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# PERFORMANCE COMPARISON OF PRECONDITIONED ITERATIVE METHODS WITH DIRECT PRECONDITIONERS<sup>†</sup>

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ABSTRACT. In this paper, we first provide comparison results of preconditioned AOR methods with direct preconditioners  $I + \beta L$ ,  $I + \beta U$  and  $I + \beta (L + U)$  for solving a linear system whose coefficient matrix is a large sparse irreducible L-matrix, where  $\beta > 0$ . Next we propose how to find a near optimal parameter  $\beta$  for which Krylov subspace method with these direct preconditioners performs nearly best. Lastly numerical experiments are provided to compare the performance of preconditioned iterative methods and to illustrate the theoretical results.

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### 1. Introduction

In this paper, we consider the following nonsingular linear system

$$Ax = b, \quad x, b \in \mathbb{R}^n, \tag{1}$$

where  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  is a large sparse irreducible L-matrix. Throughout the paper, we assume that A = I - L - U, where I is the identity matrix, and L and U are strictly lower triangular and strictly upper triangular matrices, respectively. Then the AOR iterative method [3] for solving the linear system (1) can be expressed as

$$x_{k+1} = T_{r\omega} x_k + M_{r\omega} b, \ k = 0, 1, \dots,$$
<sup>(2)</sup>

where  $x_0$  is an initial vector,  $\omega$  and r are real parameters with  $\omega \neq 0$ ,

$$T_{r\omega} = (I - rL)^{-1} ((1 - \omega)I + (\omega - r)L + \omega U),$$
(3)

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and  $M_{r\omega} = \omega (I - rL)^{-1}$ . The  $T_{r\omega}$  is called an iteration matrix of the AOR iterative method.

In order to accelerate the convergence of iterative method for solving the linear system (1), the original linear system (1) is transformed into the following preconditioned linear system

$$PAx = Pb, (4)$$

where P, called a *preconditioner*, is a nonsingular matrix. Various types of preconditioners P, which are nonnegative matrices with unit diagonal entries, have been proposed by many researchers [2, 4, 5, 6, 7, 9, 11, 14, 15, 17]. In this paper, we study comparison results of preconditioned iterative methods corresponding to direct preconditioners  $P = P_l = I + \beta L$ ,  $P = P_u = I + \beta U$ and  $P = P_b = I + \beta (L + U)$ , where  $\beta$  is a positive real number. Here, direct preconditioner means that the preconditioner can be constructed without any computational step. The preconditioner  $P_u$  was first introduced by Kotakemori et al [4], and it has been studied further in [9, 11, 15]. The preconditioner  $P_l$  for  $\beta = 1$  was first proposed by Usui et al [11], so it is worth studying further  $P_l$  for  $\beta > 0$  in this paper.

Let  $A_u = P_u A$  and  $UL = \Gamma + E + F$ , where  $\Gamma$  is a diagonal matrix, E is a strictly lower triangular matrix, and F is a strictly upper triangular matrix. Then, one obtains

$$A_u = (I + \beta U)(I - L - U) = D_u - L_u - U_u,$$
(5)

where  $D_u = I - \beta \Gamma$ ,  $L_u = L + \beta E$ , and  $U_u = (1 - \beta)U + \beta U^2 + \beta F$ .

Let  $A_l = P_l A$  and  $LU = \Gamma_1 + E_1 + F_1$ , where  $\Gamma_1$  is a diagonal matrix,  $E_1$  is a strictly lower triangular matrix, and  $F_1$  is a strictly upper triangular matrix. Then, one obtains

$$A_{l} = (I + \beta L)(I - L - U) = D_{l} - L_{l} - U_{l},$$
(6)

where  $D_l = I - \beta \Gamma_1$ ,  $L_l = (1 - \beta)L + \beta L^2 + \beta E_1$ , and  $U_l = U + \beta F_1$ .

Recently, Wang and Song [14] studied convergence of the preconditioned AOR method with preconditioner  $P_b = I + \beta(L+U)$ , where  $0 < \beta \leq 1$ . Let  $A_b = P_b A$ . Then, one obtains

$$A_b = (I + \beta (L + U))(I - L - U) = D_b - L_b - U_b,$$
(7)

where  $D_b = I - \beta \Gamma - \beta \Gamma_1$ ,  $L_b = (1 - \beta)L + \beta L^2 + \beta E + \beta E_1$ , and  $U_b = (1 - \beta)U + \beta U^2 + \beta F + \beta F_1$ .

If we apply the AOR iterative method to the preconditioned linear system (4), then we get the *preconditioned AOR iterative method* whose iteration matrix is

$$T_{u,r,\omega} = (D_u - rL_u)^{-1}((1 - \omega)D_u + (\omega - r)L_u + \omega U_u) \text{ if } P = P_u$$
  

$$T_{l,r,\omega} = (D_l - rL_l)^{-1}((1 - \omega)D_l + (\omega - r)L_l + \omega U_l) \text{ if } P = P_l \qquad (8)$$
  

$$T_{b,r,\omega} = (D_b - rL_b)^{-1}((1 - \omega)D_b + (\omega - r)L_b + \omega U_b) \text{ if } P = P_b.$$

Notice that the computational costs for constructing  $A_u$ ,  $A_l$  and  $A_b$  will not be expensive since A is assumed to be a large sparse matrix.

The purpose of this paper is to provide performance comparison of preconditioned iterative methods with direct preconditioners  $P_l$ ,  $P_u$  and  $P_b$  for solving a linear system whose coefficient matrix is a large sparse irreducible *L*-matrix satisfying some weaker conditions than those used in the existing literature. This paper is organized as follows. In Section 2, we present some notation, definitions and preliminary results. In Section 3, we provide comparison results of preconditioned AOR methods with preconditioner  $P_u$ . In Section 4, we provide comparison results of preconditioned AOR methods with preconditioners  $P_l$  and  $P_b$ . In Section 5, we propose how to find a near optimal parameter  $\beta$  for which Krylov subspace method with the direct preconditioners  $P_l$ ,  $P_u$  and  $P_b$  performs nearly best. In Section 6, numerical experiments are provided to compare the performance of preconditioned iterative methods and to illustrate the theoretical results in Sections 3 to 5. Lastly, some conclusions are drawn.

### 2. Preliminaries

A matrix  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  is called a *Z*-matrix if  $a_{ij} \leq 0$  for  $i \neq j$ , an *L*-matrix if *A* is a Z-matrix and  $a_{ii} > 0$  for i = 1, 2, ..., n, and an *M*-matrix if *A* is a Z-matrix and  $A^{-1} \geq 0$ . For a vector  $x \in \mathbb{R}^n$ ,  $x \geq 0$  (x > 0) denotes that all components of *x* are nonnegative (positive). For two vectors  $x, y \in \mathbb{R}^n, x \geq y$  (x > y) means that  $x - y \geq 0$  (x - y > 0). These definitions carry immediately over to matrices. For a square matrix *A*,  $\rho(A)$  denotes the spectral radius of *A*, and *A* is called *irreducible* if the directed graph of *A* is strongly connected [13]. Some useful results which we refer to later are provided below.

**Theorem 2.1** (Varga [13]). Let  $A \ge 0$  be an irreducible matrix. Then

- (a) A has a positive eigenvalue equal to  $\rho(A)$ .
- (b) A has an eigenvector x > 0 corresponding to  $\rho(A)$ .
- (c)  $\rho(A)$  is a simple eigenvalue of A.

**Theorem 2.2** (Berman and Plemmons [1]). Let  $A \ge 0$  be a matrix. Then the following hold.

- (a) If  $Ax \ge \beta x$  for a vector  $x \ge 0$  and  $x \ne 0$ , then  $\rho(A) \ge \beta$ .
- (b) If Ax ≤ γx for a vector x > 0, then ρ(A) ≤ γ. Moreover, if A is irreducible and if βx ≤ Ax ≤ γx, equality excluded, for a vector x ≥ 0 and x ≠ 0, then β < ρ(A) < γ and x > 0.
- (c) If  $\beta x < Ax < \gamma x$  for a vector x > 0, then  $\beta < \rho(A) < \gamma$ .

**Theorem 2.3** (Li and Sun [6]). Let A be an irreducible matrix, and let A = M - N be an M-splitting of A with  $N \neq 0$ . Then there exists a vector x > 0 such that  $M^{-1}Nx = \rho(M^{-1}N)x$  and  $\rho(M^{-1}N) > 0$ .

#### 3. Comparison results for preconditioner $P_u$

We first provide a comparison result for the preconditioned Gauss-Seidel method with the preconditioner  $P_u$ .

**Theorem 3.1.** Let  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  be an irreducible L-matrix. Suppose that  $0 < (UL)_{ii} < 1$  for all  $1 \leq i \leq n-1$ , where  $(UL)_{ii} = \sum_{j=i+1}^{n} a_{ij}a_{ji}$ . Let  $T = (I-L)^{-1}U$  and  $T_u = (D_u - L_u)^{-1}U_u$ , where  $D_u$ ,  $L_u$  and  $U_u$  are defined by (5). If  $0 < \beta \leq 1$ , then

- (a)  $\rho(T_u) < \rho(T)$  if  $\rho(T) < 1$ . (b)  $\rho(T_u) = \rho(T)$  if  $\rho(T) = 1$ .
- (c)  $\rho(T_u) > \rho(T)$  if  $\rho(T) > 1$ .

*Proof.* Since A is an L-matrix, D, L and U are nonnegative. Since  $0 < \beta \leq 1$  and  $(UL)_{ii} < 1$ ,  $D_u$ ,  $L_u$  and  $U_u$  are nonnegative and thus  $T_u \geq 0$ . Since A = (I - L) - U is an M-splitting of A and  $U \neq 0$ , from Theorem 2.3 there exists a vector x > 0 such that  $Tx = \lambda x$ , where  $\lambda = \rho(T)$ . From  $Tx = \lambda x$ , one easily obtains

$$U^2 x = \lambda (U - UL)x. \tag{9}$$

Using (5) and (9),

$$T_{u}x - \lambda x = (D_{u} - L_{u})^{-1} (U_{u} - \lambda(D_{u} - L_{u})) x$$
  

$$= \beta (D_{u} - L_{u})^{-1} (U^{2} - U + F + \lambda(\Gamma + E)) x$$
  

$$= \beta (D_{u} - L_{u})^{-1} (\lambda(U - UL) - U + F + \lambda(UL - F)) x \qquad (10)$$
  

$$= \beta (\lambda - 1) (D_{u} - L_{u})^{-1} (U - F) x$$
  

$$= \beta (\lambda - 1) (D_{u} - L_{u})^{-1} (\Gamma + E + \lambda^{-1} U^{2}) x.$$

Let  $y = (\Gamma + E + \lambda^{-1}U^2)x$ . Since  $(UL)_{ii} > 0$  for all  $1 \le i \le n-1$ ,  $y \ge 0$  is a nonzero vector whose first (n-1) components are positive and last component is zero. Since A is irreducible,  $L_u = L + \beta E \ge 0$  is a strictly lower triangular matrix whose last row vector is nonzero and thus  $(D_u - L_u)^{-1}y$  is a positive vector. It follows that  $T_u x < \lambda x$  from (10) if  $\lambda < 1$ . From Theorem 2.2,  $\rho(T_u) < \rho(T) < 1$ . For the cases of  $\lambda = 1$  and  $\lambda > 1$ ,  $T_u x = \lambda x$  and  $T_u x > \lambda x$  are obtained from (10), respectively. Hence, the theorem follows from Theorem 2.2.

We now provide a comparison result for the preconditioned AOR method with the preconditioner  $P_u$ .

**Theorem 3.2.** Let  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  be an irreducible L-matrix. Suppose that  $0 < r \le \omega \le 1$   $(r \ne 1)$  and  $0 < (UL)_{ii} < 1$  for all  $1 \le i \le n - 1$ , where  $(UL)_{ii} = \sum_{j=i+1}^{n} a_{ij}a_{ji}$ . Let  $T_{r,\omega}$  and  $T_{u,r,\omega}$  be defined by (3) and (8), respectively. If  $0 < \beta \le 1$ , then

(a)  $\rho(T_{u,r,\omega}) < \rho(T_{r,\omega})$  if  $\rho(T_{r,\omega}) < 1$ . (b)  $\rho(T_{u,r,\omega}) = \rho(T_{r,\omega})$  if  $\rho(T_{r,\omega}) = 1$ .

(c)  $\rho(T_{u,r,\omega}) > \rho(T_{r,\omega})$  if  $\rho(T_{r,\omega}) > 1$ .

*Proof.* Notice that  $T_{r,\omega}$  can be expressed as  $T_{r,\omega} = (1-\omega)I + \omega(1-r)L + \omega U + H$ , where H is a nonnegative matrix. By assumptions, it can be seen that  $T_{r,\omega} \ge 0$ 

is irreducible and  $T_{u,r,\omega} \ge 0$ . From Theorem 2.1, there exists a vector x > 0 such that  $T_{r,\omega}x = \lambda x$ , where  $\lambda = \rho(T_{r,\omega})$ . From  $T_{r,\omega}x = \lambda x$ , one easily obtains

$$\omega U^2 x = \left( (\lambda + \omega - 1)U + (r - \omega - \lambda r)UL \right) x.$$
(11)

Using (5) and (11),

$$T_{u,r,\omega}x - \lambda x = (D_u - rL_u)^{-1} \left( (1-\omega)D_u + (\omega - r)L_u + \omega U_u - \lambda(D_u - rL_u) \right) x$$
  

$$= \beta (D_u - rL_u)^{-1} \left( (\omega + \lambda - 1)\Gamma + (\omega - r + \lambda r)E - \omega U + \omega U^2 + \omega F \right) x$$
  

$$= \beta (D_u - rL_u)^{-1} \left( (r + \lambda - 1 - \lambda r)\Gamma + (\lambda - 1)U + r(1 - \lambda)F \right) x$$
  

$$= \beta (\lambda - 1)(D_u - rL_u)^{-1} ((1 - r)\Gamma + U - rF)x$$
  

$$= \beta (\lambda - 1)(D_u - rL_u)^{-1} (\Gamma + rE)$$
  

$$+ \lambda^{-1} ((1 - \omega)U + (\omega - r)UL + \omega U^2) ) x.$$
  
(12)

Let  $y = (\Gamma + rE + \lambda^{-1}((1 - \omega)U + (\omega - r)UL + \omega U^2))x$ . Since  $0 < r \le \omega \le 1$ and  $(UL)_{ii} > 0$  for all  $1 \le i \le n-1$ ,  $y \ge 0$  is a nonzero vector whose first (n-1)components are positive and last component is zero. Since A is irreducible and  $r \ne 0$ ,  $L_u = L + \beta E \ge 0$  is a strictly lower triangular matrix whose last row vector is nonzero and thus  $(D_u - rL_u)^{-1}y$  is a positive vector. It follows that  $T_{u,r,\omega}x < \lambda x$  from (12) if  $\lambda < 1$ . From Theorem 2.2,  $\rho(T_{u,r,\omega}) < \rho(T_{r,\omega}) < 1$ . For the cases of  $\lambda = 1$  and  $\lambda > 1$ ,  $T_{u,r,\omega}x = \lambda x$  and  $T_{u,r,\omega}x > \lambda x$  are obtained from (12), respectively. Hence, the theorem follows from Theorem 2.2.

If  $0 < \beta < 1$  in Theorem 3.2, the assumptions for  $(UL)_{ii}$  can be weakened.

**Theorem 3.3.** Let  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  be an irreducible L-matrix. Suppose that  $0 < r \le \omega \le 1$   $(r \ne 1)$ ,  $(UL)_{ii} < 1$   $(1 \le i \le n - 1)$  and  $(UL)_{ii} > 0$  for at least one *i*, where  $(UL)_{ii} = \sum_{j=i+1}^{n} a_{ij}a_{ji}$ . Let  $T_{r,\omega}$  and  $T_{u,r,\omega}$  be defined by (3) and (8), respectively. If  $0 < \beta < 1$ , then

- (a)  $\rho(T_{u,r,\omega}) < \rho(T_{r,\omega})$  if  $\rho(T_{r,\omega}) < 1$ . (b)  $\rho(T_{u,r,\omega}) = \rho(T_{r,\omega})$  if  $\rho(T_{r,\omega}) = 1$ .
- (c)  $\rho(T_{u,r,\omega}) > \rho(T_{r,\omega})$  if  $\rho(T_{r,\omega}) > 1$ .

*Proof.* Since A is irreducible and  $\beta < 1$ ,  $A_u$  is also irreducible. Hence it can be easily shown that both  $T_{r,\omega}$  and  $T_{u,r,\omega}$  are nonnegative and irreducible. From Theorem 2.1, there exists a vector x > 0 such that  $T_{r,\omega}x = \lambda x$ , where  $\lambda = \rho(T_{r,\omega})$ . From equation (12), one obtains

$$T_{u,r,\omega}x - \lambda x = \beta(\lambda - 1)(D_u - rL_u)^{-1}(\Gamma + rE + \lambda^{-1}((1 - \omega)U + (\omega - r)UL + \omega U^2))x.$$
(13)

If  $\lambda < 1$ , then from (13)  $T_{u,r,\omega}x \leq \lambda x$  and  $T_{u,r,\omega}x \neq \lambda x$ . Since  $T_{u,r,\omega}$  is irreducible, Theorem 2.2 implies that  $\rho(T_{u,r,\omega}) < \rho(T_{r,\omega}) < 1$ . For the cases of  $\lambda = 1$  and  $\lambda > 1$ ,  $T_{u,r,\omega}x = \lambda x$  and  $T_{u,r,\omega}x \geq \lambda x$  (with  $T_{u,r,\omega}x \neq \lambda x$ ) are obtained from (13), respectively. Hence, the theorem follows from Theorem 2.2.

By combining Theorems 3.1 and 3.2, we can obtain the following theorem.

**Theorem 3.4.** Let  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  be an irreducible L-matrix. Suppose that  $0 < r \le \omega \le 1$  and  $0 < (UL)_{ii} < 1$  for all  $1 \le i \le n - 1$ , where  $(UL)_{ii} = \sum_{j=i+1}^{n} a_{ij}a_{ji}$ . Let  $T_{r,\omega}$  and  $T_{u,r,\omega}$  be defined by (3) and (8), respectively. If  $0 < \beta \le 1$ , then

(a)  $\rho(T_{u,r,\omega}) < \rho(T_{r,\omega})$  if  $\rho(T_{r,\omega}) < 1$ . (b)  $\rho(T_{u,r,\omega}) = \rho(T_{r,\omega})$  if  $\rho(T_{r,\omega}) = 1$ . (c)  $\rho(T_{u,r,\omega}) > \rho(T_{r,\omega})$  if  $\rho(T_{r,\omega}) > 1$ .

*Proof.* If r = 1, then  $\omega = 1$  and thus the theorem follows from Theorem 3.1. If  $r \neq 1$ , then the theorem follows from Theorem 3.2.

# 4. Comparison results for preconditioner $P_l$ and $P_b$

We first provide a comparison result for the preconditioned Gauss-Seidel method with the preconditioner  $P_l$ .

**Theorem 4.1.** Let  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  be an irreducible L-matrix. Suppose that  $0 < (LU)_{ii} < 1$  for all  $2 \le i \le n$ , where  $(LU)_{ii} = \sum_{j=1}^{i-1} a_{ij}a_{ji}$ . Let  $T = (I - L)^{-1}U$  and  $T_l = (D_l - L_l)^{-1}U_l$ , where  $D_l$ ,  $L_l$  and  $U_l$  are defined by (6). If  $0 < \beta \le 1$ , then

- (a)  $\rho(T_l) < \rho(T)$  if  $\rho(T) < 1$ .
- (b)  $\rho(T_l) = \rho(T)$  if  $\rho(T) = 1$ .
- (c)  $\rho(T_l) > \rho(T)$  if  $\rho(T) > 1$ .

*Proof.* Since A is an L-matrix, D, L and U are nonnegative. Since  $0 < \beta \leq 1$  and  $(LU)_{ii} < 1$ ,  $D_l$ ,  $L_l$  and  $U_l$  are nonnegative and thus  $T_l \geq 0$ . Since A = (I-L)-U is an M-splitting of A and  $U \neq 0$ , from Theorem 2.3 there exists a vector x > 0 such that  $Tx = \lambda x$ , where  $\lambda = \rho(T)$ . From  $Tx = \lambda x$ , one easily obtains

$$Ux = \lambda (I - L)x$$
  

$$LUx = \lambda (L - L^2)x$$
(14)

Using (6) and (14),

$$T_{l}x - \lambda x = (D_{l} - L_{l})^{-1} (U_{l} - \lambda(D_{l} - L_{l})) x$$
  

$$= \beta(D_{l} - L_{l})^{-1} (F_{1} + \lambda\Gamma_{1} - \lambda L + \lambda L^{2} + \lambda E_{1}) x$$
  

$$= \beta(D_{l} - L_{l})^{-1} (F_{1} + \lambda\Gamma_{1} - LU + \lambda E_{1}) x$$
  

$$= \beta(D_{l} - L_{l})^{-1} (\lambda\Gamma_{1} - \Gamma_{1} + \lambda E_{1} - E_{1}) x$$
  

$$= \beta(\lambda - 1)(D_{l} - L_{l})^{-1} (\Gamma_{1} + E_{1}) x.$$
  
(15)

Let  $y = (\Gamma_1 + E_1)x$ . Since  $(LU)_{ii} > 0$  for all  $2 \le i \le n, y \ge 0$  is a nonzero vector whose first component is zero and last (n-1) components are positive. Thus  $z = (D_l - L_l)^{-1}y \ge 0$  is also a nonzero vector whose first component is zero and last (n-1) components are positive. Let

$$T_l = \begin{pmatrix} 0 & T_{12} \\ 0 & T_{22} \end{pmatrix}, \ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } z = \begin{pmatrix} 0 \\ z_2 \end{pmatrix}, \tag{16}$$

where  $z_2 > 0$ ,  $T_{12} \in \mathbb{R}^{1 \times (n-1)}$  and  $T_{12} \in \mathbb{R}^{(n-1) \times (n-1)}$ . From (15) and (16),

$$T_{12}x_2 - \lambda x_1 = 0$$
  

$$T_{22}x_2 - \lambda x_2 = \beta(\lambda - 1)z_2.$$
(17)

Since  $z_2 > 0$ ,  $T_{22}x_2 < \lambda x_2$  from (17) if  $\lambda < 1$ . From Theorem 2.2,  $\rho(T_{22}) < \rho(T) < 1$ . Since  $\rho(T_l) = \rho(T_{22})$ ,  $\rho(T_l) < \rho(T)$  is obtained. For the cases of  $\lambda = 1$  and  $\lambda > 1$ ,  $T_{22}x_2 = \lambda x_2$  and  $T_{22}x_2 > \lambda x_2$  are obtained from (17), respectively. Hence, the theorem follows from Theorem 2.2.

We now provide a comparison result for the preconditioned AOR method with the preconditioner  $P_l$ .

**Theorem 4.2.** Let  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  be an irreducible L-matrix. Suppose that  $0 \leq r, \omega \leq 1$  ( $\omega \neq 0, r \neq 1$ ) and  $(LU)_{ii} < 1$  for all  $2 \leq i \leq n$ , where  $(LU)_{ii} = \sum_{j=1}^{i-1} a_{ij}a_{ji}$ . Let  $T_{r,\omega}$  and  $T_{l,r,\omega}$  be defined by (3) and (8), respectively. If  $0 < \beta < 1$ , then

(a)  $\rho(T_{l,r,\omega}) < \rho(T_{r,\omega})$  if  $\rho(T_{r,\omega}) < 1$ . (b)  $\rho(T_{l,r,\omega}) = \rho(T_{r,\omega})$  if  $\rho(T_{r,\omega}) = 1$ . (c)  $\rho(T_{l,r,\omega}) > \rho(T_{r,\omega})$  if  $\rho(T_{r,\omega}) > 1$ .

*Proof.* By simple calculation, one can obtain

$$T_{r,\omega} = (1-\omega)I + \omega(I-rL)^{-1}((1-r)L+U)$$
  
=  $(1-\omega)I + \omega((1-r)L+U+H)$   
$$T_{l,r,\omega} = (1-\omega)I + \omega(I-rD_l^{-1}L_l)^{-1}((1-r)D_l^{-1}L_l+D_l^{-1}U_l)$$
  
=  $(1-\omega)I + \omega((1-r)D_l^{-1}L_l+D_l^{-1}U_l+H_l),$  (18)

where H and  $H_l$  are nonnegative matrices. Since  $0 \leq r$ ,  $\omega \leq 1$  ( $\omega \neq 0, r \neq 1$ ) and  $0 < \beta < 1$ , it can be easily shown from (18) that both  $T_{r,\omega}$  and  $T_{l,r,\omega}$ are nonnegative and irreducible. Hence, there exists a vector x > 0 such that  $T_{r,\omega}x = \lambda x$ , where  $\lambda = \rho(T_{r,\omega})$ . Using (6) and  $T_{r,\omega}x = \lambda x$ , one obtains

$$T_{l,r,\omega}x - \lambda x = (D_l - rL_l)^{-1} \left( (1 - \omega)D_l + (\omega - r)L_l + \omega U_l - \lambda(D_l - rL_l) \right) x$$
  
=  $\beta (D_l - rL_l)^{-1} \left( (\omega + \lambda - 1)\Gamma_1 + (\omega - r + \lambda r)(-L + L^2 + E_1) + \omega F_1 \right) x$  (19)  
=  $\beta (\lambda - 1)(D_l - rL_l)^{-1} (\Gamma_1 + rE_1 + (1 - r)L) x.$ 

Let  $y = (\Gamma_1 + rE_1 + (1 - r)L)x$ . Since A is irreducible and  $r \neq 1, y \geq 0$ is a nonzero vector whose first component is zero. If  $\lambda < 1$ , then from (19)  $T_{l,r,\omega}x \leq \lambda x$  and  $T_{l,r,\omega}x \neq \lambda x$ . Since  $T_{l,r,\omega}$  is irreducible, Theorem 2.2 implies that  $\rho(T_{l,r,\omega}) < \rho(T_{r,\omega}) < 1$ . For the cases of  $\lambda = 1$  and  $\lambda > 1, T_{l,r,\omega}x = \lambda x$  and  $T_{l,r,\omega}x \geq \lambda x$  (with  $T_{l,r,\omega}x \neq \lambda x$ ) are obtained from (13), respectively. Hence, the theorem follows from Theorem 2.2.

Notice that Theorem 4.2 for preconditioner  $P_l$  does not require the assumptions  $r \leq \omega$  and  $(LU)_{ii} > 0$  as compared with Theorem 3.2 for preconditioner  $P_u$ . If  $\beta = 1$ , the strict inequalities in Theorem 4.2 may not hold and only inequalities are guaranteed.

**Lemma 4.3.** Let  $A = (a_{ij}) \in \mathbb{R}^{n \times n}$  be an L-matrix. If A = I - L - U is an M-matrix, then  $(LU)_{ii} < 1$  and  $(UL)_{ii} < 1$  for all i = 1, 2, ..., n.

*Proof.* It is easy to show that (I + U)A and (I + L)A are Z-matrices. Since A is an M-matrix, there exists a vector x > 0 such that Ax > 0. Thus (I + L)Ax > 0and (I + U)Ax > 0. It follows that (I + L)A and (I + U)A are M-matrices and so all diagonal components of (I + L)A and (I + U)A are positive. Hence  $1 - (LU)_{ii} > 0$  and  $1 - (UL)_{ii} > 0$  for all i = 1, 2, ..., n, which completes the proof.

From Lemma 4.3, it can be seen that if A is an M-matrix, all theorems in Sections 3 and 4 do not require the assumptions  $(LU)_{ii} < 1$  or  $(UL)_{ii} < 1$ . Lastly, we provide a comparison result of the preconditioned AOR method for preconditioner  $P_b$ .

**Lemma 4.4** ([8]). Suppose that  $A_1 = M_1 - N_1$  and  $A_2 = M_2 - N_2$  are weak regular splittings of the monotone matrices  $A_1$  and  $A_2$ , respectively, such that  $M_2^{-1} \ge M_1^{-1}$ . If there exists a positive vector x such that  $0 \le A_1 x \le A_2 x$ , then for the monotonic norm associated with x

$$|M_2^{-1}N_2||_x \leq ||M_1^{-1}N_1||_x.$$

In particular, if  $M_1^{-1}N_1$  has a positive Perron vector, then

ŀ

$$\rho(M_2^{-1}N_2) \le \rho(M_1^{-1}N_1).$$

**Theorem 4.5.** Let A be an irreducible M-matrix. Suppose that  $0 \le r \le \omega \le 1$  ( $\omega \ne 0$ ). Let  $T_{r,\omega}$ ,  $T_{l,r,\omega}$  and  $T_{b,r,\omega}$  be defined by (3) and (8), respectively. If  $0 < \beta \le 1$ , then

$$\rho(T_{b,r,\omega}) \le \rho(T_{l,r,\omega}) \le \rho(T_{r,\omega}) < 1.$$

Proof. Since  $\rho(T_{l,r,\omega}) \leq \rho(T_{r,\omega}) < 1$  was shown in [14], it suffices to show  $\rho(T_{b,r,\omega}) \leq \rho(T_{l,r,\omega})$ . Since A is an M-matrix and  $0 < \beta \leq 1$ , there exists a vector x > 0 such that Ax > 0, and  $P_lA$  and  $P_bA$  are also Z-matrices. It follows that  $P_lA$  and  $P_bA$  are M-matrices. Notice that  $P_bAx \geq P_lAx > 0$  and  $(D_b - rL_b)^{-1} \geq (D_l - rL_l)^{-1}$ . Since A is irreducible and  $0 < \beta < 1$ ,  $P_lA$  is also irreducible. Since  $P_lA = \frac{1}{\omega}(D_l - rL_l) - \frac{1}{\omega}((1 - \omega)D_l + (\omega - r)L_l + \omega U_l)$  is an M-splitting of  $P_lA$  with  $U_l \neq 0$ ,  $T_{l,r,\omega}$  has a positive Perron vector from Theorem 2.3. Using Lemma 4.4,  $\rho(T_{b,r,\omega}) \leq \rho(T_{l,r,\omega})$  for  $0 < \beta < 1$ . By continuity of spectral radius,  $\rho(T_{b,r,\omega}) \leq \rho(T_{l,r,\omega})$  is also true for  $\beta = 1$ . Therefore, the proof is complete.

### 5. A near optimal parameter $\beta$ for Krylov subspace method

In this section, we propose how to find a near optimal parameter  $\beta$  for which Krylov subspace method with preconditioners  $P_l$ ,  $P_u$  and  $P_b$  performs nearly best. Before proceeding to the analysis for finding a near optimal parameter  $\beta$ , we define the following notations. For X and Y in  $\mathbb{R}^{n \times n}$ , we define the inner product  $\langle X, Y \rangle_F = \operatorname{tr}(X^T Y)$ , where  $\operatorname{tr}(Z)$  denotes the trace of the square matrix

Z and  $X^T$  denotes the transpose of the matrix X. The associated norm is the well-known Frobenius norm denoted by  $\|\cdot\|_F$ .

If P is a good preconditioner for Krylov subspace method, then PA is close to the identity matrix I. Thus it would be a good idea to determine a value  $\beta$  such that  $||PA - I||_F$  is minimized, where P is  $P_l$ ,  $P_u$  or  $P_b$ . We call such a value  $\beta$ a near optimal parameter for the preconditioned Krylov subspace method. We first provide a near optimal parameter  $\beta$  for the preconditioner  $P = P_l$ .

**Theorem 5.1.** Let A = I - L - U be a large sparse nonsingular matrix, and let  $P_l = I + \beta L$  be a preconditioner for Krylov subspace method. Then, a near optimal parameter  $\beta$  is given by

$$\beta = \frac{\operatorname{tr}((\mathrm{LA})^{\mathrm{T}}(\mathrm{L}+\mathrm{U}))}{\|LA\|_{F}^{2}}$$

*Proof.* Note that  $(I + \beta L)A - I = \beta LA - (L + U)$ . By definition of the Frobenius norm, one obtains

$$\|\beta LA - (L+U)\|_{F}^{2} = \operatorname{tr}\left((\beta LA - (L+U))^{T}(\beta LA - (L+U))\right)$$
  
=  $\beta^{2} \|LA\|_{F}^{2} - 2\beta \operatorname{tr}((LA)^{T}(L+U)) + \|L+U\|_{F}^{2}.$  (20)

The equation (20) is a quadratic equation in  $\beta$ , so the minimum occurs when

$$\beta = \frac{\operatorname{tr}((\operatorname{LA})^{\mathrm{T}}(\operatorname{L} + \operatorname{U}))}{\|LA\|_{F}^{2}}.$$

Hence the proof is complete.

Similarly to the proof of Theorem 5.1, one can obtain a near optimal parameter  $\beta$  for the preconditioner  $P = P_u$ .

**Theorem 5.2.** Let A = I - L - U be a large sparse nonsingular matrix, and let  $P_u = I + \beta U$  be a preconditioner for Krylov subspace method. Then, a near optimal parameter  $\beta$  is given by

$$\beta = \frac{\operatorname{tr}((\mathrm{UA})^{\mathrm{T}}(\mathrm{L}+\mathrm{U}))}{\|UA\|_{F}^{2}}$$

Lastly, we provide a near optimal parameter  $\beta$  for the preconditioner  $P = P_b$ .

**Theorem 5.3.** Let  $A = I - L - U \in \mathbb{R}^{n \times n}$  be a large sparse nonsingular matrix, and let  $P_b = I + \beta(L+U)$  be a preconditioner for Krylov subspace method. Then, a near optimal parameter  $\beta$  is given by

$$\beta = \frac{n}{n + \|L + U\|_F^2}.$$

*Proof.* Note that  $(I + \beta(L+U))A - I = (\beta - 1)(L+U) - \beta(L+U)^2$ . By property of the Frobenius norm, one obtains

$$\|(\beta - 1)(L + U) - \beta(L + U)^2\|_F = \|(L + U)((\beta - 1)I - \beta(L + U))\|_F \leq \|L + U\|_F \|(\beta - 1)I - \beta(L + U)\|_F.$$
(21)

From (21), it is sufficient to minimize  $\|(\beta-1)I-\beta(L+U)\|_F$  in order to determine a near optimal parameter  $\beta$ . Since tr(L+U) = 0, one obtains

$$\|(\beta - 1)I - \beta(L + U)\|_{F}^{2} = \operatorname{tr}\left(((\beta - 1)I - \beta(L + U))^{T}((\beta - 1)I - \beta(L + U))\right)$$
  
=  $(\beta - 1)^{2} n + \beta^{2} \|L + U\|_{F}^{2}$  (22)  
=  $(n + \|L + U\|_{F}^{2}) \beta^{2} - 2n\beta + n.$ 

The equation (22) is a quadratic equation in  $\beta$ , so the minimum occurs when

$$\beta = \frac{n}{n + \|L + U\|_F^2}.$$

Hence the proof is complete.

# 6. Numerical experiments

In this section, we provide numerical experiments to compare the performance of preconditioned iterative methods and to illustrate the theoretical results obtained in Sections 3 to 5. All numerical experiments are carried out on a PC equipped with Intel Core i5-4570 3.2GHz CPU and 8GB RAM using MATLAB R2013a. The preconditioned iterative methods used for numerical experiments are the preconditioned AOR method and the right preconditioned BiCGSTAB method [10, 12].

In Table 3, *Iter* denotes the number of iteration steps, *CPU* denotes the elapsed CPU time in seconds, *P* denotes the preconditioner to be used, ILU(0) denotes the incomplete factorization preconditioner without fill-ins, and  $\beta$  denotes a near optimal value computed by the formula given in Section 5. For numerical tests using the right preconditioned BiCGSTAB, all nonzero elements of sparse matrices are stored using sparse storage format which saves a lot of CPU time, the initial vectors are set to the zero vector, and the iterations are terminated if the current approximation  $x_k$  satisfies  $\frac{\|b-Ax_k\|_2}{\|b\|_2} < 10^{-10}$ , where  $\|\cdot\|_2$  refers to  $L_2$ -norm.

**Example 6.1.** Consider the two dimensional convection-diffusion equation [15]

$$-\Delta u + u_x + 2u_y = f(x, y) \text{ in } \Omega = (0, 1) \times (0, 1)$$
(23)

with the Dirichlet boundary condition on  $\partial\Omega$  which denotes the boundary of  $\Omega$ . When the central difference scheme on a uniform grid with  $m \times m$  interior node is applied to the discretization of the equation (23), we can obtain a linear system Ax = b whose coefficient matrix  $A \in \mathbb{R}^{n \times n}$  is of the form

$$A = I_m \otimes P_h + Q_h \otimes I_m,$$

where  $n = m^2$ ,  $I_m$  denotes the identity matrix of order m,  $\otimes$  denotes the Kronecker product,  $P_h = \operatorname{tridiag}(-\frac{2+h}{8}, 1, -\frac{2-h}{8})$  and  $Q_h = \operatorname{tridiag}(-\frac{1+h}{4}, 0, -\frac{1-h}{4})$  are  $m \times m$  tridiagonal matrices with the step size  $h = \frac{1}{m}$ . It is clear that this matrix A is an irreducible nonsymmetric L-matrix. Numerical results of the preconditioned AOR method for  $n = 50^2 = 2500$  are provided in Table 1, and

numerical results of the right preconditioned BiCGSTAB method for various values of n are listed in Table 3.

Example 6.2. Consider the two dimensional Poisson's equation

$$-\Delta u = f(x, y) \text{ in } \Omega = (0, 1) \times (0, 1)$$
(24)

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with the Dirichlet boundary condition on  $\partial\Omega$ . When the central difference scheme on a uniform grid with  $m \times m$  interior node is applied to the discretization of the equation (24), we obtain a linear system Ax = b whose coefficient matrix  $A \in \mathbb{R}^{n \times n}$  is given by

$$A = I_m \otimes P + Q \otimes I_m,$$

where  $n = m^2$ ,  $P = \text{tridiag}(-\frac{1}{4}, 1, -\frac{1}{4})$  and  $Q = \text{tridiag}(-\frac{1}{4}, 0, -\frac{1}{4})$  are  $m \times m$ tridiagonal matrices. Note that this matrix A is an irreducible symmetric Lmatrix. Numerical results of the preconditioned AOR method for  $n = 50^2 =$ 2500 are provided in Table 2.

In Figure 1, we depict the eigenvalue distributions of the preconditioned matrices corresponding to 6 different preconditioners for Examples 6.1 when  $n = 30^2$ . From Figure 1, it can be seen that eigenvalues of  $P_bA$  are more clustered than those of any other preconditioners. From Tables 1 and 2, it can be seen that  $\rho(T_{l,r,\omega}) < \rho(T_{r,\omega})$  does not hold for  $\omega > 1$  and r > 1, and  $\rho(T_{b,r,\omega}) \leq \rho(T_{l,r,\omega})$  does not hold for  $\omega > 1$  and r > 1, and  $\rho(T_{b,r,\omega}) \leq \rho(T_{l,r,\omega})$  does not hold for  $\beta > 1$ . For test problems used in this paper, the preconditioner  $P_u$  yields better optimal performance than other preconditioners  $P_l$  and  $P_b$ , and the optimal values of  $\beta$ ,  $\omega$  and r for the preconditioned AOR method with  $P_u$  are greater than or equal to 1 (see Tables 1 to 2). In other words,  $\omega = r$  is around 1.3 and  $\beta = 1$  for Examples 6.1 and 6.2. Further research is required to study how to find optimal or near optimal values of  $\beta$ ,  $\omega$  and r for the preconditioned AOR method.

From Table 3, it can be seen that the preconditioner  $P_b$  with a near optimal parameter  $\beta$  performs much better than the ILU(0) preconditioner which is one of the powerful preconditioners that are commonly used. It can be also seen that the preconditioners  $P_l$  and  $P_u$  with near optimal parameters  $\beta$  perform better than the preconditioner  $(I - L)^{-1}$ . The performance of BiCGSTAB with preconditioner  $(I - U)^{-1}$  is not provided here since its performance is similar to that with the preconditioner  $(I - L)^{-1}$ . Notice that a near optimal parameter  $\beta$  proposed in Section 5 can be easily computed by MATLAB.

### 7. Conclusions

In this paper, we provided comparison results of preconditioned AOR methods with direct preconditioners  $P_l$ ,  $P_u$  and  $P_b$  for solving a linear system whose coefficient matrix is a large sparse irreducible L-matrix, which holds under some weaker conditions than those used in the existing literature. These theoretical results are in good agreement with the numerical results (see Tables 1 and

β	ω	r	$\rho(T_{r,\omega})$	$\rho(T_{u,r,\omega})$	$\rho(T_{l,r,\omega})$	$\rho(T_{b,r,\omega})$
1	0.8	0.6	0.9977	0.9955	0.9965	0.9939
	0.8	0.8	0.9973	0.9943	0.9961	0.8928
	1.0	0.8	0.9966	0.9929	0.9952	0.9910
	1.0	1.0	0.9960	0.9903	0.9946	0.9893
	1.1	1.1	0.9951	0.9870	0.9955	0.9869
	1.2	1.2	0.9939	0.9818	1.2877	0.9839
	1.3	1.3	0.9925	0.9725	1.5853	0.9800
1.1	0.8	0.6	0.9977	0.9952	0.9964	1.3385
	0.8	0.8	0.9973	0.9939	0.9960	1.4084
	1.0	0.8	0.9966	0.9923	0.9950	1.5105
	1.0	1.0	0.9960	0.9894	0.9945	1.6288
	1.2	1.2	0.9939	0.9791	1.4964	1.9468
	1.3	1.3	0.9925	0.9666	1.8379	2.1549
0.9	0.8	0.6	0.9977	0.9958	0.9966	0.9944
	0.8	0.8	0.9973	0.9947	0.9962	0.9934
	1.0	1.0	0.9960	0.9912	0.9947	0.9901
	1.1	1.1	0.9951	0.9883	0.9938	0.9879
	1.2	1.2	0.9939	0.9841	1.0966	0.9852
	1.3	1.3	0.9925	0.9768	1.3541	0.9817

TABLE 1. Numerical results for  $\rho(T_{r,\omega})$ ,  $\rho(T_{u,r,\omega})$ ,  $\rho(T_{l,r,\omega})$  and  $\rho(T_{b,r,\omega})$  with various values of  $\beta$ , r and  $\omega$  for Example 6.1.

TABLE 2. Numerical results for  $\rho(T_{r,\omega})$ ,  $\rho(T_{u,r,\omega})$ ,  $\rho(T_{l,r,\omega})$  and  $\rho(T_{b,r,\omega})$  with various values of  $\beta$ , r and  $\omega$  for Example 6.2.

$\beta$	ω	r	$\rho(T_{r,\omega})$	$\rho(T_{u,r,\omega})$	$\rho(T_{l,r,\omega})$	$\rho(T_{b,r,\omega})$
1	0.8	0.6	0.9978	0.9958	0.9967	0.9942
	0.8	0.8	0.9975	0.9947	0.9964	0.9933
	1.0	0.8	0.9968	0.9933	0.9955	0.9916
	1.0	1.0	0.9962	0.9909	0.9949	0.9899
	1.1	1.1	0.9954	0.9878	0.9957	0.9877
	1.2	1.2	0.9943	0.9830	1.2880	0.9849
	1.3	1.3	0.9930	0.9742	1.5857	0.9813
1.1	0.8	0.6	0.9978	0.9955	0.9966	1.3390
	0.8	0.8	0.9975	0.9942	0.9963	1.4091
	1.0	0.8	0.9968	0.9928	0.9953	1.5114
	1.0	1.0	0.9962	0.9900	0.9948	1.6300
	1.2	1.2	0.9943	0.9804	1.4968	1.9485
	1.3	1.3	0.9930	0.9687	1.8384	2.1570
0.9	0.8	0.6	0.9978	0.9960	0.9968	0.9947
	0.8	0.8	0.9975	0.9950	0.9965	0.9938
	1.0	1.0	0.9962	0.9917	0.9951	0.9907
	1.1	1.1	0.9954	0.9891	0.9942	0.9887
	1.2	1.2	0.9943	0.9850	1.0968	0.9861
	1.3	1.3	0.9930	0.9782	1.3544	0.9828



-0.4

-0.6 -0.8 -1

0.5

0.4

0.3

0.2

0.1

-0.1

0.2 0.4

0.6 0.8

Spectra of (I+βU)A

1.2 1.4

1.4

0 0

-3L 0

0.5

0.4

0.3

0.2

0.1

-0.1

0.2

0.4

0.6 0.8

Spectra of (I+βL)A

-0.2 -0.2 -0.3 -0.3 -0.4 -0.4 -0.5 L -0.5 L 1.2 0.2 0.4 0.6 0.8 1 1.4 0.2 0.4 0.6 0.8 1.2

1.4

FIGURE 1. Spectra of the preconditioned matrices corresponding to several preconditioners for Example 6.1 when  $n=30^2$ 

2). However, further research is required to study how to find optimal or near optimal values of  $\beta$ ,  $\omega$  and r for the preconditioned AOR method.

P	n	Iter	CPU	$\beta$
	$128^{2}$	306	0.50	
$(I-L)^{-1}$	$250^{2}$	727	4.65	
	$400^{2}$	1529	31.2	
	$128^{2}$	95	0.20	
ILU(0)	$250^{2}$	192	1.45	
	$400^{2}$	273	6.38	
	$128^{2}$	166	0.12	0.8013
$P_b$	$250^{2}$	308	1.04	0.8006
	$400^{2}$	478	4.97	0.8004
	$128^{2}$	455	0.30	0.7286
$P_l$	$250^{2}$	794	2.53	0.7279
	$400^{2}$	2432	25.6	0.7277
	$128^{2}$	327	0.21	0.7286
$P_u$	$250^{2}$	1172	3.74	0.7279
	$400^{2}$	1926	20.3	0.7277

TABLE 3. Numerical results of preconditioned BiCGSTAB for Example 6.1.

We also proposed how to find a near optimal parameter  $\beta$  for which Krylov subspace method with preconditioners  $P_l$ ,  $P_u$  and  $P_b$  performs nearly best. Numerical experiments showed that BiCGSTAB with the preconditioner  $P_b$  with a near optimal parameter  $\beta$  performs much better than the ILU(0) preconditioner which is one of the powerful preconditioners that are commonly used. It was also seen that BiCGSTAB with the preconditioners  $P_l$  and  $P_u$  with near optimal parameters  $\beta$  perform better than the preconditioner  $(I - L)^{-1}$  (see Table 3). Notice that a near optimal parameter  $\beta$  proposed in Section 5 can be easily computed by MATLAB.

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