \leq \epsilon_n$ and

$$h(p_n,x_n,\eta(x_n,y))+\phi(x_n,y) \nleq_{intC} \left(\frac{\alpha}{2}\|x_n-y\|^2+\epsilon_n\right)e, \forall\, y\in S(p_n,x_n),\ \forall\, n\in\mathbb{N}.$$

Take $\delta_n = \|p_n - p\|$, $x_n \in \mathrm{Q}_p(\delta_n,\epsilon_n)$.
There exists a sequence $\{\bar{x}_n\} \in T_p$ such that

$$\|x_n - \bar{x}_n\| = d(x_n,T_p) \leq e(\mathrm{Q}_p(\delta_n,\epsilon_n),T_p) \to 0.$$

Since, T_p is compact, $\{\bar{x}_n\}$ has a subsequence $\{\bar{x}_{n_k}\}$ converging to $\{\bar{x}\} \in T_p$. Hence, the corresponding sequence $\{x_{n_k}\}$ of $\{x_n\}$ converges strongly to $\{\bar{x}\}$ proving that MVQVLI$_p$ is α-well-posed in the generalized sense.

On the similar lines as above, we can show that MVQVLI$_p$ is α-L-well-posed in the generalized sense if and only if T_p is nonempty compact and $e(\mathrm{L}_p(\delta,\epsilon),T_p) \to 0$ as $(\delta,\epsilon) \to (0,0)$. □

5. Conditions for α-well-posedness and α-L-well-posedness

In the following section, we will show that α-well-posedness and α-L-well-posedness of MVQVLI$_p$ is equivalent to the existence and uniqueness of its solution.

Theorem 5.1. *Let K be a nonempty compact and convex subset of a real Banach space X. Let $\phi : K \times K \to Y$ be a continuous bifunction with $\phi(x,.)$ concave and $\phi(x,x) = 0$ for fixed x. Let $\eta : K \times K \to X$ be a continuous mapping with $y \mapsto h(p,x,\eta(x,y))$ concave and $\eta(x,x) = 0$. Let $S : P \times K \to 2^K$ be a nonempty set-valued map which is convex valued, (s,w)-closed, (s,w)-subcontinuous and (s,s)-lower semicontinuous. Let $h : P \times K \times X \to Y$ be a continuous function which is positively homogeneous in the 3rd variable. Then, $MVQVLI_p$ is α-L-well-posed if and only if it has a unique solution.*

Proof. Let MVQVLI$_p$ be α-L-well-posed. Then, by definition, it has a unique solution.

Conversely, let MVQVLI$_p$ has a unique solution say z_0 and $\{x_n\}$ be an α-L-approximating sequence. Let $p_n \to p \in P$. Since, K is compact, $\{x_n\}$ has a subsequence still denoted by $\{x_n\}$ converging to $x_0 \in K$. It is sufficient to show that x_0 is a solution of MVQVLI$_p$. Then, $x_0 = z_0$ and the whole sequence $\{x_n\}$ would then converge to z_0. Following the proof of Theorem 3.1, we get that x_0 is a solution of MVQVLI$_p$. □

Theorem 5.2. *Let the assumptions be as in Theorem 5.1. Further, assume that h is C-η-pseudomonotone with respect to ϕ. Then, $MVQVLI_p$ is α-well-posed if and only if it has a unique solution.*

Proof. Necessary condition holds obviously.

For sufficient condition, let MVQVLI_p has a unique solution say x_0. Let $\{p_n\}$ be a sequence such that $p_n \to p \in P$ and $\{x_n\}$ be an α-approximating sequence for MVQVLI_p. As h is C-η-pseudomonotone with respect to ϕ. Then, $\{x_n\}$ is also an α-L-approximating sequence. By Theorem 5.1, MVQVLI_p is α-L-well-posed. Hence, $x_n \to x_0$ and so, MVQVLI_p is α-well-posed. $\square$

6. Conclusion

A mixed vector quasivariational-like inequality is considered and various results characterizing (generalized) α-well-posedness and (generalized) α-L-well-posedness for this problem have been given. For further research, Levitin–Polyak well-posedness can be investigated for the same problem.

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