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OPTIMALITY AND DUALITY IN NONDIFFERENTIABLE MULTIOBJECTIVE FRACTIONAL PROGRAMMING USING α-UNIVEXITY

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Abstract. In this paper, a multiobjective nondifferentiable fractional programming problem (MFP) is considered where the objective function contains a term involving the support function of a compact convex set. A vector valued (generalized) α-univex function is defined to extend the concept of a real valued (generalized) α-univex function. Using these functions, sufficient optimality criteria are obtained for a feasible solution of (MFP) to be an efficient or weakly efficient solution of (MFP). Duality results are obtained for a Mond-Weir type dual under (generalized) α-univexity assumptions.

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1. Introduction

Most of the real world problems which arise in the areas of portfolio selection, stock cutting, game theory and many decision making problems in management science etc. are (multiobjective) fractional programming problems. Extensive researches have been reported in the literature for the multiobjective nonlinear (nondifferentiable) fractional programming problems involving generalized convex functions by various authors, for details see ([1,4,6-13,15,18-21]) and references therein. The areas which have been explored are mainly to weaken the convexity and to relax the differentiability assumption of the functions used in developing optimality and duality of the above programming problems. Bector et al.[3] introduced univex functions by relaxing the definition of an invex

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function and obtained optimality and duality results for a nonlinear programming problem. Jayswal[8] defined α-univexity and its generalizations for a real valued function and proved duality theorems for a nondifferentiable generalized fractional programming problem.

Different authors have used different forms of nondifferentiability to obtain optimality conditions and duality theory for fractional programming problem under generalized convexity assumptions. Authors like Mond [15], Singh [19], Zhang and Mond [21] and in the references cited therein considered a class of nondifferentiable fractional programming problems containing square root terms in the objective function and derived optimality criteria and discussed duality theory. Non smooth optimization involves functions for which subderivatives exist [5]. Square root of a positive semidefinite quadratic form is one of the few types of a nondifferentiable function whose subdifferential can be written explicitly. Square root of a quadratic form can be replaced by a more general function, namely, the support function of a compact convex set, whose subdifferential can be simply expressed. For these considerations Mond and Schechter[16] considered programs which contain support function in objective function and studied symmetric duality. Kim et al. [9] established necessary and sufficient optimality conditions and proved duality results for weakly efficient solutions of multiobjective fractional programming problem containing support functions under the assumption of (V, ρ) invex functions. Later in [10], Kim et al. established duality results using (V, ρ) invexity for the same problem with cone constraints.

Motivated by the above researches, in this paper, we consider a nondifferentiable multiobjective fractional programming problem (MFP) over cones with objective function containing support function of a compact convex set. We introduce the concept of α-univexity and its various generalizations for a vector valued function. This generalizes the concept of α-univexity for a scalar valued function[8]. We also give the examples to show the existence of above defined classes of functions. Sufficient optimality conditions for a (weakly) efficient solution of (MFP) are derived using these newly defined classes of (generalized) α-univex functions. A Mond-Weir type dual is proposed for (MFP) and standard duality theorems are proved assuming the functions to be (generalized) α-univex.

2. Preliminaries

Let R^n be the n-dimensional Euclidean space and let R^n_+ be its non negative orthant. The following convention for inequalities will be used in this paper. If $x, u \in R^n$, then

$$\begin{array}{lcl} x < u & \Leftrightarrow & u - x \in int\ R^n_+; \\ x \leqq u & \Leftrightarrow & u - x \in R^n_+; \\ x \leq u & \Leftrightarrow & u - x \in R^n_+/\{0\}; \\ x \not< u & \text{is the negation of} & x < u. \end{array}$$

Note : For $x, u \in R$, we use $x \leq u$ to denote x is less than or equal to u.

Definition 2.1 ([17])**.** A non empty set C in R^n is said to be a cone if for each $x \in C$ and $\lambda \geq 0$, $\lambda x \in C$. If in addition C is convex then C is called a convex cone.

Definition 2.2 ([17])**.** Let $C \subseteq R^n$ be a cone. The set

$$C^* = \{z \in R^n \mid x^T z \leq 0,\ \forall\, x \in C\},$$

is called the polar cone of C.

We now consider the following nondifferentiable multiobjective fractional programming problem:

$$\text{(MFP)} \quad \text{Minimize } F(x) = \left(\frac{f_1(x) + s(x|D_1)}{g_1(x)}, \ldots, \frac{f_k(x) + s(x|D_k)}{g_k(x)} \right)$$

$$\text{subject to} \qquad h(x) \in C_2^*\ ,\ x \in C_1,$$

where $f : X \to R^k, g : X \to R^k$ and $h : X \to R^m$ are continuously differentiable functions over an open subset X of R^n. $C_1 \subseteq X$ and C_2 are closed convex cones with non empty interiors in R^n and R^m respectively, $D_i (i = 1, 2, \ldots, k)$ are compact convex sets in R^n and $s(x|D_i) = max\{< x, y > |y \in D_i\}$ denotes the support function of D_i. Let $X_0 = \{x \in X : x \in C_1, h(x) \in C_2^*\}$ be the set of all feasible solutions of (MFP) and

$$f_i(x) + s(x|D_i)\ \geq\ 0\ ,\ g(x)\ >\ 0,\ \ \forall\, x \in X.$$

For any $w = (w_1, w_2, \ldots, w_k) \in R^n \times R^n \times \ldots \times R^n$ and $x \in R^n$, $x^T w = (x^T w_1, x^T w_2, \ldots, x^T w_k)$.

We now review some known facts about support functions. The support function $s(x|C)$ of compact convex set $C \subseteq R^n$, being convex and everywhere finite, has a subgradient at every x in the sense of Rockafellar[17], that is, there exists $z \in C$ such that

$$s(y|C) \geq s(x|C) + z^T(y - x),\ \ \forall\, y \in C.$$

Equivalently,

$$z^T x = s(x|C).$$

The subdifferential of $s(x|C)$ is given by

$$\partial s(x|C) = \{z \in C : z^T x = s(x|C)\}.$$

For any set $D \subseteq R^n$, the normal cone to D at any point $x \in D$ is defined by

$$N_D(x) = \{y \in R^n \mid y^T(z - x) \leq 0,\ \ \forall\, z \in D\}.$$

If C is a compact convex set then $y \in N_C(x)$ iff

$$s(y|C) = x^T y,$$

or equivalently $x \in \partial s(y|C)$.

Definition 2.3. A feasible solution $\overline{x} \in X_0$ is said to be a weakly efficient solution of (MFP) if there does not exist any $x \in X_0$ such that

$$F(x) < F(\overline{x}).$$

Definition 2.4. A feasible solution $\overline{x} \in X_0$ is said to be an efficient solution of (MFP) if there does not exist any $x \in X_0$ such that

$$F(x) \leq F(\overline{x}).$$

We recall the definition of α-univexity for a differentiable real valued function $f : X \to R$.

Definition 2.5 ([8])**.** The function f is said to be **α-univex** at $\overline{x} \in X$ with respect to $\alpha : X \times X \to R_+ \setminus \{0\}$, $b : X \times X \to R_+$, $\phi : R \to R$ and $\eta : X \times X \to R^n$ if for every $x \in X$, we have

$$b(x, \overline{x})\ \phi\ (f(x) - f(\overline{x}))\ \ \geq\ < \alpha(x, \overline{x}) \bigtriangledown f(\overline{x}), \eta(x, \overline{x}) > .$$

Now to extend the above concept of α-univexity to multiobjective programming we give the following definitions for a vector valued differentiable function $f : X \to R^k$. Assume that $\alpha : X \times X \to R_+ \setminus \{0\}$, $b : X \times X \to R^k_+$, $\phi : R \to R$ and $\eta : X \times X \to R^n$.

Definition 2.6. The function $f : X \to R^k$ is said to be **α-univex** at $\overline{x} \in X$ with respect to α, b, ϕ, and η if for every $x \in X$ and for each $i = 1, 2, \ldots, k$, we have

$$b_i(x, \overline{x})\ \phi\ (f_i(x) - f_i(\overline{x}))\ \ \geq\ < \alpha(x, \overline{x}) \bigtriangledown f_i(\overline{x}), \eta(x, \overline{x}) > . \tag{2.1}$$

If (2.1) is a strict inequality for all $x \neq \overline{x}$, then f is said to be strict α-univex function.

Definition 2.7. The function $f : X \to R^k$ is said to be **pseudo α-univex** at $\overline{x} \in X$ with respect to α, b, ϕ, and η if for every $x \in X$ and for each $i = 1, 2, \ldots, k$, we have

$$b_i(x, \overline{x})\ \phi(f_i(x) - f_i(\overline{x})) < 0 \ \Rightarrow\ < \alpha(x, \overline{x}) \bigtriangledown f_i(\overline{x}), \eta(x, \overline{x}) > < 0.$$

Definition 2.8. The function $f : X \to R^k$ is said to be **strict pseudo α-univex** at $\overline{x} \in X$ with respect to α, b, ϕ, and η if for every $x \in X$ $(x \neq \overline{x})$ and for each $i = 1, 2, \ldots, k$, we have

$$b_i(x, \overline{x})\ \phi\ (f_i(x) - f_i(\overline{x})) \leq 0 \ \Rightarrow\ < \alpha(x, \overline{x}) \bigtriangledown f_i(\overline{x}), \eta(x, \overline{x}) > < 0.$$

Definition 2.9. The function $f : X \to R^k$ is said to be **quasi α-univex** at $\overline{x} \in X$ with respect to α, b, ϕ, and η if for every $x \in X$ and for each $i = 1, 2, \ldots, k$, we have

$$b_i(x, \overline{x})\ \phi\ (f_i(x) - f_i(\overline{x}))\ \leq\ 0\ \Rightarrow < \alpha(x, \overline{x}) \bigtriangledown f_i(\overline{x}), \eta(x, \overline{x}) > \leq\ 0.$$

f is said to be (strict) α-univex, (strict) pseudo α-univex and quasi α-univex on X if it is (strict) α-univex, (strict) pseudo α-univex and quasi α-univex respectively at every $x \in X$.

We now give the following example to show the existence of vector valued α-univex function.

Example 2.10. Let $X =]0, 1[$ and $f : X \to R^2$ is given by

$$f(x) = (f_1(x), f_2(x)) = (x^2, 2x + 1).$$

Also let, $b_1(x, u) = u, b_2(x, u) = x + u, \eta(x, u) = \dfrac{x - u}{3}, \alpha(x, u) = x + u$ and

$$\phi(x) = \begin{cases} 2x, & x \geq 0 \\ -2x, & x < 0. \end{cases}$$

Then f is α-univex on X with respect to α, b, ϕ and η.

We note that every α-univex function is pseudo α-univex as well as quasi α-univex but the converse is not true. To illustrate this fact, we give the following examples of pseudo α-univex and quasi α-univex functions which are not α-univex.

Example 2.11. Let $X = \left]0, \dfrac{\pi}{2}\right[$ and $f : X \to R^2$ is given by

$$f(x) = (f_1(x), f_2(x)) = (\cos x, \sin x).$$

Also let, $\phi(x) = 2x,\ \eta(x, u) = u - x,\ \alpha(x, u) = x^2 + u,$

$$b_1(x, u) = \begin{cases} 0, & x \geq u \\ xu, & x < u \end{cases} \quad \text{and} \quad b_2(x, u) = \begin{cases} 0, & u \geq x \\ x + u, & u < x. \end{cases}$$

Then f is pseudo α-univex on X with respect to α, b, ϕ and η but it is not α-univex on X because for $x = \dfrac{\pi}{3}, u = \dfrac{\pi}{6}$

$$b_1(x, u)\ \phi\ (f_1(x) - f_1(u))\ << \alpha(x, u) \bigtriangledown f_1(u), \eta(x, u) > .$$

Example 2.12. Let $X = \left]0, \dfrac{\pi}{2}\right[$ and $f : X \to R^2$ is given by

$$f(x) = (f_1(x), f_2(x)) = (\sin x, \cos x).$$

Also let, $\phi(x) = 2x,\ \alpha(x, u) = x + u,$

$$\eta(x, u) = \begin{cases} \dfrac{\sin x - \sin u}{\cos u}, & x \geq u \\ 0, & x < u \end{cases} \quad \text{and} \quad b_1(x, u) = b_2(x, u) = \begin{cases} 1, & x \geq u \\ 0, & x < u. \end{cases}$$

Then f is quasi α-univex on X with respect to α, b, ϕ and η but it is not α-univex on X because for $x = \dfrac{5\pi}{12}, u = \dfrac{\pi}{3}$

$$b_1(x, u)\ \phi\ (f_1(x) - f_1(u))\ << \alpha(x, u) \bigtriangledown f_1(u), \eta(x, u) > .$$

Now, we give the following lemma.

Lemma 2.13. *Assume that f and g are differentiable functions defined from X to R^k, where X is an open subset of R^n and $g(x) > 0$ for all $x \in X$. If for $w = (w_1, w_2, \ldots, w_k) \in R^n \times R^n \times \ldots \times R^n$, $f(\cdot) + (\cdot)^T w$ and $-g(\cdot)$ are α-univex at $\overline{x} \in X$ with respect to α, b, ϕ and η and ϕ is linear, then $\left(\dfrac{f(\cdot) + (\cdot)^T w}{g(\cdot)}\right)$ is α-univex at $\overline{x}$ with respect to $\alpha, \overline{b}, \phi$ and η, where*

$$\overline{b_i}(x, \overline{x}) = \frac{g_i(x)}{g_i(\overline{x})} b_i(x, \overline{x}), \quad \forall\, i = 1, 2, \ldots, k.$$

Proof. Consider for each $i = 1, 2, \ldots, k$ and $x \in X$,

$$\begin{aligned}
&b_i(x,\overline{x})\, \phi \left(\frac{f_i(x) + x^T w_i}{g_i(x)} \; - \; \frac{f_i(\overline{x}) + \overline{x}^T w_i}{g_i(\overline{x})} \right) \\
&= b_i(x,\overline{x})\, \phi \left(\frac{\big(f_i(x) + x^T w_i\big) - \big(f_i(\overline{x}) + \overline{x}^T w_i\big)}{g_i(x)} \; + \Big(f_i(\overline{x}) + \overline{x}^T w_i\Big) \left(\frac{1}{g_i(x)} - \frac{1}{g_i(\overline{x})} \right) \right) \\
&= b_i(x,\overline{x})\, \phi \left(\frac{\big(f_i(x) + x^T w_i\big) - \big(f_i(\overline{x}) + \overline{x}^T w_i\big)}{g_i(x)} \; + \frac{\big(f_i(\overline{x}) + \overline{x}^T w_i\big)}{g_i(x) g_i(\overline{x})} \Big(- g_i(x) + g_i(\overline{x}) \Big) \right).
\end{aligned}$$

As ϕ is linear, we have that

$$\begin{aligned}
&b_i(x,\overline{x})\, \phi \left(\frac{f_i(x) + x^T w_i}{g_i(x)} - \frac{f_i(\overline{x}) + \overline{x}^T w_i}{g_i(\overline{x})} \right) \\
&= \frac{1}{g_i(x)} b_i(x,\overline{x}) \phi \left((f_i(x) + x^T w_i) - (f_i(\overline{x}) + \overline{x}^T w_i) \right) \\
&\quad + \frac{(f_i(\overline{x}) + \overline{x}^T w_i)}{g_i(x) g_i(\overline{x})} b_i(x,\overline{x}) \phi \left(-g_i(x) + g_i(\overline{x}) \right), \ \forall i = 1, 2, \ldots, k.
\end{aligned}$$

Since $f(\cdot) + (\cdot)^T w$ and $-g(\cdot)$ are α-univex at $\overline{x}$ with respect to α, b, ϕ and η, we have for each $i = 1, 2, \ldots, k$,

$$\begin{aligned}
&b_i(x,\overline{x})\, \phi \left(\frac{f_i(x) + x^T w_i}{g_i(x)} \; - \; \frac{f_i(\overline{x}) + \overline{x}^T w_i}{g_i(\overline{x})} \right) \\
&\geq \frac{1}{g_i(x)} \left\langle \alpha(x,\overline{x}) \bigtriangledown (f_i(\overline{x}) + \overline{x}^T w_i) \ , \ \eta(x,\overline{x}) \right\rangle \\
&\quad - \frac{(f_i(\overline{x}) + \overline{x}^T w_i)}{g_i(x) g_i(\overline{x})} \left\langle \alpha(x,\overline{x}) \bigtriangledown g_i(\overline{x}) \ , \ \eta(x,\overline{x}) \right\rangle \\
&= \frac{\alpha(x,\overline{x})}{g_i(x) g_i(\overline{x})} < g_i(\overline{x}) \bigtriangledown (f_i(\overline{x}) + \overline{x}^T w_i) - (f_i(\overline{x}) + \overline{x}^T w_i) \bigtriangledown g_i(\overline{x}), \eta(x,\overline{x}) > \\
&= \alpha(x,\overline{x}) \frac{g_i(\overline{x})}{g_i(x)} \left\langle \frac{g_i(\overline{x}) \bigtriangledown (f_i(\overline{x}) + \overline{x}^T w_i) - (f_i(\overline{x}) + \overline{x}^T w_i) \bigtriangledown g_i(\overline{x})}{(g_i(\overline{x}))^2}, \eta(x,\overline{x}) \right\rangle \\
&= \alpha(x,\overline{x}) \frac{g_i(\overline{x})}{g_i(x)} \left\langle \bigtriangledown \left(\frac{f_i(\overline{x}) + \overline{x}^T w_i}{g_i(\overline{x})} \right) \ , \ \eta(x,\overline{x}) \right\rangle .
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{g_i(x)}{g_i(\overline{x})} b_i(x,\overline{x})\ \phi\left(\frac{f_i(x)+x^T w_i}{g_i(x)} \ - \ \frac{f_i(\overline{x})+\overline{x}^T w_i}{g_i(\overline{x})}\right) \\
&\geq \left\langle \alpha(x,\overline{x}) \bigtriangledown \left(\frac{f_i(\overline{x})+\overline{x}^T w_i}{g_i(\overline{x})}\right)\ ,\ \eta(x,\overline{x})\right\rangle, \quad \forall i = 1,2,\ldots,k \\
\Rightarrow \qquad & \overline{b_i}(x,\overline{x})\phi\ \left(\frac{f_i(x)+x^T w_i}{g_i(x)} \ - \ \frac{f_i(\overline{x})+\overline{x}^T w_i}{g_i(\overline{x})}\right) \\
&\geq \left\langle \alpha(x,\overline{x}) \bigtriangledown \left(\frac{f_i(\overline{x})+\overline{x}^T w_i}{g_i(\overline{x})}\right)\ ,\ \eta(x,\overline{x})\right\rangle, \quad \forall i = 1,2,\ldots,k,
\end{aligned}$$

where $\overline{b_i}(x,\overline{x}) = \dfrac{g_i(x)}{g_i(\overline{x})} b_i(x,\overline{x}), \quad \forall i = 1,2,\ldots,k$. Therefore, $\left(\dfrac{f(\cdot)+(\cdot)^T w}{g(\cdot)}\right)$ is α-univex at $\overline{x}$ with respect to $\alpha, \overline{b}, \phi$ and η. □

Remark 2.1. The following are satisfied:

(1) If in Lemma 2.13 the functions $f(\cdot)+(\cdot)^T w$ and $-g(\cdot)$ are assumed to be strict α-univex and α-univex respectively at $\overline{x}$ with respect to α, b, ϕ and η, then moving on the similar lines as in Lemma 2.13, it can be shown that $\left(\dfrac{f(\cdot)+(\cdot)^T w}{g(\cdot)}\right)$ is strict α-univex at $\overline{x}$ with respect to $\alpha, \overline{b}, \phi$ and η.

(2) Since every α-univex function is pseudo α-univex, therefore assuming all the conditions of Lemma 2.13, $\left(\dfrac{f(\cdot)+(\cdot)^T w}{g(\cdot)}\right)$ is also pseudo α-univex at $\overline{x}$.

(3) Again by using the fact that every strict α-univex function is strict pseudo α-univex, the function $\left(\dfrac{f(\cdot)+(\cdot)^T w}{g(\cdot)}\right)$ is strict pseudo α-univex at $\overline{x}$ if the functions $f(\cdot)+(\cdot)^T w$ and $-g(\cdot)$ are assumed to be strict α-univex and α-univex respectively at $\overline{x}$ in Lemma 2.13.

We now give an example which illustrates the above Lemma 2.13 and Remark 2.14(2).

Example 2.14. Let $X =]-1,1[$ and $f : X \to R^2, g : X \to R^2$ are defined by

$$\begin{aligned}
f(x) &= (f_1(x), f_2(x)) = (x+2, x^2), \\
g(x) &= (g_1(x), g_2(x)) = (-x^2+3, -x^4+4).
\end{aligned}$$

Also let, $\alpha(x,\overline{x}) = \overline{x}+2$, $b_1(x,\overline{x}) = x^2+\overline{x}^2$, $b_2(x,\overline{x}) = \overline{x}^2+1$, $\phi(x) = x$, $\eta(x,\overline{x}) = \overline{x}^2 - x^2$, $w = (w_1,w_2) = (1,0)$ and $\overline{x} = 0$. Then $\left(\dfrac{f(\cdot)+(\cdot)^T w}{g(\cdot)}\right)$ is α-univex and hence pseudo α-univex at $\overline{x} = 0$ with respect to $\alpha, \overline{b}, \phi$ and η as

$f(\cdot) + (\cdot)^T w$ and $-g(\cdot)$ are α-univex at $\overline{x} = 0$ with respect to α, b, ϕ and η, where

$$\overline{b_i}(x, \overline{x}) = \frac{g_i(x)}{g_i(\overline{x})} b_i(x, \overline{x}) \qquad \text{for } i = 1, 2.$$

3. Optimality Conditions

The following lemma giving necessary optimality conditions will be used in the sequel. The lemma is cited in [10] and can be obtained from [2] and [9].

Lemma 3.1. (*Necessary Optimality Conditions*) *Let $\overline{x}$ be a weakly efficient solution of* (*MFP*) *at which a suitable constraint qualification* [14] *be satisfied, then there exist $\overline{w} = (\overline{w_1}, \overline{w_2}, \ldots, \overline{w_k}) \in D_1 \times D_2 \times \ldots \times D_k$, $\overline{\lambda} \in R^k, \overline{\lambda} \geq 0$ and $\overline{y} \in C_2$ such that*

$$\left(\overline{\lambda}^T \bigtriangledown \left(\frac{f(\overline{x}) + \overline{x}^T \overline{w}}{g(\overline{x})}\right) + \bigtriangledown \overline{y}^T h(\overline{x})\right)^T (x - \overline{x}) \geq 0, \ \ \forall\, x \in C_1, \tag{3.1}$$

$$\overline{y}^T h(\overline{x}) = 0, \tag{3.2}$$

$$s(\overline{x}|D_i) = \overline{x}^T \overline{w_i}, \ \ i = 1, 2, \ldots, k. \tag{3.3}$$

We now establish some sufficient optimality conditions for $\overline{x} \in X_0$ to be a (weakly) efficient solution of (MFP) under (generalized) α-univexity defined in the previous section.

Theorem 3.2. *Let $\overline{x}$ be a feasible solution of* (*MFP*) *and that there exist $\overline{\lambda} \in R^k, \overline{\lambda} \geq 0$, $\overline{w} = (\overline{w_1}, \overline{w_2}, \ldots, \overline{w_k}) \in D_1 \times D_2 \times \ldots \times D_k$ and $\overline{y} \in C_2$ such that*

$$\left(\overline{\lambda}^T \bigtriangledown \left(\frac{f(\overline{x}) + \overline{x}^T \overline{w}}{g(\overline{x})}\right) + \bigtriangledown \overline{y}^T h(\overline{x})\right)^T (x - \overline{x}) \geq 0, \ \ \forall\, x \in R^n, \tag{3.4}$$

$$\overline{y}^T h(\overline{x}) = 0, \tag{3.5}$$

$$s(\overline{x}|D_i) = \overline{x}^T \overline{w_i}, \ \ i = 1, 2, \ldots, k. \tag{3.6}$$

Further assume that all the conditions of Lemma 2.13 *are satisfied at $\overline{x}$ and $\overline{w}$ with $b_i(x, \overline{x}) > 0$ for all $i = 1, 2, \ldots, k$ and $a < 0 \Rightarrow \phi(a) < 0$. Also if any one of the following conditions hold:*

(a) *$\overline{y}^T h(\cdot)$ is α-univex at $\overline{x}$ with respect to α, b_0, ϕ_0 and η with $b_0(x, \overline{x}) > 0$ and $\phi_0(a) > 0 \Rightarrow a > 0$,*
(b) *$\overline{y}^T h(\cdot)$ is quasi α-univex at $\overline{x}$ with respect to α, b_0, ϕ_0 and η with $a \leq 0 \Rightarrow \phi_0(a) \leq 0$,*

then $\overline{x}$ is a weakly efficient solution of (*MFP*).

Proof. Suppose that $\overline{x}$ is not a weakly efficient solution of (MFP). Then there exists some $x \in X_0$ such that

$$\frac{f_i(x) + s(x|D_i)}{g_i(x)} < \frac{f_i(\overline{x}) + s(\overline{x}|D_i)}{g_i(\overline{x})}, \qquad \forall i = 1, 2, \ldots, k.$$

Using the fact that $s(x|D_i) \geq x^T\overline{w_i}$ for all $i = 1, 2, \ldots, k$ and $s(\overline{x}|D_i) = \overline{x}^T\overline{w_i}$, we have for each $i = 1, 2, \ldots, k$ that

$$\frac{f_i(x) + x^T\overline{w_i}}{g_i(x)} \leq \frac{f_i(x) + s(x|D_i)}{g_i(x)} < \frac{f_i(\overline{x}) + s(\overline{x}|D_i)}{g_i(\overline{x})} = \frac{f_i(\overline{x}) + \overline{x}^T\overline{w_i}}{g_i(\overline{x})}$$

$$\Rightarrow \quad \frac{f_i(x) + x^T\overline{w_i}}{g_i(x)} - \frac{f_i(\overline{x}) + \overline{x}^T\overline{w_i}}{g_i(\overline{x})} < 0. \tag{3.7}$$

Now assume that (a) holds. By Lemma 2.13, $\left(\dfrac{f(\cdot) + (\cdot)^T\overline{w}}{g(\cdot)}\right)$ is α-univex at $\overline{x}$ with respect to $\alpha, \overline{b}, \phi$ and η where $\overline{b_i}(x, \overline{x}) = \dfrac{g_i(x)}{g_i(\overline{x})}b_i(x, \overline{x})$. Since $a < 0 \Rightarrow \phi(a) < 0$ and $\overline{b_i}(x, \overline{x}) > 0$, (3.7) gives

$$\overline{b_i}(x, \overline{x})\phi\left(\frac{f_i(x) + x^T\overline{w_i}}{g_i(x)} - \frac{f_i(\overline{x}) + \overline{x}^T\overline{w_i}}{g_i(\overline{x})}\right) < 0, \ \ \forall\, i = 1, 2, \ldots, k. \tag{3.8}$$

Using the definition of α-univexity of $\left(\dfrac{f(\cdot) + (\cdot)^T\overline{w}}{g(\cdot)}\right)$, (3.8) implies

$$\left\langle \alpha(x, \overline{x}) \bigtriangledown \left(\frac{f_i(\overline{x}) + \overline{x}^T\overline{w_i}}{g_i(\overline{x})}\right) \ , \ \eta(x, \overline{x}) \right\rangle < 0, \quad \forall\, i = 1, 2, \ldots, k.$$

Since $\overline{\lambda} \geq 0$, therefore multiplying each of the above inequalities by $\overline{\lambda_i}$ and summing over $i = 1, 2, \ldots, k$, we get that

$$\left\langle \alpha(x, \overline{x})\, \overline{\lambda}^T \bigtriangledown \left(\frac{f(\overline{x}) + \overline{x}^T\overline{w}}{g(\overline{x})}\right) \ , \ \eta(x, \overline{x}) \right\rangle < 0. \tag{3.9}$$

As (3.4) holds for all $x \in R^n$, we have

$$\overline{\lambda}^T \bigtriangledown \left(\frac{f(\overline{x}) + \overline{x}^T\overline{w}}{g(\overline{x})}\right) + \bigtriangledown\overline{y}^T h(\overline{x}) = 0. \tag{3.10}$$

Now using (3.10) in (3.9) we get that

$$\left\langle \alpha(x, \overline{x}) \ \bigtriangledown \overline{y}^T h(\overline{x}) \ , \ \eta(x, \overline{x}) \right\rangle > 0.$$

Since $\overline{y}^T h(\cdot)$ is α-univex at $\overline{x} \in X_0$, the above inequality implies

$$b_0(x, \overline{x})\ \phi_0\ (\overline{y}^T h(x) - \overline{y}^T h(\overline{x})) > 0.$$

Using (3.5), we get

$$b_0(x, \overline{x})\ \phi_0(\overline{y}^T h(x)) > 0.$$

As $b_0(x, \overline{x}) > 0$ and $\phi_0(a) > 0 \Rightarrow a > 0$, we have

$$\overline{y}^T h(x) > 0. \tag{3.11}$$

But as x is feasible for (MFP) and $\overline{y} \in C_2$, we get that

$$\overline{y}^T h(x) \leq 0,$$

which contradicts (3.11). Hence $\overline{x}$ is a weakly efficient solution of (MFP).
Assume that (b) holds. Using Remark 2.14(2), we have that $\left(\dfrac{f(\cdot)+(\cdot)^T\overline{w}}{g(\cdot)}\right)$ is pseudo α-univex at $\overline{x}$ with respect to $\alpha, \overline{b}, \phi$ and η where $\overline{b_i}(x,\overline{x}) = \dfrac{g_i(x)}{g_i(\overline{x})} b_i(x,\overline{x}) > 0$. Also as $a < 0 \Rightarrow \phi(a) < 0$, (3.7) gives

$$\overline{b_i}(x,\overline{x})\phi\left(\frac{f_i(x)+x^T\overline{w_i}}{g_i(x)} - \frac{f_i(\overline{x})+\overline{x}^T\overline{w_i}}{g_i(\overline{x})}\right) < 0, \quad \forall\, i = 1,2,\ldots,k. \tag{3.12}$$

Now $\left(\dfrac{f(\cdot)+(\cdot)^T\overline{w}}{g(\cdot)}\right)$ being pseudo α-univex at $\overline{x}$, (3.12) implies

$$\left\langle \alpha(x,\overline{x}) \bigtriangledown \left(\frac{f_i(\overline{x})+\overline{x}^T\overline{w_i}}{g_i(\overline{x})}\right) \;,\; \eta(x,\overline{x}) \right\rangle \;<\; 0, \qquad \forall\, i = 1,2,\ldots,k.$$

Since $\overline{\lambda} \geq 0$, therefore multiplying each of the above inequalities by $\overline{\lambda_i}$ and summing over $i = 1,2,\ldots,k$, we get

$$\left\langle \alpha(x,\overline{x})\, \overline{\lambda}^T \bigtriangledown \left(\frac{f(\overline{x})+\overline{x}^T\overline{w}}{g(\overline{x})}\right) \;,\; \eta(x,\overline{x}) \right\rangle \;<\; 0. \tag{3.13}$$

As (3.4) holds for all $x \in R^n$, we have

$$\overline{\lambda}^T \bigtriangledown \left(\frac{f(\overline{x})+\overline{x}^T\overline{w}}{g(\overline{x})}\right) + \bigtriangledown \overline{y}^T h(\overline{x}) = 0. \tag{3.14}$$

Because x is feasible for (MFP) and $\overline{y} \in C_2$, we get that

$$\overline{y}^T h(x) \leq 0.$$

Using (b), (3.5) and above inequality, we obtain

$$b_0(x,\overline{x})\; \phi_0(\overline{y}^T h(x) - \overline{y}^T h(\overline{x})) \leq 0.$$

As $\overline{y}^T h(\cdot)$ is quasi α-univex at $\overline{x}$, we obtain from above inequality that

$$\left\langle \alpha(x,\overline{x}) \;\; \bigtriangledown \overline{y}^T h(\overline{x}) \;,\; \eta(x,\overline{x}) \right\rangle \;\leq\; 0. \tag{3.15}$$

Adding (3.13) and (3.15), we get that

$$\left\langle \alpha(x,\overline{x}) \;\left(\overline{\lambda}^T \bigtriangledown \left(\frac{f(\overline{x})+\overline{x}^T\overline{w}}{g(\overline{x})}\right) + \bigtriangledown \overline{y}^T h(\overline{x})\right) \;,\; \eta(x,\overline{x}) \right\rangle \;<\; 0,$$

which contradicts (3.14). Hence $\overline{x}$ is a weakly efficient solution of (MFP). □

Theorem 3.3. *Let $\overline{x}$ be a feasible solution of (MFP) and that there exist $\overline{\lambda} \in R^k, \overline{\lambda} > 0$, $\overline{w} = (\overline{w_1}, \overline{w_2}, \ldots, \overline{w_k}) \in D_1 \times D_2 \times \ldots \times D_k$ and $\overline{y} \in C_2$ such that the conditions* (3.4) - (3.6) *hold. Assume that all the conditions of Lemma* 2.13 *are satisfied at $\overline{x}$ and $\overline{w}$ with $b_i(x,\overline{x}) > 0$ for all $i = 1,2,\ldots,k$ and $a < 0 \Rightarrow \phi(a) < 0$. Also assume that $\overline{y}^T h(\cdot)$ is α-univex at $\overline{x}$ with respect to α, b_0, ϕ_0 and η with $b_0(x,\overline{x}) > 0$ and $\phi_0(a) > 0 \Rightarrow a > 0$. Then $\overline{x}$ is an efficient solution of (MFP).*

Proof. Suppose that $\overline{x}$ is not an efficient solution of (MFP). Then there exists some $x \in X_0$ such that

$$\frac{f_i(x)+s(x|D_i)}{g_i(x)} \leq \frac{f_i(\overline{x})+s(\overline{x}|D_i)}{g_i(\overline{x})}, \quad \forall i=1,2,\ldots,k,\ \ i\neq j$$
$$\text{and} \quad \frac{f_j(x)+s(x|D_j)}{g_j(x)} < \frac{f_j(\overline{x})+s(\overline{x}|D_j)}{g_j(\overline{x})}.$$

Using (3.6) and the fact that $s(x|D_i) \geq x^T\overline{w_i}$, $i=1,2,\ldots,k$, we have for all $i=1,2,\ldots,k$, $i\neq j$ that

$$\frac{f_i(x)+x^T\overline{w_i}}{g_i(x)} \leq \frac{f_i(x)+s(x|D_i)}{g_i(x)} \leq \frac{f_i(\overline{x})+s(\overline{x}|D_i)}{g_i(\overline{x})} = \frac{f_i(\overline{x})+\overline{x}^T\overline{w_i}}{g_i(\overline{x})}$$

and

$$\frac{f_j(x)+x^T\overline{w_j}}{g_j(x)} \leq \frac{f_j(x)+s(x|D_j)}{g_j(x)} < \frac{f_j(\overline{x})+s(\overline{x}|D_j)}{g_j(\overline{x})} = \frac{f_j(\overline{x})+\overline{x}^T\overline{w_j}}{g_j(\overline{x})}.$$

That is,

$$\frac{f_i(x)+x^T\overline{w_i}}{g_i(x)} - \frac{f_i(\overline{x})+\overline{x}^T\overline{w_i}}{g_i(\overline{x})} \leq 0, \ \ \forall i=1,2,\ldots,k,\ i\neq j \tag{3.16}$$
$$\text{and} \quad \frac{f_j(x)+x^T\overline{w_j}}{g_j(x)} - \frac{f_j(\overline{x})+\overline{x}^T\overline{w_j}}{g_j(\overline{x})} < 0. \tag{3.17}$$

Since $f(\cdot)+(\cdot)^T\overline{w}$ and $-g(\cdot)$ are α-univex at $\overline{x}$ with respect to α, b, ϕ and η, therefore by Lemma 2.13, $\left(\dfrac{f(\cdot)+(\cdot)^T\overline{w}}{g(\cdot)}\right)$ is α-univex at $\overline{x}$ with respect to $\alpha, \overline{b}, \phi$ and η where $\overline{b_i}(x,\overline{x}) = \dfrac{g_i(x)}{g_i(\overline{x})} b_i(x,\overline{x}) > 0$ for all $i=1,2,\ldots,k$. Using the assumption that $a<0 \Rightarrow \phi(a)<0$ where ϕ is linear, (3.16) and (3.17) give

$$\overline{b_i}(x,\overline{x})\ \phi\left(\frac{f_i(x)+x^T\overline{w_i}}{g_i(x)} - \frac{f_i(\overline{x})+\overline{x}^T\overline{w_i}}{g_i(\overline{x})}\right) \leq 0, \ \ \forall i=1,2,\ldots,k,\ i\neq j$$
$$\text{and} \ \ \overline{b_j}(x,\overline{x})\ \phi\left(\frac{f_j(x)+x^T\overline{w_j}}{g_j(x)} - \frac{f_j(\overline{x})+\overline{x}^T\overline{w_j}}{g_j(\overline{x})}\right) < 0.$$

Using α-univexity of $\left(\dfrac{f(\cdot)+(\cdot)^T\overline{w}}{g(\cdot)}\right)$ in last two inequalities, we get

$$\left\langle \alpha(x,\overline{x}) \bigtriangledown \left(\frac{f_i(\overline{x})+\overline{x}^T\overline{w_i}}{g_i(\overline{x})}\right)\ ,\ \eta(x,\overline{x})\right\rangle \leq 0, \ \ \forall\, i=1,2,\ldots,k,\ i\neq j$$
$$\text{and} \ \ \left\langle \alpha(x,\overline{x}) \bigtriangledown \left(\frac{f_j(\overline{x})+\overline{x}^T\overline{w_j}}{g_j(\overline{x})}\right)\ ,\ \eta(x,\overline{x})\right\rangle < 0.$$

Since $\overline{\lambda} > 0$, therefore multiplying each of the above inequalities by $\overline{\lambda_i}$ and summing over $i = 1, 2, \ldots, k$, we get

$$\left\langle \alpha(x, \overline{x})\, \overline{\lambda}^T \bigtriangledown \left(\frac{f(\overline{x}) + \overline{x}^T \overline{w}}{g(\overline{x})} \right) \, , \, \eta(x, \overline{x}) \right\rangle \; < \; 0.$$

Rest of the proof follows on the lines of proof of part (a) of Theorem 3.2. □

Theorem 3.4. *Let $\overline{x}$ be a feasible solution of (MFP) and that there exist $\overline{\lambda} \in R^k, \overline{\lambda} \geq 0$, $\overline{w} = (\overline{w_1}, \overline{w_2}, \ldots, \overline{w_k}) \in D_1 \times D_2 \times \ldots \times D_k$ and $\overline{y} \in C_2$ such that the conditions* (3.4) - (3.6) *hold. Assume that all the conditions of Lemma* 2.13 *are satisfied at $\overline{x}$ and $\overline{w}$ with $f(\cdot) + (\cdot)^T \overline{w}$ being strict α-univex at $\overline{x}$. Further assume that $\overline{y}^T h(\cdot)$ is quasi α-univex at $\overline{x}$ with respect to α, b_0, ϕ_0 and η. Also let $a \leq 0 \Rightarrow \phi(a) \leq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$. Then $\overline{x}$ is an efficient solution of (MFP).*

Proof. Suppose that $\overline{x}$ is not an efficient solution of (MFP). Then there exists some $x \in X_0$ such that

$$\frac{f_i(x) + s(x|D_i)}{g_i(x)} \leq \frac{f_i(\overline{x}) + s(\overline{x}|D_i)}{g_i(\overline{x})}, \quad \forall i = 1, 2, \ldots, k, \;\; i \neq j$$

$$\text{and} \;\; \frac{f_j(x) + s(x|D_j)}{g_j(x)} < \frac{f_j(\overline{x}) + s(\overline{x}|D_j)}{g_j(\overline{x})}.$$

Using (3.6) and the fact that $s(x|D_i) \geq x^T \overline{w_i}$, $i = 1, 2, \ldots, k$, we have for all $i = 1, 2, \ldots, k$, $i \neq j$ that

$$\frac{f_i(x) + x^T \overline{w_i}}{g_i(x)} \leq \frac{f_i(x) + s(x|D_i)}{g_i(x)} \leq \frac{f_i(\overline{x}) + s(\overline{x}|D_i)}{g_i(\overline{x})} = \frac{f_i(\overline{x}) + \overline{x}^T \overline{w_i}}{g_i(\overline{x})}$$

and

$$\frac{f_j(x) + x^T \overline{w_j}}{g_j(x)} \leq \frac{f_j(x) + s(x|D_j)}{g_j(x)} < \frac{f_j(\overline{x}) + s(\overline{x}|D_j)}{g_j(\overline{x})} = \frac{f_j(\overline{x}) + \overline{x}^T \overline{w_j}}{g_j(\overline{x})}.$$

That is,

$$\frac{f_i(x) + x^T \overline{w_i}}{g_i(x)} - \frac{f_i(\overline{x}) + \overline{x}^T \overline{w_i}}{g_i(\overline{x})} \leq 0, \;\; \forall i = 1, 2, \ldots, k, \; i \neq j \tag{3.18}$$

$$\text{and} \;\; \frac{f_j(x) + x^T \overline{w_j}}{g_j(x)} - \frac{f_j(\overline{x}) + \overline{x}^T \overline{w_j}}{g_j(\overline{x})} < 0. \tag{3.19}$$

Since $f(\cdot) + (\cdot)^T \overline{w}$ is strict α-univex and $-g(\cdot)$ is α-univex at $\overline{x}$ with respect to α, b, ϕ and η, therefore by Remark 2.14(3), $\left(\frac{f(\cdot) + (\cdot)^T \overline{w}}{g(\cdot)} \right)$ is strict pseudo α-univex at $\overline{x}$ with respect to $\alpha, \overline{b}, \phi$ and η where $\overline{b_i}(x, \overline{x}) = \dfrac{g_i(x)}{g_i(\overline{x})} b_i(x, \overline{x}) \geq 0$

for all $i = 1, 2, \ldots, k$. Thus by using the assumption that $a \leq 0 \Rightarrow \phi(a) \leq 0$, (3.18) and (3.19) give

$$\overline{b_i}(x, \overline{x})\, \phi \left(\frac{f_i(x) + x^T \overline{w_i}}{g_i(x)} - \frac{f_i(\overline{x}) + \overline{x}^T \overline{w_i}}{g_i(\overline{x})} \right) \leq 0, \ \forall i = 1, 2, \ldots, k. \tag{3.20}$$

Using the definition of strict pseudo α-univexity of $\left(\frac{f(\cdot) + (\cdot)^T \overline{w}}{g(\cdot)} \right)$, (3.20) implies

$$\left\langle \alpha(x, \overline{x}) \bigtriangledown \left(\frac{f_i(\overline{x}) + \overline{x}^T \overline{w_i}}{g_i(\overline{x})} \right) \ , \ \eta(x, \overline{x}) \right\rangle \ < \ 0, \quad \forall\, i = 1, 2, \ldots, k.$$

Since $\overline{\lambda} \geq 0$, therefore multiplying each of the above inequalities by $\overline{\lambda_i}$ and summing over $i = 1, 2, \ldots, k$, we get

$$\left\langle \alpha(x, \overline{x})\, \overline{\lambda}^T \bigtriangledown \left(\frac{f(\overline{x}) + \overline{x}^T \overline{w}}{g(\overline{x})} \right) \ , \ \eta(x, \overline{x}) \right\rangle \ < \ 0.$$

Rest of the proof follows on the lines of proof of part (b) of Theorem 3.2. $\square$

4. Duality

Now we consider the following Mond-Weir type dual of (MFP).

(MFD) Maximize $\left(\frac{f(u) + u^T w}{g(u)} \right)$

subject to

$$-\left(\lambda^T \bigtriangledown \left(\frac{f(u) + u^T w}{g(u)} \right) + \bigtriangledown y^T h(u) \right) \in C_1^*, \tag{4.1}$$

$$y^T h(u) - u^T \left(\lambda^T \bigtriangledown \left(\frac{f(u) + u^T w}{g(u)} \right) + \bigtriangledown y^T h(u) \right) \geq 0, \tag{4.2}$$

$$y \in C_2, \ \lambda \in R^k, \ \lambda \geq 0, \ w = (w_1, \ldots, w_k) \in D_1 \times \ldots \times D_k, \tag{4.3}$$

where w_i $(i = 1, 2, \ldots, k)$ is a vector in R^n and $u^T w = (u^T w_1, \ldots, u^T w_k)$.

Theorem 4.1. (*Weak Duality*) *Let x be feasible for* (*MFP*) *and* (u, y, λ, w) *be feasible for* (*MFD*). *Assume that*

(a) $f(\cdot) + (\cdot)^T w$ *and* $-g(\cdot)$ *are α-univex at u with respect to α, b, ϕ, η with ϕ linear and $y^T h(\cdot) + v^T(\cdot)$ is α-univex at u with respect to α, b_0, ϕ_0 and η for all $v \in C_1^*$,*

(b) $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, $a < 0 \Rightarrow \phi(a) < 0$ *and* $b_i(x, u) > 0$ *for all* $i = 1, 2, \ldots, k$,

then
$$F(x) \not< \frac{f(u) + u^T w}{g(u)}.$$

Proof. Let us suppose on the contrary that

$$F(x) \ < \ \frac{f(u)+u^T w}{g(u)},$$

that is,

$$\frac{f_i(x)+s(x|D_i)}{g_i(x)} \ < \ \frac{f_i(u)+u^T w_i}{g_i(u)}, \qquad \forall i=1,2,\ldots,k.$$

Using the fact that $s(x|D_i) \geq x^T w_i$ for all $i=1,2,\ldots,k$, we get that

$$\frac{f_i(x)+x^T w_i}{g_i(x)} \ < \ \frac{f_i(u)+u^T w_i}{g_i(u)}, \qquad \forall\, i=1,2,\ldots,k$$

$$\Rightarrow \quad \frac{f_i(x)+x^T w_i}{g_i(x)} \ - \ \frac{f_i(u)+u^T w_i}{g_i(u)} \ < \ 0, \ \ \forall\, i=1,2,\ldots,k. \tag{4.4}$$

By (a) as $f(\cdot\,)+(\cdot\,)^T w$ and $-g(\cdot\,)$ are α-univex at u with respect to α, b, ϕ and η, therefore by Lemma 2.13, $\left(\dfrac{f(\cdot\,)+(\cdot\,)^T w}{g(\cdot\,)}\right)$ is α-univex at u with respect to $\alpha, \bar{b}, \phi$ and η where $\overline{b_i}(x,u) = \dfrac{g_i(x)}{g_i(u)} b_i(x,u) > 0$ for all $i=1,2,\ldots,k$. Thus by using the assumption that $a<0 \Rightarrow \phi(a)<0$, (4.4) gives

$$\overline{b_i}(x,u)\ \phi\ \left(\frac{f_i(x)+x^T w_i}{g_i(x)} \ - \ \frac{f_i(u)+u^T w_i}{g_i(u)}\right) \ < \ 0, \ \ \forall\, i=1,2,\ldots,k.$$

Now as $\left(\dfrac{f(\cdot\,)+(\cdot\,)^T w}{g(\cdot\,)}\right)$ is α - univex at u, we get

$$\left\langle \alpha(x,u) \bigtriangledown \left(\frac{f_i(u)+u^T w_i}{g_i(u)}\right) \ , \ \eta(x,u) \right\rangle \ < \ 0, \qquad \forall\, i=1,2,\ldots,k.$$

Since $\lambda \geq 0$ by (4.3), therefore multiplying above inequality for each $i = 1,2,\ldots,k$ by λ_i and summing over $i=1,2,\ldots,k$, we get that

$$\left\langle \alpha(x,u)\ \lambda^T \bigtriangledown \left(\frac{f(u)+u^T w}{g(u)}\right) \ , \ \eta(x,u) \right\rangle \ < \ 0. \tag{4.5}$$

From the dual constraint (4.1), we have

$$-\left(\lambda^T \bigtriangledown \left(\frac{f(u)+u^T w}{g(u)}\right) + \bigtriangledown y^T h(u)\right) \in C_1^*,$$

therefore there exist $v \in C_1^*$ such that

$$v = -\left(\lambda^T \bigtriangledown \left(\frac{f(u)+u^T w}{g(u)}\right) + \bigtriangledown y^T h(u)\right). \tag{4.6}$$

On using (4.6) in (4.2), we obtain

$$y^T h(u) + v^T u \geq 0. \tag{4.7}$$

Since x is feasible for (MFP), $y \in C_2$ and $v \in C_1^*$, we have

$$y^T h(x) + v^T x \leq 0, \tag{4.8}$$

therefore (4.7) and (4.8) together give

$$\begin{aligned} y^T h(u) + v^T u \ &\geq \ y^T h(x) + v^T x \\ \Rightarrow \quad y^T h(x) + v^T x - y^T h(u) - v^T u \ &\leq \ 0. \end{aligned}$$

From assumption (b) as $a \leq 0 \Rightarrow \phi_0(a) \leq 0$ and $b_0(x,u) \geq 0$, therefore

$$b_0(x,u)\ \phi_0\ \left(y^T h(x) + v^T x - y^T h(u) - v^T u\right) \ \leq \ 0.$$

Since $y^T h(\cdot) + v^T(\cdot)$ is α-univex at u with respect to α, b_0, ϕ_0 and η, therefore the above inequality gives

$$< \alpha(x,u)\ (\triangledown y^T h(u) + v)\ ,\ \eta(x,u) > \ \leq \ 0,$$

which on using (4.6) reduces to

$$< \alpha(x,u)\ \lambda^T \ \triangledown \left(\frac{f(u) + u^T w}{g(u)}\right)\ ,\ \eta(x,u) > \geq \ 0.$$

This contradicts (4.5). Hence,

$$F(x) \ \not\leq \ \frac{f(u) + u^T w}{g(u)}.$$

□

Theorem 4.2. (*Strong Duality*) *Let $\overline{x}$ be a weakly efficient solution of* (*MFP*) *at which a suitable constraint qualification* [14] *be satisfied. Then there exist* $\overline{\lambda} \in R^k, \overline{\lambda} \geq 0$, $\overline{y} \in C_2$ *and* $\overline{w} = (\overline{w_1}, \overline{w_2}, \ldots, \overline{w_k}) \in D_1 \times D_2 \times \ldots \times D_k$ *such that* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ *is feasible for* (*MFD*) *and the objective function values of* (*MFP*) *and* (*MFD*) *are equal. Furthermore, if the assumptions of weak duality Theorem* 4.1 *hold for all the feasible solutions of* (*MFP*) *and* (*MFD*), *then* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ *is weakly efficient for* (*MFD*).

Proof. Since $\overline{x}$ is a weakly efficient solution of (MFP), therefore there exist $\overline{\lambda} \in R^k, \overline{\lambda} \geq 0$, $\overline{y} \in C_2$ and $\overline{w} = (\overline{w_1}, \overline{w_2}, \ldots, \overline{w_k}) \in D_1 \times D_2 \times \ldots \times D_k$ such that (3.1),(3.2) and (3.3) hold. Since $\overline{x} \in C_1$ and C_1 is a closed convex cone, therefore for any $x \in C_1$, we have $x + \overline{x} \in C_1$. Thus replacing x by $x + \overline{x}$ in (3.1), we get

$$\left(\overline{\lambda}^T \ \triangledown \left(\frac{f(\overline{x}) + \overline{x}^T \overline{w}}{g(\overline{x})}\right) + \triangledown \overline{y}^T h(\overline{x})\right)^T x \geq 0, \qquad \forall\, x \in C_1.$$

That is,

$$-\left(\overline{\lambda}^T \ \triangledown \left(\frac{f(\overline{x}) + \overline{x}^T \overline{w}}{g(\overline{x})}\right) + \triangledown \overline{y}^T h(\overline{x})\right) \in C_1^*. \tag{4.9}$$

Also by letting $x = 0$ and $x = 2\overline{x}$ in (3.1), we get

$$\left(\overline{\lambda}^T \ \triangledown \left(\frac{f(\overline{x}) + \overline{x}^T \overline{w}}{g(\overline{x})}\right) + \triangledown \overline{y}^T h(\overline{x})\right)^T \overline{x} = 0. \tag{4.10}$$

Therefore by using (3.2) and (4.10), we have that

$$\overline{y}^T h(\overline{x}) - \overline{x}^T \left(\overline{\lambda}^T \bigtriangledown \left(\frac{f(\overline{x}) + \overline{x}^T \overline{w}}{g(\overline{x})} \right) + \bigtriangledown \overline{y}^T h(\overline{x}) \right) = 0. \tag{4.11}$$

Thus (4.9) and (4.11) imply that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is feasible for (MFD) and the objective function values of (MFP) and (MFD) are equal. Since the assumptions of weak duality hold for all the feasible solutions of (MFP) and (MFD), we get that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is a weakly efficient solution of (MFD). □

5. Conclusion

This paper generalizes the concept of α-univexity for a real valued function by defining the concept of α-univexity, pseudo α-univexity and quasi α-univexity for a vector valued function. Examples have been included to show the existence of these functions. Sufficient optimality criteria have been obtained for (MFP) by using the above defined classes of (generalized) α-univex functions. Assuming the functions to be α-univex duality is established between (MFP) and its Mond-Weir type dual (MFD).

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