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LA-SEMIGROUPS CHARACTERIZED BY THE PROPERTIES OF INTERVAL VALUED (α, β)-FUZZY IDEALS

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ABSTRACT. The concept of interval-valued (α, β)-fuzzy ideals, interval-valued (α, β)-fuzzy generalized bi-ideals are introduced in LA-semigroups, using the ideas of belonging and quasi-coincidence of an interval-valued fuzzy point with an interval-valued fuzzy set and some related properties are investigated. We define the lower and upper parts of interval-valued fuzzy subsets of an LA-semigroup. Regular LA-semigroups are characterized by the properties of the lower part of interval-valued $(\in, \in \vee q)$-fuzzy left ideals, interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideals and interval-valued $(\in, \in \vee q)$-fuzzy generalized bi-ideals. *Main Facts.*

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Key words and phrases : interval-valued (α, β)-fuzzy sub LA-semigroups, interval-valued (α, β)-fuzzy ideals, interval-valued (α, β)-fuzzy bi-ideals, interval-valued (α, β)-fuzzy quasi-ideals.

1. Introduction

The concept of fuzzy sets was first introduced by Zadeh [16] and then the fuzzy sets have been used in the reconsideration of classical mathematics. Fuzzy set theory has been shown to be a useful tool to describe situations in which data is imprecise or vague. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. The fuzzy algebraic structures play a prominent role in mathematics with wide applications in many other branches such as theoratical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, logic, set theory, group theory, real analysis, measure theory etc. The notion of fuzzy subgroups was defined by Rosenfled [12]. A systematic exposition of fuzzy semigroup was given by Mordeson et. al. [7], and they have find theoratical results on fuzzy semigroups and their use in finite state machine, fuzzy languages and fuzzy coding. Using the

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notions "belong to" relation ($\in$) introduced by Pu and Liu [11], in [8] Morali proposed the concept of a fuzzy point belonging to a fuzzy subset under natural equivalence on fuzzy subsets. Bhakat and Das introduced the concept of (α, β)-fuzzy subgroups by using the *belong to* relation ($\in$) and *quasi-coincident with* relation (q) between a fuzzy point and a fuzzy subgroup, and defined an $(\in, \in \vee q)$-fuzzy subgroup of a group [1]. Kazanci and Yamak [4] studied generalized types fuzzy bi-ideals of semigroups and defined $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideals of semigroups. In [14] Shabir et. al. characterized regular semigroups by the properties of (α, β)-fuzzy ideals, bi-ideals and quasi-ideals. In [15], Shabir and Yasir characterized regular semigroups by the properties of $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy ideals, generalized bi-ideals and quasi-ideals of a semigroup.

Interval-valued fuzzy subsets were proposed about thirty years ago as a natural extension of fuzzy sets by Zadeh [17]. Interval-valued fuzzy subsets have many applications in several areas such as the method of approximate inference. In [10] Al Narayanan and T. Manikantan introduced the notions of interval-valued fuzzy ideals generated by an interval-valued fuzzy subset in semigroups.

In this paper we introduced the concepts of interval-valued $(\in, \in \vee q)$-fuzzy ideals, interval-valued $(\in, \in \vee q)$-fuzzy bi-ideals and interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideals of an LA-semigroup, where α, β are any one of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$, by using *belong to* relation ($\in$) and *quasi-coincidence with* relation (q) between interval-valued fuzzy set and an interval-valued fuzzy point, and investigated related properties.

2. Preliminaries

We first recall some basic concepts. A groupoid $(S, *)$ is called a left almost semigroup, abbreviated as an LA-semigroup, if it satisfies left invertive law:

$$(a * b) * c = (c * b) * a \quad \forall a, b, c \in S.$$

A non-empty subset A of S is called a sub LA-semigroup of S if $AA \subseteq A$ and is called left (resp. right) ideal of S if $SA \subseteq A$ ($AS \subseteq A$). By a two-sided ideal or simply an ideal we mean a non-empty subset of S which is both a left and a right ideal of S. A non-empty subset B of S is called a generalized bi-ideal of S if $(BS)B \subseteq B$. A sub LA-semigroup B of S is called a bi-ideal of S if $(BS)B \subseteq B$. A non-empty subset Q of S is called a quasi-ideal of S if $QS \cap SQ \subseteq Q$. Obviously every one-sided ideal of an LA-semigroup S is a quasi-ideal, every quasi-ideal is a bi-ideal and every bi-ideal is a generalized bi-ideal. An LA-semigroup S is called regular if for each element a of S, there exists an element x in S such that $a = (ax)a$. It is well known that for a regular LA-semigroup the concepts of generalized bi-ideal, bi-ideal and quasi-ideal coincide.

We now review some concepts of fuzzy subsets. A fuzzy subset λ of an LA-semigroup S is a mapping $\lambda : S \to [0, 1]$. Throughout this paper S denotes an LA-semigroup.

Definition 1. A fuzzy subset λ of S is called a fuzzy sub LA-semigroup of S if for all $x, y \in S$

$$\lambda(xy) \geq \lambda(x) \wedge \lambda(y).$$

Definition 2. A fuzzy subset λ of S is called a fuzzy left (resp. right) ideal of S if for all $x, y \in S$

$$\lambda(xy) \geq \lambda(y) \text{ (resp. } \lambda(xy) \geq \lambda(x)).$$

A fuzzy subset λ of S is called a fuzzy two-sided ideal or simply fuzzy ideal of S if it is both a fuzzy left and a fuzzy right ideal of S.

Definition 3. A fuzzy subsemigroup λ of S is called a fuzzy bi-ideal of S if for all $x, y, z \in S$,

$$\lambda((xy)z) \geq \lambda(x) \wedge \lambda(z).$$

We will describe some results of an interval number. By an interval number on $[0,1]$, say $\widetilde{a}$, is a closed subinterval of $[0,1]$, that is, $\widetilde{a} = [a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. Let $D[0,1]$ denote the family of all closed subintervals of $[0,1]$, $\widetilde{0} = [0,0]$ and $\widetilde{1} = [1,1]$. Now we define $\leq, =, \wedge, \vee$ in case of two elements in $D[0,1]$.

Consider two elements $\widetilde{a} = [a^-, a^+]$ and $\widetilde{b} = [b^-, b^+]$ in $D[0,1]$. Then,
$\widetilde{a} \leq \widetilde{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$.
$\widetilde{a} = \widetilde{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$.
$\min\{\widetilde{a}, \widetilde{b}\} = [\min\{a^-, b^-\}, \min\{a^+, b^+\}]$.
$\max\{\widetilde{a}, \widetilde{b}\} = [\max\{a^-, b^-\}, \max\{a^+, b^+\}]$.

Let X be a set. A mapping $\widetilde{\lambda} : X \to D[0,1]$ is called an interval-valued fuzzy subset (briefly, an i-v fuzzy subset) of X, where $\widetilde{\lambda}(x) = [\lambda^-(x), \lambda^+(x)]$ for all $x \in X$, where $\lambda^-$and λ^+ are fuzzy subsets in X such that $\lambda^-(x) \leq \lambda^+(x)$ for all $x \in X$.

Let $\widetilde{\lambda}$ and $\widetilde{\mu}$ be two interval-valued fuzzy subsets of X. Define the relation $\subseteq$ between λ and μ as follows:

$\widetilde{\lambda} \subseteq \widetilde{\mu}$ if and only if $\widetilde{\lambda}(x) \leq \widetilde{\mu}(x)$ for all $x \in X$, that is, $\lambda^-(x) \leq \mu^-(x)$ and $\lambda^+(x) \leq \mu^+(x)$. An interval-valued fuzzy subset $\widetilde{\lambda}$ in a universe X of the form

$$\widetilde{\lambda}(y) = \begin{cases} \widetilde{t}(\neq [0,0]) & \text{if } y = x \\ \widetilde{0} & \text{if } y \neq x \end{cases}$$

for all $y \in X$, is said to be an interval-valued fuzzy point with support x and interval value $\widetilde{t}$, and is denoted by $x_{\widetilde{t}}$.

J. Zhan et al [19] gave meaning to the symbol $x_{\widetilde{t}}\alpha\widetilde{\lambda}$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. An interval-valued fuzzy point $x_{\widetilde{t}}$ is said to *belongs to* (resp. *quasi-coincident with*) an interval-valued fuzzy set $\widetilde{\lambda}$ written $x_{\widetilde{t}} \in \widetilde{\lambda}$ (resp. $x_{\widetilde{t}}q\widetilde{\lambda}$) if $\widetilde{\lambda}(x) \geq \widetilde{t}$ (resp. $\widetilde{\lambda}(x) + \widetilde{t} > \widetilde{1}$), and in this case $x_{\widetilde{t}} \in \vee q\widetilde{\lambda}$ (resp. $x_{\widetilde{t}} \in \wedge q\widetilde{\lambda}$) means that $x_{\widetilde{t}} \in \widetilde{\lambda}$ or $x_{\widetilde{t}}q\widetilde{\lambda}$ (resp. $x_{\widetilde{t}} \in \widetilde{\lambda}$ and $x_{\widetilde{t}}q\widetilde{\lambda}$). To say that $x_{\widetilde{t}}\overline{\alpha}\widetilde{\lambda}$ means that $x_{\widetilde{t}}\alpha\widetilde{\lambda}$ does not hold.

Let $\widetilde{\lambda}$ and $\widetilde{\mu}$ be two interval-valued fuzzy subsets of an LA-semigroup S. The product $\widetilde{\lambda} \circ \widetilde{\mu}$ is defined by

$$(\widetilde{\lambda} \circ \widetilde{\mu})(x) = \begin{cases} \vee_{x=yz}\{\widetilde{\lambda}(y) \wedge \widetilde{\mu}(z)\},\ \exists\ y, z \in S,\ x = yz \\ [0,0] \quad \text{otherwise.} \end{cases}$$

3. Main results

Definition 4. An interval-valued fuzzy subset $\widetilde{\lambda}$ of an LA-semigroup S is called an interval-valued fuzzy sub LA-semigroup of S if for all $x, y \in S$

$$\widetilde{\lambda}(xy) \geq \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(y).$$

Definition 5. An interval-valued fuzzy subset $\widetilde{\lambda}$ of an LA-semigroup S is called an interval-valued fuzzy left (resp. right) ideal of S if for all $x, y \in S$

$$\widetilde{\lambda}(xy) \geq \widetilde{\lambda}(y) \text{ (resp. } \widetilde{\lambda}(xy) \geq \widetilde{\lambda}(x)).$$

Definition 6. An interval-valued fuzzy subset $\widetilde{\lambda}$ of an LA-semigroup S is called an interval-valued fuzzy generalized bi-ideal of S if for all $x, y, z \in S$

$$\widetilde{\lambda}((xy)z) \geq \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(z).$$

Definition 7. An interval-valued fuzzy sub LA-semigroup $\widetilde{\lambda}$ of S is called an interval-valued fuzzy bi-ideal of S if for all $x, y, z \in S$

$$\widetilde{\lambda}((xy)z) \geq \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(z).$$

Definition 8. An interval-valued fuzzy subset $\widetilde{\lambda}$ of an LA-semigroup S is called an interval-valued fuzzy quasi-ideal of S if

$$(\widetilde{\lambda} \circ \widetilde{\delta}) \wedge (\widetilde{\delta} \circ \widetilde{\lambda}) \leq \widetilde{\lambda}$$

where $\widetilde{\delta} : S \to [1, 1]$.

Theorem 1 ([18])**.** *Let S be an LA-semigroup with left identity e such that $(xe)S = xS$ for all $x \in S$. Then, the following are equivalent*

(1) S is regular.
(2) $R \cap L = RL$ for every right ideal R and left ideal L of S.
(3) $A = (AS)A$ for every quasi-ideal A of S.

4. INTERVAL-VALUED (α, β)-FUZZY IDEALS OF LA-SEMIGROUPS

Let S be an LA-semigroup and α, β denote any one of $\in, q, \in \vee q$, or $\in \wedge q$ unless otherwise specified.

Definition 9. An interval-valued fuzzy subset $\widetilde{\lambda}$ of an LA-semigroup S is called an interval-valued (α, β)-fuzzy sub LA-semigroup of S, where $\alpha \neq \in \wedge q$, if $x_{\widetilde{t}_1} \alpha \widetilde{\lambda}$ and $y_{\widetilde{t}_2} \alpha \widetilde{\lambda}$ implies that $(xy)_{\min\{\widetilde{t}_1, \widetilde{t}_2\}} \beta \widetilde{\lambda}$.

Let $\widetilde{\lambda}$ be an interval-valued fuzzy subset of S such that, $\widetilde{\lambda}(x) \leq [\widetilde{0.5}, \widetilde{0.5}]$ for all $x \in S$. Let $x \in S$ and $\widetilde{t} \in D[0,1]$ such that, $x_{\widetilde{t}} \in \wedge q\widetilde{\lambda}$. Then, $x_{\widetilde{t}} \in \widetilde{\lambda}$ and $x_{\widetilde{t}} q\widetilde{\lambda}$, that is, $\widetilde{\lambda}(x) \geq \widetilde{t}$ and $\widetilde{\lambda}(x) + \widetilde{t} > \widetilde{1}$. It follows that $\widetilde{1} < \widetilde{\lambda}(x) + \widetilde{t} \leq \widetilde{\lambda}(x) + \widetilde{\lambda}(x) = 2\widetilde{\lambda}(x)$, so that $\widetilde{\lambda}(x) > [\widetilde{0.5}, \widetilde{0.5}]$. This means that $\{x_{\widetilde{t}} : x_{\widetilde{t}} \in \wedge q\widetilde{\lambda}\} = \phi$. Therefore, the case $\alpha = \in \wedge q$ in above definition is omitted.

Definition 10. An interval-valued fuzzy subset $\widetilde{\lambda}$ of an LA-semigroup S is called an interval-valued (α, β)-fuzzy left (resp. right) ideal of S, where $\alpha \neq \in \wedge q$, if it satisfies, $y_{\widetilde{t}}\alpha\widetilde{\lambda}$ and $x \in S$ implies that $(xy)_{\widetilde{t}}\beta\widetilde{\lambda}$ (resp. $(yx)_{\widetilde{t}}\beta\widetilde{\lambda}$) for all $x, y \in S$.

An interval-valued fuzzy subset $\widetilde{\lambda}$ of an LA-semigroup S is called an interval-valued (α, β)-fuzzy ideal of S if it is both an interval-valued (α, β)-fuzzy left ideal and an interval-valued (α, β)-fuzzy right ideal of S.

Definition 11. An interval-valued fuzzy subset $\widetilde{\lambda}$ of an LA-semigroup S is called an interval-valued (α, β)-fuzzy generalized bi-ideal of S, where $\alpha \neq \in \wedge q$, if it satisfies

for all $x, y, z \in S$ and for all $\widetilde{t}_3, \widetilde{t}_4 \in D[0,1]$, where $\widetilde{t}_3, \widetilde{t}_4 \neq \widetilde{0}$, $x_{\widetilde{t}_3}\alpha\widetilde{\lambda}$ and $z_{\widetilde{t}_4}\alpha\widetilde{\lambda}$ implies that $((xy)z)_{\min\{\widetilde{t}_3, \widetilde{t}_4\}}\beta\widetilde{\lambda}$.

Definition 12. An interval-valued fuzzy subset $\widetilde{\lambda}$ of an LA-semigroup S is called an interval-valued (α, β)-fuzzy bi-ideal of S, where $\alpha \neq \in \wedge q$, if it satisfies, the following two conditions.

(i) For all $x, y \in S$ and for all $\widetilde{t}_1, \widetilde{t}_2 \in D[0,1]$, where $\widetilde{t}_1, \widetilde{t}_2 \neq \widetilde{0}$, $x_{\widetilde{t}_1}\alpha\widetilde{\lambda}$ and $y_{\widetilde{t}_2}\alpha\widetilde{\lambda}$ implies that $(xy)_{\min\{\widetilde{t}_1, \widetilde{t}_2\}}\beta\widetilde{\lambda}$.

(ii) For all $x, y, z \in S$ and for all $\widetilde{t}_3, \widetilde{t}_4 \in D[0,1]$, where $\widetilde{t}_3, \widetilde{t}_4 \neq \widetilde{0}$, $x_{\widetilde{t}_3}\alpha\widetilde{\lambda}$ and $z_{\widetilde{t}_4}\alpha\widetilde{\lambda}$ implies that $((xy)z)_{\min\{\widetilde{t}_3, \widetilde{t}_4\}}\beta\widetilde{\lambda}$.

Lemma 1. *An interval-valued fuzzy subset $\widetilde{\lambda}$ of an LA-semigroup S is an interval-valued fuzzy sub LA-semigroup of S if and only if it satisfies,*

For all $x, y \in S$ and for all $\widetilde{t}_1, \widetilde{t}_2 \in D[0,1]$, where $\widetilde{t}_1, \widetilde{t}_2 \neq \widetilde{0}$, such that $x_{\widetilde{t}_1} \in \widetilde{\lambda}$ and $y_{\widetilde{t}_2} \in \widetilde{\lambda}$ implies that $(xy)_{\min\{\widetilde{t}_1, \widetilde{t}_2\}} \in \widetilde{\lambda}$.

Proof. Suppose that $\widetilde{\lambda}$ is an interval-valued fuzzy sub LA-semigroup of an LA-semigroup S. Let $x, y \in S$ and $\widetilde{t}_1, \widetilde{t}_2 \in D[0,1]$, where $\widetilde{t}_1, \widetilde{t}_2 \neq \widetilde{0}$, such that, $x_{\widetilde{t}_1} \in \widetilde{\lambda}$ and $y_{\widetilde{t}_2} \in \widetilde{\lambda}$. Then $\widetilde{\lambda}(x) \geq \widetilde{t}_1$ and $\widetilde{\lambda}(y) \geq \widetilde{t}_2$. Since $\widetilde{\lambda}$ is an interval-valued fuzzy sub LA-semigroup of S. So $\widetilde{\lambda}(xy) \geq \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\} \geq \min\{\widetilde{t}_1, \widetilde{t}_2\}$. Hence, $(xy)_{\min\{\widetilde{t}_1, \widetilde{t}_2\}} \in \widetilde{\lambda}$.

Conversely, assume that $\widetilde{\lambda}$ satisfies the given condition. We show that $\widetilde{\lambda}(xy) \geq \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\}$. On contrary, assume that there exist $x, y \in S$ such that $\widetilde{\lambda}(xy) < \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\}$. Let $\widetilde{t} \in D[0,1]$, where $\widetilde{t} \neq \widetilde{0}$, such that $\widetilde{\lambda}(xy) < \widetilde{t} < \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\}$. Then, $x_{\widetilde{t}} \in \widetilde{\lambda}$ and $y_{\widetilde{t}} \in \widetilde{\lambda}$ but $(xy)_{\widetilde{t}} \overline{\in} \widetilde{\lambda}$. This contradicts our hypothesis. Thus, $\widetilde{\lambda}(xy) \geq \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\}$. $\square$

Lemma 2. *An interval-valued fuzzy subset $\widetilde{\lambda}$ of an LA-semigroup S is an interval-valued fuzzy left (resp. right) ideal of S if and only if it satisfies*

for all $x, y \in S$ and for all $\widetilde{t} \in D[0,1]$, where $\widetilde{t} \neq \widetilde{0}$, such that $y_{\widetilde{t}} \in \widetilde{\lambda}$ implies that $(xy)_{\widetilde{t}} \in \widetilde{\lambda}$ (resp. $(yx)_{\widetilde{t}} \in \widetilde{\lambda}$).

Proof. Suppose that $\widetilde{\lambda}$ is an interval-valued fuzzy left ideal of an LA-semigroup S. If $y_{\widetilde{t}} \in \widetilde{\lambda}$ then $\widetilde{\lambda}(y) \geq \widetilde{t}$. Since $\widetilde{\lambda}$ is an interval-valued fuzzy left ideal of S, so $\widetilde{\lambda}(xy) \geq \widetilde{\lambda}(y) \geq \widetilde{t}$. Hence, $(xy)_{\widetilde{t}} \in \widetilde{\lambda}$.

Conversely, suppose that $\widetilde{\lambda}$ satisfies the given condition. We show that $\widetilde{\lambda}(xy) \geq \widetilde{\lambda}(y)$. On contrary, assume that there exist $x, y \in S$ such that $\widetilde{\lambda}(xy) < \widetilde{\lambda}(y)$. Let $\widetilde{t} \in D[0,1]$, where $\widetilde{t} \neq \widetilde{0}$, be such that $\widetilde{\lambda}(xy) < \widetilde{t} < \widetilde{\lambda}(y)$. Then, $y_{\widetilde{t}} \in \widetilde{\lambda}$ but $(xy)_{\widetilde{t}} \overline{\in} \widetilde{\lambda}$. Which contradicts our hypothesis. Hence, $\widetilde{\lambda}(xy) \geq \widetilde{\lambda}(y)$. □

Remark 1. The above lemma shows that every fuzzy left (resp. right) ideal of S is an $(\in, \in)$-fuzzy left (resp. right) ideal of S.

Lemma 3. *An interval-valued fuzzy subset $\widetilde{\lambda}$ of an LA-semigroup S is an interval-valued fuzzy generalized bi-ideal of S if and only if it satisfies,*

For all $x, y, z \in S$ and for all $\widetilde{t}_1, \widetilde{t}_2 \in D[0,1]$, where $\widetilde{t}_1, \widetilde{t}_2 \neq \widetilde{0}$, $x_{\widetilde{t}_1} \in \widetilde{\lambda}$ and $z_{\widetilde{t}_2} \in \widetilde{\lambda}$ implies that $((xy)z)_{\min\{\widetilde{t}_1, \widetilde{t}_2\}} \in \widetilde{\lambda}$.

Proof. Suppose that $\widetilde{\lambda}$ is an interval-valued fuzzy generalized bi-ideal of S. Let $x, y, z \in S$ and $\widetilde{t}_1, \widetilde{t}_2 \in D[0,1]$, where $\widetilde{t}_1, \widetilde{t}_2 \neq \widetilde{0}$, such that $x_{\widetilde{t}_1} \in \widetilde{\lambda}$ and $z_{\widetilde{t}_2} \in \widetilde{\lambda}$. Then, $\widetilde{\lambda}(x) \geq \widetilde{t}_1$ and $\widetilde{\lambda}(z) \geq \widetilde{t}_2$. Since $\widetilde{\lambda}$ is an interval-valued fuzzy generalized bi-ideal of S. So $\widetilde{\lambda}((xy)z) \geq \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\} \geq \min\{\widetilde{t}_1, \widetilde{t}_2\}$. Hence, $((xy)z))_{\min\{\widetilde{t}_1, \widetilde{t}_2\}} \in \widetilde{\lambda}$.

Conversely, suppose that $\widetilde{\lambda}$ satisfies the given condition. We show that $\widetilde{\lambda}((xy)z) \geq \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\}$. Suppose contrary that $\widetilde{\lambda}((xy)z) < \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\}$. Let $\widetilde{t} \in D[0,1]$, where $\widetilde{t} \neq \widetilde{0}$ be such that $\widetilde{\lambda}((xy)z) < \widetilde{t} < \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\}$. Then, $x_{\widetilde{t}} \in \widetilde{\lambda}$ and $z_{\widetilde{t}} \in \widetilde{\lambda}$ but $((xy)z)_{\min\{\widetilde{t}, \widetilde{t}\}} \overline{\in} \widetilde{\lambda}$. Which contradicts our supposition. Hence, $\widetilde{\lambda}((xy)z) \geq \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(z)\}$. □

Lemma 4. *An interval-valued fuzzy subset $\widetilde{\lambda}$ of an LA-semigroup S is an interval-valued fuzzy bi-ideal of S if and only if it satisfy,*

(1) for all $x, y \in S$ and $\widetilde{t}_1, \widetilde{t}_2 \in D[0,1]$, where $\widetilde{t}_1, \widetilde{t}_2 \neq \widetilde{0}$, $x_{\widetilde{t}_1} \in \widetilde{\lambda}$ and $y_{\widetilde{t}_2} \in \widetilde{\lambda}$ implies that $(xy)_{\min\{\widetilde{t}_1, \widetilde{t}_2\}} \in \widetilde{\lambda}$.

(2) for all $x, y, z \in S$ and for all $\widetilde{t}_3, \widetilde{t}_4 \in D[0,1]$, where $\widetilde{t}_3, \widetilde{t}_4 \neq \widetilde{0}$, $x_{\widetilde{t}_3} \in \widetilde{\lambda}$ and $z_{\widetilde{t}_4} \in \widetilde{\lambda}$ implies that $((xy)z)_{\min\{\widetilde{t}_3, \widetilde{t}_4\}} \in \widetilde{\lambda}$.

Proof. Follows from Lemma 1 and Lemma 3. □

Theorem 2. *Let $\widetilde{\lambda}$ be a non-zero interval-valued (α, β)-fuzzy sub LA-semigroup of S. Then, the set $\widetilde{\lambda}_\circ = \{x \in S \mid \widetilde{\lambda}(x) > \widetilde{0}\}$ is a sub LA-semigroup of S.*

Proof. Let $x, y \in \widetilde{\lambda}_\circ$. Then, $\widetilde{\lambda}(x) > \widetilde{0}$ and $\widetilde{\lambda}(y) > \widetilde{0}$. Let $\widetilde{\lambda}(xy) > \widetilde{0}$. If $\alpha \in \{\in, \in \vee q\}$, then $x_{\widetilde{\lambda}(x)} \alpha \widetilde{\lambda}$ and $y_{\widetilde{\lambda}(y)} \alpha \widetilde{\lambda}$ but $\widetilde{\lambda}(xy) = \widetilde{0} < \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\}$ and $\widetilde{\lambda}(xy) + \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\} \leq \widetilde{0} + \widetilde{1} = \widetilde{1}$. So, $(xy)_{\min\{\widetilde{\lambda}(x), \widetilde{\lambda}(y)\}} \overline{\beta} \widetilde{\lambda}$ for every $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$, a contradiction. Hence, $\widetilde{\lambda}(xy) > \widetilde{0}$, that is, $xy \in \widetilde{\lambda}_\circ$. Also, $x_{\widetilde{1}} q \widetilde{\lambda}$ and $y_{\widetilde{1}} q \widetilde{\lambda}$ but $(xy)_{\widetilde{1}} \overline{\beta} \widetilde{\lambda}$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$. Hence, $\widetilde{\lambda}(xy) > \widetilde{0}$, that is, $xy \in \widetilde{\lambda}_\circ$. Thus, $\widetilde{\lambda}_\circ$ is a sub LA-semigroup of S. □

Theorem 3. *Let $\widetilde{\lambda}$ be a non-zero interval-valued (α, β)-fuzzy left (resp. right) ideal of S. Then, the set $\widetilde{\lambda}_\circ = \{x \in S \mid \widetilde{\lambda}(x) > \widetilde{0}\}$ is a left (resp. right) ideal of S.*

Proof. Similar to the proof of Theorem 2. □

Theorem 4. *Let $\widetilde{\lambda}$ be a non-zero interval-valued (α, β)-fuzzy generalized bi-ideal of S. Then the set $\widetilde{\lambda}_\circ = \{x \in S \mid \widetilde{\lambda}(x) > \widetilde{0}\}$ is a generalized bi-ideal of S.*

Proof. Similar to the proof of Theorem 2. □

Theorem 5. *Let $\widetilde{\lambda}$ be a non-zero interval-valued (α, β)-fuzzy bi-ideal of S. Then, the set $\widetilde{\lambda}_\circ = \{x \in S \mid \widetilde{\lambda}(x) > \widetilde{0}\}$ is a bi-ideal of S.*

Proof. Proof follows from Theorem 1 and Theorem 4. □

5. INTERVAL-VALUED $(\in, \in \vee q)$-FUZZY IDEALS

Theorem 6. *Let A be a sub LA-semigroup of an LA-semigroup S and let $\widetilde{\lambda}$ be an interval-valued fuzzy subset in S such that*

$$\widetilde{\lambda}(x) = \begin{cases} \widetilde{0} & \text{if } x \in S \backslash A \\ \geq \widetilde{0.5} & \text{if } x \in S \end{cases}$$

Then,

(1) $\widetilde{\lambda}$ is an interval-valued $(q, \in \vee q)$-fuzzy sub LA-semigroup of S.

(2) $\widetilde{\lambda}$ is an interval-valued $(\in, \in \vee q)$-fuzzy sub LA-semigroup of S.

Proof. (1) Let $x, y \in S$ and $\widetilde{t}_1, \widetilde{t}_2 \in D[0,1]$, where $\widetilde{t}_1, \widetilde{t}_2 \neq \widetilde{0}$, such that, $x_{\widetilde{t}_1} q \widetilde{\lambda}$ and $y_{\widetilde{t}_2} q \widetilde{\lambda}$. Then, $\widetilde{\lambda}(x) + \widetilde{t}_1 > \widetilde{1}$ and $\widetilde{\lambda}(y) + \widetilde{t}_2 > 1$. So, $x \in A$. Therefore, $xy \in A$. Thus, if $\min\{\widetilde{t}_1, \widetilde{t}_2\} \leq \widetilde{0.5}$, then $\widetilde{\lambda}(xy) \geq \widetilde{0.5} \geq \min\{\widetilde{t}_1, \widetilde{t}_2\}$. Then, $(xy)_{\min\{\widetilde{t}_1, \widetilde{t}_2\}} \in \widetilde{\lambda}$. If $\min\{\widetilde{t}_1, \widetilde{t}_2\} > \widetilde{0.5}$, then $\widetilde{\lambda}(xy) + \min\{\widetilde{t}_1, \widetilde{t}_2\} > \widetilde{0.5} + \widetilde{0.5} = \widetilde{1}$. So, $(xy)_{\min\{\widetilde{t}_1, \widetilde{t}_2\}} q \widetilde{\lambda}$. Therefore, $(xy)_{\min\{\widetilde{t}_1, \widetilde{t}_2\}} \in \vee q \widetilde{\lambda}$. Then, $\widetilde{\lambda}$ is an interval-valued $(q, \in \vee q)$-fuzzy sub LA-semigroup of S.

(2) Let $x, y \in S$ and $\widetilde{t}_1, \widetilde{t}_2 \in D[0,1]$, where$\widetilde{t}_1, \widetilde{t}_2 \neq \widetilde{0}$, such that, $x_{\widetilde{t}_1} \in \widetilde{\lambda}$ and $y_{\widetilde{t}_2} \in \widetilde{\lambda}$. Then, $\widetilde{\lambda}(x) \geq \widetilde{t}_1$ and $\widetilde{\lambda}(y) \geq \widetilde{t}_1$. So, $x \in A$ and $y \in A$. Therefore, $xy \in A$. If $\min\{\widetilde{t}_1, \widetilde{t}_2\} \leq \widetilde{0.5}$, then $\widetilde{\lambda}(xy) \geq \widetilde{0.5} \geq \min\{\widetilde{t}_1, \widetilde{t}_2\}$. So, $(xy)_{\min\{\widetilde{t}_1, \widetilde{t}_2\}} \in \widetilde{\lambda}$. If $\min\{\widetilde{t}_1, \widetilde{t}_2\} > \widetilde{0.5}$, then $\widetilde{\lambda}(xy) +$ $\min\{\widetilde{t}_1, \widetilde{t}_2\} > \widetilde{0.5} + \widetilde{0.5} = \widetilde{1}$. So, $(xy)_{\min\{\widetilde{t}_1, \widetilde{t}_2\}} q \widetilde{\lambda}$.

Therefore, $(xy)_{\min\{\widetilde{t_1},\widetilde{t_2}\}} \in \vee q\widetilde{\lambda}$. Hence, $\widetilde{\lambda}$ is an interval-valued $(\in,\in\vee q)$-fuzzy sub LA-semigroup of S. □

Theorem 7. *Let L be a left (resp. right) ideal of S and let $\widetilde{\lambda}$ be an interval-valued fuzzy subset in S such that*

$$\widetilde{\lambda}(x) = \begin{cases} \widetilde{0} & \text{if } x \in S\backslash L \\ \geq \widetilde{0.5} & \text{if } x \in L \end{cases}$$

Then,
(1) $\widetilde{\lambda}$ is an interval-valued $(q,\in\vee q)$-fuzzy left (resp. right) ideal of S.
(2) $\widetilde{\lambda}$ is an interval-valued $(\in,\in\vee q)$-fuzzy left (resp. right) ideal of S.

Proof. Similar to the proof of Theorem 6. □

Theorem 8. *Let B be a generalized bi-ideal of S and let $\widetilde{\lambda}$ be an interval-valued fuzzy subset in S such that*

$$\widetilde{\lambda}(x) = \begin{cases} \widetilde{0} & \text{if } x \in S \setminus B \\ \geq \widetilde{0.5} & \text{if } x \in B \end{cases}$$

Then,
(1) $\widetilde{\lambda}$ is an interval-valued $(q,\in\vee q)$-fuzzy generalized bi-ideal of S.
(2) $\widetilde{\lambda}$ is an interval-valued $(\in,\in\vee q)$-fuzzy generalized bi-ideal of S.

Proof. Similar to the proof of Theorem 6. □

Theorem 9. *Let B be a bi-ideal of S and let $\widetilde{\lambda}$ be an interval-valued fuzzy subset in S such that*

$$\widetilde{\lambda}(x) = \begin{cases} \widetilde{0} & \text{if } x \in S - B \\ \geq \widetilde{0.5} & \text{if } x \in B \end{cases}$$

Then,
(1) $\widetilde{\lambda}$ is an interval-valued $(q,\in\vee q)$-fuzzy bi-ideal of S.
(2) $\widetilde{\lambda}$ is an interval-valued $(\in,\in\vee q)$-fuzzy bi-ideal of S.

Proof. Follows from Theorem 6 and Theorem 8. □

Lemma 5. *Let $\widetilde{\lambda}$ be an interval-valued fuzzy subset of an LA-semigroup S. Then $\widetilde{\lambda}$ is an interval-valued $(\in,\in\vee q)$-fuzzy left (right) ideal of S if and only if*
$\widetilde{\lambda}(xy) \geq \min\{\widetilde{\lambda}(y), \widetilde{0.5}\}$ ($\widetilde{\lambda}(xy) \geq \min\{\widetilde{\lambda}(x), \widetilde{0.5}\}$)

Proof. Let $\widetilde{\lambda}$ be an interval-valued $(\in,\in\vee q)$-fuzzy left (right) ideal of S. On contrary, suppose that $\widetilde{\lambda}(xy) < \min\{\widetilde{\lambda}(y), \widetilde{0.5}\}$. Choose, $\widetilde{t} \in D[0,1]$, where $\widetilde{t} \neq \widetilde{0}$, such that, $\widetilde{\lambda}(xy) < \widetilde{t} < \min\{\widetilde{\lambda}(y), \widetilde{0.5}\}$. Then, $y_{\widetilde{t}} \in \widetilde{\lambda}$ but $(xy)_{\widetilde{t}} \overline{\in\vee q} \widetilde{\lambda}$. Which is contradiction. Hence, $\widetilde{\lambda}(xy) \geq \min\{\widetilde{\lambda}(y), \widetilde{0.5}\}$. Conversely, assume that $\widetilde{\lambda}(xy) \geq$

$\min\{\widetilde{\lambda}(y), \widetilde{0.5}\}$. Let $y_{\widetilde{t}} \in \widetilde{\lambda}$ then $\widetilde{f}(y) \geq \widetilde{t}$. Now, $\widetilde{\lambda}(xy) \geq \min\{\widetilde{\lambda}(y), \widetilde{0.5}\} \geq \min\{\widetilde{t}, \widetilde{0.5}\}$. If $\widetilde{t} \leq \widetilde{0.5}$, then $\widetilde{\lambda}(xy) \geq \widetilde{t}$. So, $(xy)_{\widetilde{t}} \in \widetilde{\lambda}$. If $\widetilde{t} > \widetilde{0.5}$, then $\widetilde{\lambda}(xy) \geq \widetilde{0.5}$. So, $(xy)_{\widetilde{t}} \in \widetilde{\lambda}$. If $\widetilde{t} > \widetilde{0.5}$, then $\widetilde{\lambda}(xy) \geq \widetilde{0.5}$. So, $\widetilde{\lambda}(xy) + \widetilde{t} > \widetilde{0.5} + \widetilde{0.5} = \widetilde{1}$. So, $(xy)_{\widetilde{t}} q \widetilde{\lambda}$. Therefore, $(xy)_{\widetilde{t}} \in \vee q \widetilde{\lambda}$. □

Corollary 1. *Let $\widetilde{\lambda}$ be an interval-valued fuzzy subset of an LA-semigroup S. Then, $\widetilde{\lambda}$ is an interval-valued $(\in, \in \vee q)$-fuzzy two-sided ideal of S if and only if $\widetilde{\lambda}(xy) \geq \min\{\widetilde{\lambda}(y), \widetilde{0.5}\}$ and $\widetilde{\lambda}(xy) \geq \min\{\widetilde{\lambda}(x), \widetilde{0.5}\}$.*

Lemma 6. *Intersection of interval-valued $(\in, \in \vee q)$-fuzzy left ideals of an LA-semigroup S is an $(\in, \in \vee q)$-fuzzy left ideal of S.*

Proof. Let $\{\widetilde{\lambda}_i\}_{i \in I}$ be a family of interval-valued $(\in, \in \vee q)$-fuzzy left ideals of S. Let $x, y \in S$. Then,

$$(\wedge_{i \in I} \widetilde{\lambda}_i)(xy) = \wedge_{i \in I}(\widetilde{\lambda}_i(xy))$$

Since each $\widetilde{\lambda}_i$ is an interval-valued $(\in, \in \vee q)$-fuzzy left ideal of S. So, $\widetilde{\lambda}_i(xy) \geq \wedge_{i \in I}\{\widetilde{\lambda}_i(y), \widetilde{0.5}\}$. Thus,

$$\begin{aligned}(\wedge_{i \in I} \widetilde{\lambda}_i)(xy) &= \wedge_{i \in I}(\widetilde{\lambda}_i(xy)) \\ &\geq \wedge_{i \in I}\{\widetilde{\lambda}_i(y) \wedge \widetilde{0.5}\} \\ &= (\wedge_{i \in I} \widetilde{\lambda}_i(y)) \wedge \widetilde{0.5} \\ &= (\wedge_{i \in I} \widetilde{\lambda}_i)(y) \wedge \widetilde{0.5}\end{aligned}$$

Hence, $\wedge_{i \in I} \widetilde{\lambda}_i$ is an interval-valued $(\in, \in \vee q)$ fuzzy left ideal of S. □

Similarly, we can prove that intersection of interval-valued $(\in, \in \vee q)$-fuzzy right ideals of an LA-semigroup S is an interval-valued $(\in, \in \vee q)$-fuzzy right ideal of S. Thus, the intersection of interval-valued $(\in, \in \vee q)$-fuzzy two-sided ideals of an LA-semigroup S is an interval-valued $(\in, \in \vee q)$-fuzzy two-sided ideal of S.

Now, we show that If $\widetilde{\lambda}$ and $\widetilde{\mu}$ are interval-valued $(\in, \in \vee q)$-fuzzy ideals of an LA-semigroup S, then $\widetilde{\lambda} \circ \widetilde{\mu} \nleq \min\{\widetilde{\lambda}, \widetilde{\mu}\}$.

Example 1. Consider an LA-semigroup $S = \{1, 2, 3, 4\}$,

.	1	2	3	4
1	3	3	3	4
2	1	3	3	4
3	3	3	3	4
4	4	4	4	4

Let $\widetilde{\lambda}$ and $\widetilde{\mu}$ be interval-valued fuzzy subsets of S such that $\widetilde{\lambda}(1) = [0.8, 0.9]$, $\widetilde{\lambda}(2) = [0.7, 0.75]$, $\widetilde{\lambda}(3) = [0.6, 0.65]$, $\widetilde{\lambda}(4) = [0.5, 0.55]$, $\widetilde{\mu}(1) = [0.2, 0.3]$,

$\widetilde{\mu}(2) = [0.6, 0.65]$, $\widetilde{\mu}(3) = [0.5, 0.55]$, $\widetilde{\mu}(4) = [0.7, 0.75]$. Then, $\widetilde{\lambda}$ and $\widetilde{\mu}$ are interval-valued $(\in, \in \vee q)$-fuzzy ideals of S. Now,

$$\begin{aligned}(\widetilde{\lambda} \circ \widetilde{\mu})(3) &= \vee_{3=xy}\{\widetilde{\lambda}(x) \wedge \widetilde{\mu}(y)\} \\ &= \vee\{[0.2, 0.3], [0.5, 0.55], [0.6, 0.65]\} \\ &= [0.6, 0.65] \nleq \min\{(\widetilde{\lambda}, \widetilde{\mu})(3)\} \\ &= [0.5, 0.55]\end{aligned}$$

Hence, $\widetilde{\lambda} \circ \widetilde{\mu} \nleq \min\{\widetilde{\lambda}, \widetilde{\mu}\}$ in general.

Theorem 10. *An interval-valued fuzzy subset $\widetilde{\lambda}$ of an LA-semigroup S is an interval-valued $(\in, \in \vee q)$-fuzzy sub LA-semigroup of S if and only if $\widetilde{\lambda}(xy) \geq \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(y) \wedge \widetilde{0.5}\}$ for all $x, y \in S$.*

Proof. Suppose that $\widetilde{\lambda}$ is an interval-valued $(\in, \in \vee q)$-fuzzy sub LA-semigroup of S. On contrary, suppose that there exist $x, y \in S$ such that, $\widetilde{\lambda}(xy) < \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(y) \wedge \widetilde{0.5}\}$. Choose, $\widetilde{t} \in D[0, 1]$, where$\widetilde{t} \neq \widetilde{0}$, such that, $\widetilde{\lambda}(xy) < \widetilde{t} < \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(y) \wedge \widetilde{0.5}\}$. Then, $x_{\widetilde{t}} \in \widetilde{\lambda}$ and $y_{\widetilde{t}} \in \widetilde{\lambda}$ but $(xy)_{\widetilde{t}} \overline{\in \vee q} \widetilde{\lambda}$, which is contradiction. Thus, $\widetilde{\lambda}(xy) \geq \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(y) \wedge \widetilde{0.5}\}$ for all $x, y \in S$. Conversely, assume that $\widetilde{\lambda}(xy) \geq \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(y) \wedge \widetilde{0.5}\}$ for all $x, y \in S$. Let $x_{\widetilde{t}_1} \in \widetilde{\lambda}$ and $y_{\widetilde{t}_2} \in \widetilde{\lambda}$ for $\widetilde{t}_1, \widetilde{t}_2 \in D[0, 1]$. Then, $\widetilde{\lambda}(x) \geq \widetilde{t}_1$ and $\widetilde{\lambda}(y) \geq \widetilde{t}_2$. So, $\widetilde{\lambda}(xy) \geq \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \widetilde{0.5}\} \geq \min\{\widetilde{t}_1, \widetilde{t}_2, \widetilde{0.5}\}$. Now, if $\min\{\widetilde{t}_1 \widetilde{t}_2\} \leq \widetilde{0.5}$, then $\widetilde{\lambda}(xy) \geq \min\{\widetilde{t}_1, \widetilde{t}_2\}$. Then, $(xy)_{\min\{\widetilde{t}_1, \widetilde{t}_2\}} \in \widetilde{\lambda}$. If $\min\{\widetilde{t}_1, \widetilde{t}_2\} > \widetilde{0.5}$, then $\widetilde{\lambda}(xy) \geq \widetilde{0.5}$. Then, $\widetilde{\lambda}(xy) + \min\{\widetilde{t}_1, \widetilde{t}_2\} > \widetilde{0.5} + \widetilde{0.5} = \widetilde{1}$. So, $(xy)_{\min\{\widetilde{t}_1, \widetilde{t}_2\}} q \widetilde{\lambda}$. Therefore, $(xy)_{\min\{\widetilde{t}_1, \widetilde{t}_2\}} \in \vee q \widetilde{\lambda}$. Hence, $\widetilde{\lambda}$ is an interval-valued $(\in, \in \vee q)$-fuzzy sub LA-semigroup of S. □

Theorem 11. *An interval-valued fuzzy subset $\widetilde{\lambda}$ of an LA-semigroup S is an interval-valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S if and only if $\widetilde{\lambda}((xy)z) \geq \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(z) \wedge \widetilde{0.5}$ for all $x, y \in S$.*

Proof. Straightforward. □

Theorem 12. *An interval-valued fuzzy subset $\widetilde{\lambda}$ of an LA-semigroup S is an interval-valued $(\in, \in \vee q)$-fuzzy bi-ideal of S if and only if it satisfy the following conditions*

(i) $\widetilde{\lambda}(xy) \geq \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(y) \wedge \widetilde{0.5}$ for all $x, y \in S$.
(ii) $\widetilde{\lambda}((xy)z) \geq \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(y) \wedge \widetilde{0.5}\}$ for all $x, y, z \in S$.

Proof. Follows from Theorem 10 and Theorem 11. □

Definition 13. An LA-semigroup S is called a left regular if for each element a of S, there exists an element x in S such that $a = (aa)x$.

Lemma 7. *Every interval-valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal of a left regular LA-semigroup S, with left identity e, is an interval-valued $(\in, \in \vee q)$-fuzzy bi-ideal of S.*

Proof. Let $\widetilde{\lambda}$ be any interval-valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal of a left regular LA-semigroup S, with left identity e. Let $a, b \in S$. Then, there exists an element $x \in S$ such that $a = (aa)x$. Thus, we have

$$\begin{aligned} ab &= ((aa)x)b = ((aa)(ex))b \\ &= ((ae)(ax))b = (a((ae)x))b \\ &= (a((ae)x))b = a((ae)x)b. \end{aligned}$$

Thus, we have $\widetilde{\lambda}(ab) = \widetilde{\lambda}(a((ae)x)b) \geq \widetilde{\lambda}(a) \wedge \widetilde{\lambda}(b) \wedge \widetilde{0.5}\}$. This shows that $\widetilde{\lambda}$ is an interval-valued $(\in, \in \vee q)$-fuzzy sub LA-semigroup of S and so $\widetilde{\lambda}$ is an interval-valued $(\in, \in \vee q)$-fuzzy bi-ideal of S. □

Definition 14. An interval-valued fuzzy subset $\widetilde{\lambda}$ of an LA-semigroup S is called an interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal of S, if it satisfies $\widetilde{\lambda}(x) \geq (\widetilde{\lambda} \circ \widetilde{\delta})(x) \wedge (\widetilde{\delta} \circ \widetilde{\lambda})(x) \wedge \widetilde{0.5}$ where $\widetilde{\delta}$ is an interval-valued fuzzy subset of S mapping every element of S on $\widetilde{1}$, that is, $\widetilde{\delta}(x) = \widetilde{1}$.

Theorem 13. *Let $\widetilde{\lambda}$ be an interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal of an LA-semigroup S. Then the set $\widetilde{\lambda}_{\circ} = \{x \in S : \widetilde{\lambda}(x) > \widetilde{0}\}$ is a quasi-ideal of S.*

Proof. In order to show that $\widetilde{\lambda}_{\circ}$ is a quasi-ideal of S, we have to show that $S\widetilde{\lambda}_{\circ} \cap \widetilde{\lambda}_{\circ}S \subseteq \widetilde{\lambda}_{\circ}$. Let $a \in S\widetilde{\lambda}_{\circ} \cap \widetilde{\lambda}_{\circ}S$. This implies that $a \in S\widetilde{\lambda}_{\circ}$ and $a \in \widetilde{\lambda}_{\circ}S$. So, $a = sx$ and $a = yr$ for some $s, r \in S$ and $x, y \in \widetilde{\lambda}_{\circ}$. Thus, $\widetilde{\lambda}(x) > \widetilde{0}$ and $\widetilde{\lambda}(y) > \widetilde{0}$. Now,

$$\widetilde{\lambda}(a) \geq (\widetilde{\lambda} \circ \widetilde{\delta})(a) \wedge (\widetilde{\delta} \circ \widetilde{\lambda})(a) \wedge \widetilde{0.5}\}$$

Since

$$\begin{aligned} (\widetilde{\delta} \circ \widetilde{\lambda})(a) &= \vee_{a=pq}\{\widetilde{\delta}(p) \wedge \widetilde{\lambda}(q)\} \quad \text{because } a = sx \\ &\geq \widetilde{\delta}(s) \wedge \widetilde{\lambda}(x) = \widetilde{\lambda}(x) \end{aligned}$$

Similarly,

$$\begin{aligned} (\widetilde{\lambda} \circ \widetilde{\delta}) &= \vee_{a=pq}\{\widetilde{\lambda}(p) \wedge \widetilde{\delta}(q)\} \quad \text{because } a = yr \\ &\geq \widetilde{\lambda}(y) \wedge \widetilde{\delta}(r) = \widetilde{\lambda}(y) \end{aligned}$$

Thus,

$$\begin{aligned} \widetilde{\lambda}(a) &\geq (\widetilde{\lambda} \circ \widetilde{\delta})(a) \wedge (\widetilde{\delta} \circ \widetilde{\lambda})(a) \wedge \widetilde{0.5} \\ &\geq \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(y) \wedge \widetilde{0.5} \\ &> \widetilde{0} \quad (\text{ because } \widetilde{\lambda}(x) > \widetilde{0} \text{ and } \widetilde{\lambda}(y) > \widetilde{0}) \end{aligned}$$

Thus, $a \in \widetilde{\lambda}$. So, $S\widetilde{\lambda}_{\circ} \cap \widetilde{\lambda}_{\circ}S \subseteq \widetilde{\lambda}_{\circ}$. Hence, $\widetilde{\lambda}_{\circ}$ is a quasi-ideal of S. □

Remark 2. Every interval-valued fuzzy quasi-ideal of S is an interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal of S.

Definition 15. Let S be a non-empty LA-semigroup. Let A be any subset of S. Then, an interval-valued characteristic function of A, is the function $\widetilde{C}_A$ of S into $D[0,1]$ defined by

$$\widetilde{C}_A(x) = \begin{cases} [1,1] & \text{if } x \in A \\ [0,0] & \text{if } x \notin A \end{cases}.$$

Lemma 8. *A non-empty subset Q of an LA-semigroup S is a quasi-ideal of S if and only if interval-valued characteristic function $\widetilde{C}_Q$ is an interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal of S.*

Proof. Suppose that Q is a quasi-ideal of S. Let $\widetilde{C}_Q$ be an interval-valued characteristic function of Q. Let $x \in S$. If $x \notin Q$, then $x \notin SQ$. If $x \notin SQ$, then $(\widetilde{\delta} \circ \widetilde{C}_Q)(x) = \widetilde{0}$. So, $(\widetilde{C}_Q \circ \widetilde{\delta})(x) \wedge (\widetilde{\delta} \circ \widetilde{C})(x) \wedge \widetilde{0.5}\} = \widetilde{0} = \widetilde{C}_Q(x)$. Hence, $\widetilde{C}_Q$ is an interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal of S.

Conversely, assume that $\widetilde{C}_Q$ is an interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal of S. Let $a \in QS \cap SQ$. Then, there exist $b, c \in S$ and $x, y \in Q$ such that, $a = xb$ and $a = cy$. Then,

$$(\widetilde{C}_Q \circ \widetilde{\delta})(a) = \vee_{a=pq}\{\widetilde{C}_Q(p) \wedge \widetilde{\delta}(q)\} \geq \widetilde{C}_Q(x) \wedge \widetilde{\delta}(b) = \widetilde{1} \wedge \widetilde{1} = \widetilde{1}$$

So, $(\widetilde{C}_Q \circ \widetilde{\delta})(a) = \widetilde{1}$. Similarly,

$$(\widetilde{\delta} \circ \widetilde{C}_Q)(a) = \vee_{a=pq}\{\widetilde{\delta}(p) \wedge \widetilde{C}_Q(q)\} \geq \widetilde{\delta}(c) \wedge \widetilde{C}_Q(y) = \widetilde{1} \wedge \widetilde{1} = \widetilde{1}$$

So, $(\widetilde{\delta} \circ \widetilde{C}_Q)(a) = \widetilde{1}$. Hence,

$$\widetilde{C}_Q(a) \geq (\widetilde{C}_Q \circ \widetilde{\delta})(x) \wedge (\widetilde{\delta} \circ \widetilde{C}_Q)(x) \wedge \widetilde{0.5} = \widetilde{0.5}$$

Thus, $\widetilde{C}_Q(a) = \widetilde{1}$, which implies that $a \in Q$. Hence, $SQ \cap QS \subseteq Q$, that is, Q is a quasi-ideal of S. □

Lemma 9. *An interval-valued characteristic function $\widetilde{C}_L$ is an interval-valued $(\in, \in \vee q)$-fuzzy left ideal of S if and only if L is a left ideal of S.*

Proof. Suppose that L is a left ideal of S and $x, y \in S$. If $y \in L$, then $xy \in L$. So, $\widetilde{C}_L(y) = \widetilde{C}_L(xy) = \widetilde{1}$.

$$\widetilde{C}_L(xy) = \widetilde{1} \geq \widetilde{0.5} = \widetilde{C}_L(y) \wedge \widetilde{0.5}$$

If $y \notin L$, then $\widetilde{C}_L(y) = \widetilde{0}$. So,

$$\widetilde{C}_L(y) \wedge \widetilde{0.5} = \widetilde{0} \leq \widetilde{C}_L(xy)$$

Thus, $\widetilde{C}_L$ is an interval-valued $(\in, \in \vee q)$-fuzzy left ideal of S.

Conversely, assume that $\widetilde{C}_L$ is an interval-valued $(\in, \in \vee q)$-fuzzy left ideal of S. Let $y \in L$ and $x \in S$. As

$$\widetilde{C}_L(xy) \geq \widetilde{C}_L(y) \wedge \widetilde{0.5} = \widetilde{0.5}.$$

This implies that $\widetilde{C}_L(xy) = \widetilde{1}$, that is, $xy \in L$. So, L is a left ideal of S. $\square$

Similarly, an interval-valued characteristic function $\widetilde{C}_R$ is an interval-valued $(\in, \in \vee q)$-fuzzy right ideal of S if and only if R is a right ideal of S. Hence, it follows that interval-valued characteristic function $\widetilde{C}_I$ is an interval-valued $(\in, \in \vee q)$-fuzzy two-sided ideal of S if and only if I is two-sided ideal of S.

Theorem 14. *Every $(\in, \in \vee q)$-fuzzy left ideal of S is an $(\in, \in \vee q)$-fuzzy quasi-ideal of S.*

Proof. Let $x \in S$. Then

$$(\widetilde{\delta} \circ \widetilde{\lambda})(x) = \vee_{x=yz}\{\widetilde{\delta}(y) \wedge \widetilde{\lambda}(z)\} = \vee_{x=yz}\widetilde{\lambda}(z)$$

This implies that

$$\begin{aligned}(\widetilde{\delta} \circ \widetilde{\lambda})(x) \wedge \widetilde{0.5} &= \vee_{x=yz}\{\widetilde{\delta}(x) \wedge \widetilde{\lambda}(z)\} \wedge \widetilde{0.5} \\ &= \vee_{x=yz}\{\widetilde{\lambda}(z) \wedge \widetilde{0.5}\} \\ &\leq \widetilde{\lambda}(yz) \\ &= \widetilde{\lambda}(x)\end{aligned}$$

Thus, $(\widetilde{\delta} \circ \widetilde{\lambda})(x) \wedge \widetilde{0.5}\} \leq \widetilde{\lambda}(x)$. Hence, $\widetilde{\lambda}(x) \geq (\widetilde{\delta} \circ \widetilde{\lambda})(x) \wedge \widetilde{0.5}\} \geq (\widetilde{\lambda} \circ \widetilde{\delta})(x) \wedge (\widetilde{\delta} \circ \widetilde{\lambda})(x) \wedge \widetilde{0.5}\}$. Thus, $\widetilde{\lambda}$ is an interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal of S. Similarly, we have this result for interval-valued $(\in, \in \vee q)$-fuzzy right ideals. $\square$

Similarly, we can show that every interval-valued $(\in, \in \vee q)$-fuzzy right ideal of S is an interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal of S.

Lemma 10. *Let S be an LA-semigroup with left identity e, such that, $(xe)S = xS$. Then every interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal of S is an interval-valued $(\in, \in \vee q)$-fuzzy bi-ideal of S.*

Proof. Suppose that $\widetilde{\lambda}$ is an interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal of an LA-semigroup S. Let $x, y \in S$. Then

$$\begin{aligned}\widetilde{\lambda}(xy) &\geq (\widetilde{\lambda} \circ \widetilde{\delta})(xy) \wedge (\widetilde{\delta} \circ \widetilde{\lambda})(xy) \wedge \widetilde{0.5} \\ &= (\vee_{xy=ab}\{\widetilde{\lambda}(a) \wedge \widetilde{\delta}(b)\}) \wedge (\vee_{xy=pq}\{\widetilde{\delta}(p) \wedge \widetilde{\lambda}(q)\}) \wedge \widetilde{0.5} \\ &\geq \{\widetilde{\lambda}(x) \wedge \widetilde{\delta}(y)\} \wedge \{\widetilde{\delta}(x) \wedge \widetilde{\lambda}(y) \wedge \widetilde{0.5}\} \\ &\geq \{\widetilde{\lambda}(x) \wedge \widetilde{1}\} \wedge \{\widetilde{1} \wedge \widetilde{\lambda}(y) \wedge \widetilde{0.5}\} \\ &= \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(y) \wedge \widetilde{0.5}\end{aligned}$$

So,

$$\widetilde{\lambda}(xy) \geq \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(y) \wedge \widetilde{0.5}\}$$

Also, if $x, y, z \in S$ then

$$\widetilde{\lambda}((xy)z) \geq (\widetilde{\lambda} \circ \widetilde{\delta})((xy)z) \wedge (\widetilde{\delta} \circ \widetilde{\lambda})((xy)z) \wedge \widetilde{0.5}\}$$

Now,

$$\begin{aligned}(\widetilde{\delta}\circ\widetilde{\lambda})((xy)z) &= \vee_{(xy)z=pq}\{\widetilde{\delta}(p)\wedge\widetilde{\lambda}(q)\}\}\\ &= \vee_{(xy)z=pq}\{\widetilde{1}\vee\widetilde{\lambda}(q)\}\\ &\geq \widetilde{\lambda}(z)\end{aligned}$$

So,

$$(\widetilde{\delta}\circ\widetilde{\lambda})((xy)z)\geq\widetilde{\lambda}(z)$$

Also,

$$(\widetilde{\lambda}\circ\widetilde{\delta})((xy)z)=\vee_{(xy)z=pq}\{\widetilde{\lambda}(a)\wedge\widetilde{\delta}(b)\}=\vee_{(xy)z=pq}\{\widetilde{\lambda}(a)\wedge\widetilde{1}\}.$$

Since $(xy)z=(xy)(ez)=(xe)(yz)\in(xe)S=xS$. So, $(xy)z=xr$ for some $r\in S$. Then,

$$(\widetilde{\lambda}\circ\widetilde{\delta})((xy)z)=\vee_{xr=ab}\{\widetilde{\lambda}(a)\wedge\widetilde{1}\}\geq\widetilde{\lambda}(x)$$

Thus,

$$\widetilde{\lambda}((xy)z)\geq\widetilde{\lambda}(x)\wedge\widetilde{\lambda}(z)\wedge\widetilde{0.5}$$

Hence, $\widetilde{\lambda}$ is an interval-valued $(\in,\in\vee q)$-fuzzy bi-ideal of S. □

6. LOWER AND UPPER PARTS OF INTERVAL VALUED $(\in,\in\vee q)$-FUZZY IDEALS

Definition 16. Let λ be an interval-valued fuzzy subset of an LA-semigroup S. We define upper and lower parts $\widetilde{\lambda}^{+}$ and $\widetilde{\lambda}^{-}$ of $\widetilde{\lambda}$ as follows.

$$\widetilde{\lambda}^{+}(x)=\widetilde{\lambda}(x)\vee\widetilde{0.5},\quad \widetilde{\lambda}^{-}(x)=\widetilde{\lambda}(x)\wedge\widetilde{0.5}$$

Lemma 11. *Let $\widetilde{\lambda}$ and $\widetilde{\mu}$ be interval-valued fuzzy subsets of an LA-semigroup S. Then, the following holds.*
(i) $(\widetilde{\lambda}\wedge\widetilde{\mu})^{-}=\widetilde{\lambda}^{-}\wedge\widetilde{\mu}^{-}$ (ii) $(\widetilde{\lambda}\vee\widetilde{\mu})^{-}=(\widetilde{\lambda}^{-}\vee\widetilde{\mu}^{-}\})$ (iii) $(\widetilde{\lambda}\circ\widetilde{\mu})^{-}=(\widetilde{\lambda}^{-}\circ\widetilde{\mu}^{-})$

Proof. (i) For all $a\in S$

$$\begin{aligned}(\widetilde{\lambda}\wedge\widetilde{\mu})^{-}(a) &= (\widetilde{\lambda}\wedge\widetilde{\mu})^{-})(a)\wedge\widetilde{0.5}\\ &= \widetilde{\lambda}(a)\wedge\widetilde{\mu}(a)\wedge\widetilde{0.5}\\ &= (\widetilde{\lambda}(a)\wedge\widetilde{0.5}\}\wedge\{\widetilde{\mu}(a)\wedge\widetilde{0.5}\}\\ &= \widetilde{\lambda}^{-}(a)\wedge\widetilde{\mu}^{-}(a)\\ &= (\widetilde{\lambda}^{-}\wedge\widetilde{\mu}^{-})(a)\end{aligned}$$

Thus,

$$(\widetilde{\lambda}\wedge\widetilde{\mu})^{-}=\widetilde{\lambda}^{-}\wedge\widetilde{\mu}^{-}$$

(ii)

$$\begin{aligned}(\widetilde{\lambda}\vee\widetilde{\mu})^{-}(a) &= (\widetilde{\lambda}\vee\widetilde{\mu})(a)\wedge\widetilde{0.5}\\ &= \{\widetilde{\lambda}(a)\vee\widetilde{\mu}(a)\}\wedge\widetilde{0.5}\end{aligned}$$

$$= \{\widetilde{\lambda}(a) \wedge \widetilde{0.5}\} \vee \{\widetilde{\mu}(a) \wedge \widetilde{0.5}\}$$
$$= \widetilde{\lambda}^{-}(a) \vee \widetilde{\mu}^{-}(a)$$
$$= (\widetilde{\lambda}^{-} \vee \widetilde{\mu}^{-})(a)$$

Thus,

$$(\widetilde{\lambda} \vee \widetilde{\mu})^{-} = (\widetilde{\lambda}^{-} \vee \widetilde{\mu}^{-}\}).$$

(iii) If a is not expressible as $a = bc$ for some $b, c \in S$, then $(\widetilde{\lambda} \circ \widetilde{\mu})(a) = \widetilde{0}$. Thus,

$$(\widetilde{\lambda} \circ \widetilde{\mu})^{-}(a) = (\widetilde{\lambda} \circ \widetilde{\mu}) \wedge \widetilde{0.5}\} = \widetilde{0}$$

Since a is not expressible as $a = bc$, so $(\widetilde{\lambda}^{-} \circ \widetilde{\mu}^{-})(a) = \widetilde{0}$. Thus, in this case

$$(\widetilde{\lambda} \circ \widetilde{\mu})^{-} = (\widetilde{\lambda}^{-} \circ \widetilde{\mu}^{-})$$

If a is expressible as $a = xy$ for some $x, y \in S$,then

$$\begin{aligned}
(\widetilde{\lambda} \circ \widetilde{\mu})^{-}(a) &= (\widetilde{\lambda} \circ \widetilde{\mu})(a) \wedge \widetilde{0.5} \\
&= (\vee_{a=xy}\{\widetilde{\lambda}(x) \wedge \widetilde{\mu}(y)\}) \wedge \widetilde{0.5} \\
&= \vee_{a=xy}\{\widetilde{\lambda}(x) \wedge \widetilde{\mu}(y) \wedge \widetilde{0.5}\} \\
&= \vee_{a=xy}\{\{(\widetilde{\lambda}(x) \wedge \widetilde{0.5})\} \wedge \{(\widetilde{\mu}(y) \wedge \widetilde{0.5})\}\} \\
&= \vee_{a=xy}\{\widetilde{\lambda}^{-}(x) \wedge \widetilde{\mu}^{-}(y)\} \\
&= (\widetilde{\lambda}^{-} \circ \widetilde{\mu}^{-})(a)
\end{aligned}$$

Thus,

$$(\widetilde{\lambda} \circ \widetilde{\mu})^{-} = (\widetilde{\lambda}^{-} \circ \widetilde{\mu}^{-})$$

□

Lemma 12. *Let $\widetilde{\lambda}$ and $\widetilde{\mu}$ be an interval-valued fuzzy subsets of S. Then, the following hold.*

(i) $(\widetilde{\lambda} \wedge \widetilde{\mu})^{+} = \widetilde{\lambda}^{+} \wedge \widetilde{\mu}^{+}$ (ii) $(\widetilde{\lambda} \vee \widetilde{\mu})^{+} = \widetilde{\lambda}^{+} \vee \widetilde{\mu}^{+}$ (iii) $(\widetilde{\lambda} \circ \widetilde{\mu})^{+} \geq (\widetilde{\lambda}^{+} \circ \widetilde{\mu}^{+})$

If every element x of S is expressible as $x = bc$, then $(\widetilde{\lambda} \circ \widetilde{\mu})^{+} = (\widetilde{\lambda}^{+} \circ \widetilde{\mu}^{+})$

Proof. (i) For all $a \in S$

$$\begin{aligned}
(\widetilde{\lambda} \wedge \widetilde{\mu})^{+}(a) &= (\widetilde{\lambda} \wedge \widetilde{\mu})(a) \vee \widetilde{0.5}\} \\
&= \{\widetilde{\lambda}(a) \wedge \widetilde{\mu}(a)\} \vee \widetilde{0.5} \\
&= \{\widetilde{\lambda}(a) \vee \widetilde{0.5}\} \wedge \{\widetilde{\mu}(a) \vee \widetilde{0.5}\} \\
&= \widetilde{\lambda}^{+}(a) \wedge \widetilde{\mu}^{+}(a) \\
&= (\widetilde{\lambda}^{+} \wedge \widetilde{\mu}^{+})(a)
\end{aligned}$$

Thus,

$$(\widetilde{\lambda} \vee \widetilde{\mu})^{+} = \widetilde{\lambda}^{+} \vee \widetilde{\mu}^{+}.$$

(ii)

$$\begin{aligned}(\widetilde{\lambda} \vee \widetilde{\mu})^{+}(a) &= (\widetilde{\lambda} \vee \widetilde{\mu})(a) \vee \widetilde{0.5}\} \\ &= \widetilde{\lambda}(a) \vee \widetilde{\mu}(a) \vee \widetilde{0.5}\} \\ &= \{\widetilde{\lambda}(a) \vee \widetilde{0.5}\} \vee \{\widetilde{\mu}(a) \vee \widetilde{0.5}\} \\ &= \widetilde{\lambda}^{+}(a) \vee \widetilde{\mu}^{+}(a) \\ &= (\widetilde{\lambda}^{+} \vee \widetilde{\mu}^{+})(a)\end{aligned}$$

Thus,

$$(\widetilde{\lambda} \vee \widetilde{\mu})^{+} = \widetilde{\lambda}^{+} \vee \widetilde{\mu}^{+}$$

(iii) If a is not expressible as $a = bc$ for some $b, c \in S$, then $(\widetilde{\lambda} \circ \widetilde{\mu})(a) = \widetilde{0}$. Thus,

$$(\widetilde{\lambda} \circ \widetilde{\mu})^{+}(a) = (\widetilde{\lambda} \circ \widetilde{\mu})(a) \vee \widetilde{0.5} = \widetilde{0.5}$$

But $(\widetilde{\lambda}^{+} \circ \widetilde{\mu}^{+})(a) = \widetilde{0}$. So, $\widetilde{\lambda}^{+} \circ \widetilde{\mu}^{+} \leq (\widetilde{\lambda} \circ \widetilde{\mu})^{+}$.If a is expressible as $a = xy$ for some $x, y \in S$, then

$$\begin{aligned}(\widetilde{\lambda} \circ \widetilde{\mu})^{+}(a) &= (\widetilde{\lambda} \circ \widetilde{\mu})(a) \vee \widetilde{0.5} \\ &= \vee_{a=xy}\{\widetilde{\lambda}(x) \wedge \widetilde{\mu}(y)\} \vee \widetilde{0.5}\} \\ &= \vee_{a=xy}(\{\widetilde{\lambda}(x) \vee \widetilde{0.5}\} \wedge \{\widetilde{\mu}(y), \widetilde{0.5}\}) \\ &= \vee_{a=xy}\{\widetilde{\lambda}^{+}(x) \wedge \widetilde{\mu}^{+}(y)\} \\ &= (\widetilde{\lambda}^{+} \circ \widetilde{\mu}^{+})(a)\end{aligned}$$

Thus,

$$(\widetilde{\lambda} \circ \widetilde{\mu})^{+} = \widetilde{\lambda}^{+} \circ \widetilde{\mu}^{+}.$$

□

Definition 17. Let A be a non-empty subset of an LA-semigroup S. Then the lower and upper parts of an interval-valued characteristic function is

$$\widetilde{C}_{A}^{-}(a) = \begin{cases} [0.5, 0.5] & \text{if } a \in A \\ [0, 0] & \text{if } a \notin A \end{cases} \quad \text{and} \quad \widetilde{C}_{A}^{+}(a) = \begin{cases} [1, 1] & \text{if } a \in A \\ [0.5, 0.5] & \text{if } a \notin A \end{cases}.$$

Lemma 13. *Let A and B be non-empty subsets of an LA-semigroup S. Then the following properties hold*

(i) $(\widetilde{C}_{A} \wedge \widetilde{C}_{B})^{-} = \widetilde{C}_{A\cap B}^{-}$ (ii) $(\widetilde{C}_{A} \vee \widetilde{C}_{B}\})^{-} = \widetilde{C}_{A\cup B}^{-}$ (iii) $(\widetilde{C}_{A} \circ \widetilde{C}_{B})^{-} = \widetilde{C}_{AB}^{-}$

Proof. The proof is obvious. □

Lemma 14. *The lower part of an interval-valued characteristic function $\widetilde{C}_{L}^{-}$ is an interval-valued $(\in, \in \vee q)$-fuzzy left ideal of S if and only if L is a left ideal of S.*

Proof. The proof is obvious. □

Similarly, we can prove that the lower part of an interval-valued characteristic function $\widetilde{C}_R^-$ is an interval-valued $(\in, \in \vee q)$-fuzzy right ideal of S if and only if R is a right ideal of S. Thus, lower part of an interval-valued characteristic function $\widetilde{C}_I$ is an interval-valued $(\in, \in \vee q)$-fuzzy two-sided ideal of S if and only if I is a two-sided ideal of S.

Lemma 15. *Let Q be a non-empty subset of an LA-semigroup S. Then, Q is a quasi-ideal of S if and only if lower part of an interval-valued characteristic function $\widetilde{C}_Q^-$ is an interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal of S.*

Proof. Suppose that Q is a quasi-ideal of S. Let $\widetilde{C}_Q^-$ be lower part of an interval-valued characteristic function of Q. Let $x \in S$. If $x \notin Q$ then $x \notin SQ$ or $x \notin QS$. If $x \notin SQ$, then $(\widetilde{\delta} \circ \widetilde{C}_Q^-)(x) = \widetilde{0}$ and so,

$$(\widetilde{C}_Q^- \circ \widetilde{\delta})(x) \wedge (\widetilde{\delta} \circ \widetilde{C}_Q^-)(x) \wedge \widetilde{0.5}\} = \widetilde{0} = \widetilde{C}_Q^-(x)$$

If $x \in Q$, then

$$\widetilde{C}_Q^-(x) = \widetilde{0.5} \geq (\widetilde{C}_Q^- \circ \widetilde{\delta})(x) \wedge (\widetilde{\delta} \circ \widetilde{C}_Q^-)(x) \wedge \widetilde{0.5}\}$$

Hence, $\widetilde{C}_Q^-$ is an interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal of S. Conversely, assume that $\widetilde{C}_Q^-$ is an interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal of S. Let $a \in SQ \cap QS$ then there exist $b, c \in S$ and $x, y \in Q$ such that $a = xb$ and $a = cy$. Then,

$$(\widetilde{C}_Q^- \circ \widetilde{\delta})(a) = \vee_{a=pq}\{\widetilde{C}_Q^-(p) \wedge \widetilde{\delta}(q)\} \geq \{\widetilde{C}_Q^-(x) \wedge \widetilde{\delta}(b)\} = \widetilde{0.5} \wedge \widetilde{1} = \widetilde{0.5}$$

So,

$$(\widetilde{C}_Q^- \circ \widetilde{\delta})(a) = \widetilde{0.5}$$

Similarly,

$$(\widetilde{\delta} \circ \widetilde{C}_Q^-)(x) = \vee_{x=pq}\{\widetilde{\delta}(p) \wedge \widetilde{C}_Q^-(q)\} \geq \widetilde{\delta}(c) \wedge \widetilde{C}_Q^-(y) = \widetilde{1} \wedge \widetilde{0.5} = \widetilde{0.5}$$

So,

$$(\widetilde{\delta} \circ \widetilde{C}_Q^-)(x) = \widetilde{0.5}$$

Hence,

$$\widetilde{C}_Q^-(x) \geq (\widetilde{C}_Q^- \circ \widetilde{\delta})(a) \wedge (\widetilde{\delta} \circ \widetilde{C}_Q^-)(a) \wedge \widetilde{0.5} = \widetilde{0.5}$$

Thus, $\widetilde{C}_Q^-(a) = \widetilde{0.5}$. This implies that $a \in Q$. Hence, $QS \cap SQ \subseteq Q$, that is, a quasi-ideal of S. $\square$

We have shown in Lemma 1 that every fuzzy left (right) ideal of an LA-semigroup S is an $(\in, \in)$-fuzzy left (right) ideal of S. Obviously, every $(\in, \in)$-fuzzy left (right) ideal of S is an interval-valued $(\in, \in \vee q)$-fuzzy left (right) ideal of S. But interval-valued $(\in, \in \vee q)$-fuzzy left (right) ideal of S need not to be interval-valued fuzzy left (right) ideal of S.

Example 2. Consider the LA-semigroup given in Example 1. Then, the interval-valued fuzzy subset $\widetilde{\lambda}$ of S defined by $\widetilde{\lambda}(1) = [0.8, 0.9]$, $\widetilde{\lambda}(2) = [0.7, 0.75]$, $\widetilde{\lambda}(3) = [0.6, 0.65]$ and $\widetilde{\lambda}(4) = [0.5, 0.55]$ is an $(\in, \in \vee q)$-fuzzy left ideal of S but $\widetilde{\lambda}$ is not an interval-valued fuzzy left ideal of S, because $\widetilde{\lambda}(3) = \widetilde{\lambda}(1.2) = [0.6, 0.65]$ and $\widetilde{\lambda}(2) = \widetilde{0.7}$. So, $\widetilde{\lambda}(1.2) \not\geq \widetilde{\lambda}(2)$.

Theorem 15. *For an LA-semigroup S the following conditions are equivalent.*
(1) *S is regular.*
(2) $(\widetilde{\lambda} \wedge \widetilde{\mu})^{-} = (\widetilde{\lambda} \circ \widetilde{\mu})^{-}$ *for every interval-valued $(\in, \in \vee q)$-fuzzy right ideal $\widetilde{\lambda}$ and every interval-valued $(\in, \in \vee q)$-fuzzy left ideal $\widetilde{\mu}$ of S.*

Proof. First we assume that (1) holds. Let $\widetilde{\lambda}$ be an interval-valued $(\in, \in \vee q)$-fuzzy right ideal and $\widetilde{\mu}$ an interval-valued $(\in, \in \vee q)$-fuzzy left ideal of S. Let, $a \in S$ then

$$\begin{aligned}
(\widetilde{\lambda} \circ \widetilde{\mu})^{-}(a) &= (\widetilde{\lambda} \circ \widetilde{\mu})(a) \wedge \widetilde{0.5} \\
&= (\vee_{a=yz}\{\widetilde{\lambda}(y) \wedge \widetilde{\mu}(z)) \wedge \widetilde{0.5}\} \\
&= \vee_{a=yz}(\widetilde{\lambda}(y) \wedge \widetilde{\mu}(z) \wedge \widetilde{0.5}) \\
&= \vee_{a=yz}(\{\widetilde{\lambda}(y) \wedge \widetilde{0.5}\} \wedge \{\widetilde{\mu}(z) \wedge \widetilde{0.5}\} \wedge \widetilde{0.5}) \\
&\leq \vee_{a=yz}(\widetilde{\lambda}(yz) \wedge \widetilde{\mu}(yz) \wedge \widetilde{0.5}) \\
&= \widetilde{\lambda}(a) \wedge \widetilde{\mu}(a) \wedge \widetilde{0.5} \\
&= (\widetilde{\lambda} \wedge \widetilde{\mu})(a) \wedge \widetilde{0.5} \\
&= (\widetilde{\lambda} \wedge \widetilde{\mu})^{-}(a)
\end{aligned}$$

So, $(\widetilde{\lambda} \circ \widetilde{\mu})^{-} \leq (\widetilde{\lambda} \wedge \widetilde{\mu})^{-}$.

Since S is regular, so there exists an element $x \in S$ such that, $a = (ax)a$.
So,

$$\begin{aligned}
(\widetilde{\lambda} \circ \widetilde{\mu})^{-} &= (\widetilde{\lambda} \circ \widetilde{\mu})(a) \wedge \widetilde{0.5} \\
&= (\vee_{a=yz}\{\widetilde{\lambda}(y) \wedge \widetilde{\mu}(z)) \wedge \widetilde{0.5} \\
&\geq \{\widetilde{\lambda}(ax) \wedge \widetilde{\mu}(a)\} \wedge \widetilde{0.5} \\
&\geq \{\widetilde{\lambda}(a) \wedge \widetilde{0.5} \wedge \widetilde{\mu}(a)\} \wedge \widetilde{0.5}\} \\
&= \widetilde{\lambda}(a) \wedge \widetilde{\mu}(a) \wedge \widetilde{0.5} \\
&= (\widetilde{\lambda} \wedge \widetilde{\mu}\})(a) \wedge \widetilde{0.5} \\
&= (\widetilde{\lambda} \wedge \widetilde{\mu})^{-}(a)
\end{aligned}$$

So,

$$(\widetilde{\lambda} \circ \widetilde{\mu})^{-} \geq (\widetilde{\lambda} \wedge \widetilde{\mu})^{-}$$

Thus,

$$(\widetilde{\lambda} \circ \widetilde{\mu})^{-} = (\widetilde{\lambda} \wedge \widetilde{\mu})^{-}$$

Conversely, assume that (2) holds. Let R and L be right ideal and left ideal of S respectively. In order to see that $R \cap L = RL$ holds. Let a be any element of $R \cap L$. Then, by Lemma 14, the lower part of an interval-valued characteristic functions $\widetilde{C}_R^-$ and $\widetilde{C}_L^-$ of R and L are interval-valued $(\in, \in \vee q)$-fuzzy right ideal and interval-valued $(\in, \in \vee q)$-fuzzy left ideal of S, respectively. Thus, we have

$$\begin{aligned}\widetilde{C}_{RL}^- &= (\widetilde{C}_R^- \circ \widetilde{C}_L)^- \quad \text{(by Lemma 13)}\\ &= (\widetilde{C}_R \wedge \widetilde{C}_L)^- \quad \text{(by (1))}\\ &= \widetilde{C}_{R\cap L}^-\end{aligned}$$

Thus, $R \cap L = RL$. Hence, by Theorem 1, S is regular and so (2)⇒(1). □

Theorem 16. *Let S be an LA-semigroup with left identity e such that $(xe)S = xS$ for all $x \in S$. Then, the following conditions are equivalent.*

(1) *S is regular.*

(2) $((\widetilde{\rho} \wedge \widetilde{\lambda}) \wedge \widetilde{\mu})^- \leq ((\widetilde{\rho} \circ \widetilde{\lambda}) \circ \widetilde{\mu})^-$ *for every interval-valued $(\in, \in \vee q)$-fuzzy right ideal $\widetilde{\rho}$ every interval-valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal $\widetilde{\lambda}$ and every interval-valued $(\in, \in \vee q)$-fuzzy left ideal $\widetilde{\mu}$ of S.*

(3) $((\widetilde{\rho} \wedge \widetilde{\lambda}) \wedge \widetilde{\mu})^- \leq ((\widetilde{\rho} \circ \widetilde{\lambda}) \circ \widetilde{\mu})^-$ *for every interval-valued $(\in, \in \vee q)$-fuzzy right ideal $\widetilde{\rho}$, every interval-valued $(\in, \in \vee q)$-fuzzy bi-ideal $\widetilde{\lambda}$ and every interval-valued $(\in, \in \vee q)$-fuzzy left ideal $\widetilde{\mu}$ of S.*

(4) $((\widetilde{\rho} \wedge \widetilde{\lambda}) \wedge \widetilde{\mu})^- \leq ((\widetilde{\rho} \circ \widetilde{\lambda}) \circ \widetilde{\mu})^-$ *for every interval-valued $(\in, \in \vee q)$-fuzzy right ideal $\widetilde{\rho}$, every interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal $\widetilde{\lambda}$ and every interval-valued $(\in, \in \vee q)$-fuzzy left ideal $\widetilde{\mu}$ of S.*

Proof. (1)⇒(2)

Let $\widetilde{\rho}, \widetilde{\lambda}$ and $\widetilde{\mu}$ be any interval-valued $(\in, \in \vee q)$-fuzzy right ideal, interval-valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal and for any interval-valued $(\in, \in \vee q)$-fuzzy left ideal of S, respectively. Let a be any element of S. Since S is regular, so there exists an element $x \in S$ such that $a = (ax)a$. Hence, we have

$$((\widetilde{\rho} \circ \widetilde{\lambda}) \circ \widetilde{\mu})^-(a) = (\vee_{a=yz}\{(\widetilde{\rho} \circ \widetilde{\lambda})(y) \wedge \widetilde{\mu}(z)) \wedge \widetilde{0.5}$$

Now, $a = (ax)a = (ax)(ea) = (ae)(xa) = a(xa)$ because $(xe)S = xS$ for all $x \in S$.

$$\begin{aligned}((\widetilde{\rho} \circ \widetilde{\lambda}) \circ \widetilde{\mu})^-(a) &= (\vee_{a=yz}\{(\widetilde{\rho} \circ \widetilde{\lambda})(y) \wedge \widetilde{\mu}(z)) \wedge \widetilde{0.5}\\ &\geq (\widetilde{\rho} \circ \widetilde{\lambda})(a) \wedge \widetilde{\mu}(xa) \wedge \widetilde{0.5}\\ &\geq (\vee_{a=pq}\{\widetilde{\rho}(p) \wedge \widetilde{\lambda}(q)), \{\widetilde{\mu}(a) \wedge \widetilde{0.5}\} \wedge \widetilde{0.5}\\ &\geq \{\widetilde{\rho}(ax) \wedge \widetilde{\lambda}(a)\} \wedge \{\widetilde{\mu}(a) \wedge \widetilde{0.5}\}\\ &\geq \{(\widetilde{\rho}(a) \wedge \widetilde{0.5}) \wedge \widetilde{\lambda}(a)\} \wedge \widetilde{\mu}(a) \wedge \widetilde{0.5}\\ &= \{\widetilde{\rho}(a) \wedge \widetilde{\lambda}(a)\} \wedge \widetilde{\mu}(a) \wedge \widetilde{0.5}\\ &= ((\widetilde{\rho} \wedge \widetilde{\lambda}) \wedge \widetilde{\mu})^-(a)\end{aligned}$$

Thus,

$$((\widetilde{\rho} \wedge \widetilde{\lambda}) \wedge \widetilde{\mu})^{-}(a) \leq ((\widetilde{\rho} \circ \widetilde{\lambda}) \circ \widetilde{\mu})^{-}$$

Hence, (1)$\Rightarrow$ (2).

(2) $\Longrightarrow$(3) is straight forward because every interval-valued $(\in, \in \vee q)$-fuzzy bi-ideal is an interval-valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S.

(3)$\Longrightarrow$(4) is also straight forward because every interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal is an interval-valued $(\in, \in \vee q)$-fuzzy bi-ideal of S.

(4)$\Longrightarrow$(1) Let $\widetilde{\rho}$ and $\widetilde{\mu}$ be any interval-valued $(\in, \in \vee q)$-fuzzy right ideal and any interval-valued $(\in, \in \vee q)$-fuzzy left ideal of S respectively. Since $\widetilde{\delta}$ is an $(\in, \in \vee q)$-fuzzy quasi-ideal of S, by the assumption, we have

$$\begin{aligned}
(\widetilde{\rho} \wedge \widetilde{\mu})^{-}(a) &= (\widetilde{\rho} \wedge \widetilde{\mu})(a) \wedge \widetilde{0.5} \\
&= \{\{\widetilde{\rho} \wedge \widetilde{\delta}\} \wedge \widetilde{\mu}\} \wedge \widetilde{0.5} \\
&= (\{\widetilde{\rho} \wedge \widetilde{\delta}\} \wedge \widetilde{\mu}\})^{-}(a) \\
&\leq ((\widetilde{\rho} \circ \widetilde{\delta}) \circ \widetilde{\mu})^{-}(a) \\
&= ((\widetilde{\rho} \circ \widetilde{\delta}) \circ \widetilde{\mu})(a) \wedge \widetilde{0.5} \\
&= \vee_{a=bc}\{(\widetilde{\rho} \circ \widetilde{\delta})(b) \wedge \widetilde{\mu}(c)\} \wedge \widetilde{0.5} \\
&= \vee_{a=bc}\{\vee_{b=pq}\{\widetilde{\rho}(p) \wedge \widetilde{\delta}(q)\} \wedge \widetilde{\mu}(c)\} \wedge \widetilde{0.5} \\
&= \vee_{a=bc}\{\vee_{b=pq}\{\widetilde{\rho}(p)\} \wedge \widetilde{\mu}(c)\} \wedge \widetilde{0.5} \\
&= \vee_{a=bc}\{\vee_{b=pq}\{\{\widetilde{\rho}(p) \wedge 0.5\}\} \wedge \widetilde{\mu}(c)\} \wedge \widetilde{0.5} \\
&\leq \vee_{a=bc}\{\vee_{b=pq}\{\widetilde{\rho}(pq)\} \wedge \widetilde{\mu}(c)\} \wedge \widetilde{0.5} \\
&= \vee_{a=bc}\{\{\widetilde{\rho}(b) \wedge \widetilde{\mu}(c)\}\} \wedge \widetilde{0.5} \\
&= (\widetilde{\rho} \circ \widetilde{\mu})(a) \wedge \widetilde{0.5} \\
&= (\widetilde{\rho} \circ \widetilde{\mu})^{-}(a)
\end{aligned}$$

Thus, it follows that $(\widetilde{\rho} \wedge \widetilde{\mu})^{-} \leq (\widetilde{\rho} \circ \widetilde{\mu})^{-}$ for every interval-valued $(\in, \in \vee q)$-fuzzy right ideal $\widetilde{\rho}$ and every interval-valued $(\in, \in \vee q)$-fuzzy left ideal $\widetilde{\mu}$ of S. But $(\widetilde{\rho} \circ \widetilde{\mu})^{-} \leq (\widetilde{\rho} \wedge \widetilde{\mu})^{-}$ always. So, $(\widetilde{\rho} \circ \widetilde{\mu})^{-} = (\widetilde{\rho} \wedge \widetilde{\mu})^{-}$. Hence, it follows from Theorem 15 that S is regular. □

Theorem 17. *Let S be an LA-semigroup with left identity e such that, $(xe)S = xS$, for all $x \in S$. Then, the following conditions are equivalent.*

(1) *S is regular.*

(2) $(\widetilde{\lambda} \wedge \widetilde{\mu})^{-} \leq (\widetilde{\lambda} \circ \widetilde{\mu})^{-}$ *for every interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal $\widetilde{\lambda}$ and every interval-valued $(\in, \in \vee q)$-fuzzy left ideal $\widetilde{\mu}$ of S.*

(3) $(\widetilde{\lambda} \wedge \widetilde{\mu})^{-} \leq (\widetilde{\lambda} \circ \widetilde{\mu})^{-}$ *for every interval-valued $(\in, \in \vee q)$-fuzzy bi-ideal $\widetilde{\lambda}$ and every interval-valued $(\in, \in \vee q)$-fuzzy left ideal $\widetilde{\mu}$ of S.*

(4) $(\widetilde{\lambda} \wedge \widetilde{\mu})^{-} \leq (\widetilde{\lambda} \circ \widetilde{\mu})^{-}$ *for every interval-valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal $\widetilde{\lambda}$ and every interval-valued $(\in, \in \vee q)$-fuzzy left ideal $\widetilde{\mu}$ of S.*

Proof. (1)⇒(4)

Let $\widetilde{\lambda}$ and $\widetilde{\mu}$ be any interval-valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal and any interval-valued $(\in, \in \vee q)$-fuzzy left ideal of S respectively. Let a be any element of S. Then, there exist an element $x \in S$ such that $a = (ax)a$. Thus, we have

$$(\widetilde{\lambda} \circ \widetilde{\mu})^{-}(a) = (\widetilde{\lambda} \circ \widetilde{\mu})(a) \wedge \widetilde{0.5} = \vee_{a=yz}\{\widetilde{\lambda}(y) \wedge \widetilde{\mu}(z)\} \wedge \widetilde{0.5}$$

Now, $a = (ax)a = (ax)(ea) = (ae)(xa) = a(xa)$ because $(xe)S = xS$ for all $x \in S$. So,

$$\begin{aligned}(\widetilde{\lambda} \circ \widetilde{\mu})^{-}(a) &\geq \widetilde{\lambda}(a) \wedge \widetilde{\mu}(xa) \wedge \widetilde{0.5} \\ &\geq \widetilde{\lambda}(a) \wedge \{\widetilde{\mu}(a) \wedge \widetilde{0.5}\} \wedge \widetilde{0.5} \\ &= (\widetilde{\lambda} \wedge \widetilde{\mu})(a) \wedge \widetilde{0.5}\} \\ &= (\widetilde{\lambda} \wedge \widetilde{\mu})^{-}(a)\end{aligned}$$

Hence, $(\widetilde{\lambda} \circ \widetilde{\mu})^{-} \geq (\widetilde{\lambda} \wedge \widetilde{\mu})^{-}$.

(4)⇒(3) is obvious because every interval-valued $(\in, \in \vee q)$-fuzzy bi-ideal is an interval-valued $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S.

(3)⇒(2) is obvious because every interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal is an interval-valued $(\in, \in \vee q)$-fuzzy bi-ideal of S.

(2)⇒(1)

Let $\widetilde{\lambda}$ be an interval-valued $(\in, \in \vee q)$-fuzzy right ideal and $\widetilde{\mu}$ be an interval-valued $(\in, \in \vee q)$-fuzzy left ideal of S. Since every interval-valued $(\in, \in \vee q)$-fuzzy right ideal of S is an interval-valued $(\in, \in \vee q)$-fuzzy quasi-ideal of S. So, $(\widetilde{\lambda} \circ \widetilde{\mu})^{-} \geq (\widetilde{\lambda} \wedge \widetilde{\mu}\})^{-}$. Now,

$$\begin{aligned}(\widetilde{\lambda} \circ \widetilde{\mu})^{-}(a) &= (\widetilde{\lambda} \circ \widetilde{\mu})(a) \wedge \widetilde{0.5} \\ &= \vee_{a=yz}\{\widetilde{\lambda}(y) \wedge \widetilde{\mu}(z)\} \wedge \widetilde{0.5} \\ &= \vee_{a=yz}\{\widetilde{\lambda}(y) \wedge \widetilde{\mu}(z) \wedge \widetilde{0.5}\} \\ &= \vee_{a=yz}\{\widetilde{\lambda}(y) \wedge \widetilde{0.5}\} \wedge \{\widetilde{\mu}(z) \wedge \widetilde{0.5}\} \wedge \widetilde{0.5}\} \\ &\leq \vee_{a=yz}\{\widetilde{\lambda}(yz) \wedge \widetilde{0.5}\} \wedge \{\widetilde{\mu}(yz) \wedge \widetilde{0.5}\} \\ &= \{\widetilde{\lambda}(a) \wedge \widetilde{\mu}(a)\} \wedge \widetilde{0.5} \\ &= (\widetilde{\lambda} \wedge \widetilde{\mu})(a) \wedge \widetilde{0.5} \\ &= (\widetilde{\lambda} \wedge \widetilde{\mu})^{-}(a)\end{aligned}$$

So, $(\widetilde{\lambda} \circ \widetilde{\mu})^{-} \leq (\widetilde{\lambda} \wedge \widetilde{\mu})^{-}$. Hence, $(\widetilde{\lambda} \circ \widetilde{\mu})^{-} = (\widetilde{\lambda} \wedge \widetilde{\mu})^{-}$ for every interval-valued $(\in, \in \vee q)$-fuzzy right ideal $\widetilde{\lambda}$ of S and every interval-valued $(\in, \in \vee q)$-fuzzy left ideal $\widetilde{\mu}$ of S. Thus, by Theorem 15, S is regular. □

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