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ON SOME GENERALIZED I-CONVERGENT DOUBLE SEQUENCE SPACES DEFINED BY A MODULUS FUNCTION†

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ABSTRACT. In this article we introduce the sequence spaces ${}_2c_0^I(f,p)$, ${}_2c^I(f,p)$ and ${}_2l_\infty^I(f,p)$ for a modulus function f, where $p = (p_k)$ is a sequence of positive reals and study some of the properties of these spaces.

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1. Introduction

The notion of I-Convergence is a generalization of the concept of statistical convergence which was first introduced by H.Fast [5] and later on studied by various mathematicians like J.A.Fridy [6,7], Kostyrko, Salat and Wilezynski [19], Salat, Tripathy, Ziman [29] and Demirci [3].

Also a double sequence is a double infinite array of elements $x_{kl} \in \mathbb{R}$ for all $k, l \in \mathbb{N}$ (see [14,15]). The initial works on double sequences is found in Bromwich [1], Basarir and Solancan [2] and many others. Throughout this article a double sequence is denoted by $x = (x_{ij})$.
Next we discuss some preliminaries about I-convergence (see [12],[30]).

Let X be a non empty set. Then a family of sets I$\subseteq 2^X$ (power set of X) is said to be an ideal if I is additive i.e A,B$\in$I $\Rightarrow$A$\cup$ B$\in$I and hereditary i.e A$\in$I, B$\subseteq$A$\Rightarrow$B$\in$I.

A non-empty family of sets $\pounds(I) \subseteq 2^X$ is said to be filter on X if and only if $\Phi \notin \pounds$(I), for A,B$\in \pounds$(I) we have A$\cap$B$\in \pounds$(I) and for each A $\in \pounds$(I)and A$\subseteq$B implies B$\in \pounds$(I). An Ideal I$\subseteq 2^X$ is called non-trivial if I$\neq 2^X$. A non-trivial ideal I$\subseteq 2^X$ is called admissible if $\{x : \{x\} \in X\} \subseteq$I.

A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal J$\neq$I containing I as a subset. For each ideal I, there is a filter $\pounds$(I) corresponding to I. i.e $\pounds$(I) $= \{K \subseteq N : K^c \in I\}$,where K^c = N-K.

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Definition 1.1. A double sequence $(x_{ij}) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$. $\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in$ I. In this case we write I-$\lim x_{ij} = L$. (see [17])

The space ${}_2c^I$ of all I-convergent sequences to L is given by

$${}_2c^I = \{(x_{ij}) \in \omega : \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in I, \text{for some L} \in \mathbb{C}\}$$

Definition 1.2. A sequence $(x_{ij}) \in \omega$ is said to be I-null if L = 0. In this case we write I-$\lim x_{ij} = 0$.

Definition 1.3. A sequence $(x_{ij}) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number m = m(ϵ) and n = n(ϵ) such that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{mn}| \geq \epsilon\} \in I$$

.

Definition 1.4. A sequence $(x_{ij}) \in \omega$ is said to be I-bounded if there exists M >0 such that $\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij}| > M\}$.

Definition 1.5. Let $(x_{ij}), (y_{ij})$ be two sequences. We say that $(x_{ij}) = (y_{ij})$ for *almost all (i,j) relative to I (a.a.k.r.I)*, if $\{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \neq y_{ij}\} \in I$

Definition 1.6. For any set E of sequences the space of multipliers of E, denoted by $M(E)$ is given by

$$M(E) = \{a \in \omega : ax \in E \text{ for all } x \in E\}\text{(see[28])}.$$

Definition 1.7. A map $\hbar$ defined on a domain $D \subset X$ i.e $\hbar : D \subset X \to \boldsymbol{R}$ is said to satisfy Lipschitz condition if $|\hbar(x) - \hbar(y)| \leq K|x - y|$ where $K$is known as the Lipschitz constant.The class of K-Lipschitz functions defined on D is denoted by $\hbar \in (D, K)$.

Definition 1.8. A convergence field of I-convergence is a set

$$F(I) = \{x = (x_{ij}) \in l_\infty : \text{there exists } I - \lim x \in \boldsymbol{R}\}.$$

The convergence field $F(I)$ is a closed linear subspace of l_∞ with respect to the supremum norm, $F(I) = l_\infty \cap {}_2c^I$(See[23]).

Define a function $\hbar : F(I) \to \boldsymbol{R}$ such that $\hbar(x) = I - \lim x$, for all $x \in F(I)$, then the function $\hbar : F(I) \to \boldsymbol{R}$ is a Lipschitz function ([11,4,13]).

Definition 1.9. The concept of paranorm is closely related to linear metric spaces [16]. It is a generalization of that of absolute value.
Let X be a linear space. A function $g : X \longrightarrow R$ is called paranorm, if for all $x, y, z \in X$,
(PI) $g(x) = 0 \; if \; x = \theta$, (P2) $g(-x) = g(x)$, (P3) $g(x + y) \leq g(x) + g(y)$,
(P4) If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and $x_n, a \in X$ with $x_n \to a$ $(n \to \infty)$, in the sense that $g(x_n - a) \to 0$ $(n \to \infty)$, in the sense that $g(\lambda_n x_n - \lambda a) \to 0$ $(n \to \infty)$.

A paranorm g for which $g(x) = 0$ implies $x = \theta$ is called a total paranorm on X, and the pair (X, g) is called a totally paranormed space.(See[23]). The idea of modulus was structured in 1953 by Nakano.(See[24]). A function f : [0,∞)⟶[0,∞) is called a modulus if
(1) f(t) = 0 if and only if t = 0, (2) f(t+u)≤ f(t)+ f(u) for all t,u≥0,
(3) f is increasing and (4) f is continuous from the right at zero.

Ruckle in [25,26,27] used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}$$

This space is an FK space ,and Ruckle proved that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences.
The space $X(f)$ is closely related to the space l_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle proved that,for any modulus f

$$X(f) \subset l_1 \text{ and } X(f)^{\alpha} = l_{\infty}$$

Where

$$X(f)^{\alpha} = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}$$

The space $X(f)$ is a Banach space with respect to the norm

$$||x|| = \sum_{k=1}^{\infty} f(|x_k|) < \infty.\text{(See[22])}.$$

Spaces of the type $X(f)$ are a special case of the spaces structured by B.Gramsch in[10]. From the point of view of local convexity, spaces of the type $X(f)$ are quite interesting.

Symmetric sequence spaces, which are locally convex have been frequently studied by D.J.H Garling [8,9], G.Köthe [18].

The following subspaces of ω were first introduced and discussed by Maddox [22,23].

$$\begin{aligned} l(p) &= \{x \in \omega : \sum_k |x_k|^{p_k} < \infty\}, \\ l_{\infty}(p) &= \{x \in \omega : \sup_k |x_k|^{p_k} < \infty\}, \\ c(p) &= \{x \in \omega : \lim_k |x_k - l|^{p_k} = 0, \text{ for some } l \in \mathbb{C}\}, \\ c_0(p) &= \{x \in \omega : \lim_k |x_k|^{p_k} = 0, \}, \end{aligned}$$

where $p = (p_k)$ is a sequence of striclty positive real numbers.

After then Lascarides[20,21] defined the following sequence spaces

$$l_{\infty}\{p\} = \left\{ x \in \omega : \text{ there exists } r > 0 \text{ such that } \sup_k |x_k r|^{p_k} t_k < \infty \right\},$$

$$c_0\{p\} = \left\{ x \in \omega : \text{ there exists } r > 0 \text{ such that } \lim_k |x_k r|^{p_k} t_k = 0 \right\},$$

$$l\{p\} = \left\{ x \in \omega : \text{ there exists } r > 0 \text{ such that } \sum_{k=1}^{\infty} |x_k r|^{p_k} t_k < \infty \right\},$$

where $t_k = p_k^{-1}$, for all $k \in \mathbb{N}$.

We need the following lemmas in order to establish some results of this article.

Lemma 1.10. *Let $h = \inf_k p_k$ and $H = \sup_k p_k$. Then the following conditions are equivalent.(See[18]).*
(a) $H < \infty$ and $h > 0$. (b) $c_0(p) = c_0$ or $l_\infty(p) = l_\infty$. (c) $l_\infty\{p\} = l_\infty(p)$.
(d) $c_0\{p\} = c_0(p)$. (e) $l\{p\} = l(p)$.

Lemma 1.11. *Let $K \in \pounds(I)$ and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$.(See[29,30]).*

Lemma 1.12. *If $I \subset 2^X$ and $M \subseteq X$. If $M \notin I$, then $M \cap K \notin I$.(See[29,30]).*

Throughout the article $l_\infty, c^I, c_0^I, m^I$ and m_0^I represent the bounded , I-convergent, I-null, bounded I-convergent and bounded I-null sequence spaces respectively. In this article we introduce the following classes of sequence spaces.

$$_2c^I(f,p) = \{(x_{ij}) \in \omega : f(|x_{ij} - L|^{p_{ij}}) \geq \epsilon \text{ for some L}\} \in I$$

$$_2c_0^I(f,p) = \{(x_{ij}) \in \omega : f(|x_{ij}|^{p_{ij}}) \geq \epsilon\} \in I$$

$$_2l_\infty^I(f,p) = \{(x_{ij}) \in \omega : \sup_{i,j} f(|x_{ij}|^{p_{ij}}) < \infty\} \in I$$

Also we write

$$_2m^I(f,p) = {}_2c^I(f,p) \cap {}_2l_\infty(f,p) \quad \text{and} \quad {}_2m_0^I(f,p) = {}_2c_0^I(f,p) \cap {}_2l_\infty(f,p).$$

2. Main results

Theorem 2.1. *Let $(p_{ij}) \in {}_2l_\infty$. Then ${}_2c^I(f,p)$, ${}_2c_0^I(f,p)$, ${}_2m^I(f,p)$ and ${}_2m_0^I(f,p)$ are linear spaces.*

Proof. Let $(x_{ij}), (y_{ij}) \in_2 c^I(f,p)$ and α, β be two scalars. Then for a given $\epsilon > 0$. We have

$$\left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : f(|x_{ij} - L_1|^{p_{ij}}) \geq \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in \mathbb{C} \right\} \in I$$

$$\left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : f(|y_{ij} - L_2|^{p_{ij}}) \geq \frac{\epsilon}{2M_2}, \text{ for some } L_2 \in \mathbb{C} \right\} \in I$$

where

$$M_1 = D.max\left\{1, \sup_{i,j} |\alpha|^{p_{ij}}\right\},\ M_2 = D.max\left\{1, \sup_{i,j} |\beta|^{p_{ij}}\right\} \text{ and } D = max\left\{1, 2^{H-1}\right\},$$

where $H = \sup_{i,j} p_{ij} \geq 0$. Let

$$A_1 = \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : f(|x_{ij} - L_1|^{p_{ij}}) < \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in \text{C} \right\} \in I$$

$$A_2 = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|y_{ij} - L_2|^{p_{ij}}) < \frac{\epsilon}{2M_2},\ \text{for some } L_2 \in \mathrm{C}\ \right\} \in I$$

be such that $A_1^c, A_2^c \in I$. Then

$$\begin{aligned} A_3 &= \{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|(\alpha x_{ij} + \beta y_{ij}) - f(\alpha L_1 + \beta L_2)|^{p_{ij}}) < \epsilon\} \\ &\supseteq \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : |\alpha|^{p_{ij}} f(|x_{ij} - L_1|^{p_{ij}}) < \frac{\epsilon}{2M_1}|\alpha|^{p_{ij}}.D\right\} \\ &\quad \cap \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : |\beta|^{p_{ij}} f(|y_{ij} - L_2|^{p_{ij}}) < \frac{\epsilon}{2M_2}|\beta|^{p_{ij}}.D\right\} \end{aligned}$$

Thus $A_3^c = A_1^c \cap A_2^c \in I$. Hence $(\alpha x_{ij} + \beta y_{ij}) \in {}_2c^I(f,p)$. Therefore ${}_2c^I(f,p)$ is a linear space. The rest of the result follows similarly. □

Theorem 2.2. *Let $(p_{ij}) \in {}_2l_\infty$. Then ${}_2m^I(f,p)$ and ${}_2m_0^I(f,p)$ are paranormed spaces, paranormed by $g(x_{ij}) = \sup_{i,j} f(|x_{ij}|^{\frac{p_{ij}}{M}})$ where $M = max\{1, \sup_{i,j} p_{ij}\}$*

Proof. Let $x = (x_{ij}), y = (y_{ij}) \in {}_2m^I(f,p)$.
(1) Clearly,$g(x) = 0$ if and only if $x = 0$.
(2) $g(x) = g(-x)$ is obvious.
(3) Since $\frac{p_{ij}}{M} \le 1$ and $M > 1$,using Minkowski's inequality and the definition of f we have

$$\sup_{i,j} f\left(|x_{ij} + y_{ij}|^{\frac{p_{ij}}{M}}\right) \le \sup_{i,j} f\left(|x_{ij}|^{\frac{p_{ij}}{M}}\right) + \sup_{i,j} f\left(|y_{ij}|^{\frac{p_{ij}}{M}}\right)$$

(4) Now for any complex λ we have (λ_{ij}) such that $\lambda_{ij} \to \lambda$, $(i,j \to \infty)$.
Let $x_{ij} \in {}_2m^I(f,p)$ such that $f(|x_{ij} - L|^{p_{ij}}) \ge \epsilon$.
Therefore,

$$g(x_{ij} - L) = \sup_{i,j} f\left(|x_{ij} - L|^{\frac{p_{ij}}{M}}\right) \le \sup_{i,j} f\left(|x_{ij}|^{\frac{p_{ij}}{M}}\right) + \sup_{i,j} f\left(|L|^{\frac{p_{ij}}{M}}\right).$$

Hence $g(\lambda_{ij} x_{ij} - \lambda L) \le g(\lambda_{ij} x_{ij}) + g(\lambda L) = \lambda_{ij} g(x_{ij}) + \lambda g(L)$ as $(i,j \to \infty)$.
Hence ${}_2m^I(f,p)$ is a paranormed space. The rest of the result follows similarly.
□

Theorem 2.3. *A sequence $x = (x_{ij}) \in {}_2m^I(f,p)$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N} \times \mathbb{N}$ where $N_\epsilon = (m,n)$, m and n depending upon ϵ such that*

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|x_{ij} - x_{N_\epsilon}|^{p_{ij}}) < \epsilon\} \in {}_2m^I(f,p) \tag{1}$$

Proof. Suppose that $L = I - \lim x$. Then

$$B_\epsilon = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L|^{p_{ij}} < \frac{\epsilon}{2}\right\} \in m^I(f,p),\ \text{for all } \epsilon > 0.$$

Fixing some $N_\epsilon \in B_\epsilon$, we get

$$|x_{N_\epsilon} - x_{ij}|^{p_{ij}} \le |x_{N_\epsilon} - L|^{p_{ij}} + |L - x_{ij}|^{p_{ij}} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $(i,j) \in B_\epsilon$. Hence

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|x_{ij} - x_{N_\epsilon}|^{p_{ij}}) < \epsilon\} \in \ {}_2m^I(f,p).$$

Conversely, suppose that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|x_{ij} - x_{N_\epsilon}|^{p_{ij}}) < \epsilon\} \in \ {}_2m^I(f,p).$$

That is $\{(i,j) \in \mathbb{N} \times \mathbb{N} : (|x_{ij} - x_{N_\epsilon}|^{p_{ij}}) < \epsilon\} \in \ {}_2m^I(f,p)$ for all $\epsilon > 0$. Then the set

$$C_\epsilon = \{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in \ {}_2m^I(f,p) \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in \ {}_2m^I(f,p)$ as well as $C_{\frac{\epsilon}{2}} \in \ {}_2m^I(f,p)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in \ {}_2m^I(f,p)$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$$

that is

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \in J\} \in \ {}_2m^I(f,p)$$

that is

$$diam J \leq diam J_\epsilon$$

where the diam of J denotes the length of interval J. In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq \ldots\ldots \supseteq I_k \supseteq \ldots\ldots\ldots\ldots$$

with the property that $diam I_k \leq \frac{1}{2} diam I_{k-1}$ for (k=2,3,4,.....) and $\{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \in I_k\} \in \ {}_2m^I(f,p)$ for (k=1,2,3,4,......).
Then there exists a $\xi \in \cap I_k$ where $(i,j) \in \mathbb{N} \times \mathbb{N}$ such that $\xi = I - \lim x$. So that $f(\xi) = I - \lim f(x)$, that is $L = I - \lim f(x)$. ☐

Theorem 2.4. *Let $H = \sup_{i,j} p_{ij} < \infty$ and I be an admissible ideal. Then the following are equivalent.*

(a) $(x_{ij}) \in \ {}_2c^I(f,p)$;

(b) there exists$(y_{ij}) \in \ {}_2c(f,p)$ such that $x_{ij} = y_{ij}$, for a.a.k.r.I;

(c) there exists$(y_{ij}) \in \ {}_2c(f,p)$ and $(x_{ij}) \in \ {}_2c_0^I(f,p)$ such that $x_{ij} = y_{ij} + z_{ij}$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$ and $\{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|y_{ij} - L|^{p_{ij}}) \geq \epsilon\} \in I$;

(d) there exists a subset $J \times K$ where $J = \{j_1, j_2, ...\}$ and $K = \{k_1 < k_2 ....\}$ of $\mathbb{N} \times \mathbb{N}$ such that $J \times K \in \mathcal{L}(I)$ and $\lim_{n\to\infty} f(|x_{j_n k_n} - L|^{p_{j_n k_n}}) = 0$.

Proof. (a) implies (b)
Let $(x_{ij}) \in \ {}_2c^I(f,p)$. Then there exists $L \in \mathbb{C}$ such that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|x_{ij} - L|^{p_{ij}}) \geq \epsilon\} \in I.$$

Let (m_t) *and* (n_t) be increasing sequences with m_t *and* $n_t \in \mathbb{N}$ such that

$$\{(i,j) \leq (m_t, n_t) : f(|x_{ij} - L|^{p_{ij}}) \geq \epsilon\} \in I.$$

Define a sequence (y_{ij}) as

$$y_{ij} = x_{ij}, \text{ for all } (i,j) \leq (m_1, n_1).$$

For $m_t < k \le m_{t+1}, t \in \mathbb{N}$.

$$y_{ij} = \begin{cases} x_{ij}, & \text{if } |x_{ij} - L|^{p_{ij}} < t^{-1}, \\ L, & otherwise. \end{cases}$$

Then $(y_{ij}) \in {}_2c(f,p)$ and form the following inclusion

$$\{k \le m_t : x_{ij} \ne y_{ij}\} \subseteq \{(i,j) \le m_t : f(|x_{ij} - L|^{p_{ij}}) \ge \epsilon\} \in I.$$

We get $x_{ij} = y_{ij}$, for a.a.k.r.I.

(b) implies (c)

For $(x_{ij}) \in {}_2c^I(f,p)$. Then there exists $(y_{ij}) \in {}_2c(f,p)$ such that $x_{ij} = y_{ij}$, for a.a.k.r.I. Let $K = \{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \ne y_{ij}\}$, then $(i,j) \in I$.
Define a sequence (z_{ij}) as

$$z_{ij} = \begin{cases} x_{ij} - y_{ij}, & \text{if } (i,j) \in K, \\ 0, & otherwise. \end{cases}$$

Then $z_{ij} \in {}_2c_0^I(f,p)$ and $y_{ij} \in {}_2c(f,p)$.

(c) implies (d)

Let $P_1 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|x_{ij}|^{p_{ij}}) \ge \epsilon\} \in I$ and

$$K = P_1^c = \{(i_1,j_1) < (i_2,j_2) < (i_3,j_3) < ...\} \in \pounds(I).$$

Then we have $\lim\limits_{n\to\infty} f(|x_{i_nj_n} - L|^{p_{i_nj_n}}) = 0$.

(d) implies (a)

Let $K = \{(i_1,j_1) < (i_2,j_2) < (i_3,j_3) < ...\} \in \pounds(I)$ and $\lim\limits_{n\to\infty} f(|x_{i_nj_n} - L|^{p_{i_nj_n}}) = 0$. Then for an $\epsilon > 0$, and Lemma 1.10, we have

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|x_{ij} - L|^{p_{ij}}) \ge \epsilon\} \subseteq K^c \cup \{(i,j) \in K : f(|x_{ij} - L|^{p_{ij}}) \ge \epsilon\}.$$

Thus $(x_{ij}) \in {}_2c^I(f,p)$. □

Theorem 2.5. *Let (p_{ij}) and (q_{ij}) be a sequence of positive real numbers. Then ${}_2m_0^I(f,p) \supseteq {}_2m_0^I(f,q)$ if and only if $\lim\limits_{(i,j)\in K} \inf \frac{p_{ij}}{q_{ij}} > 0$, where $K^c \subseteq \mathbb{N} \times \mathbb{N}$ such that $K \in I$.*

Proof. Let $\lim\limits_{(i,j)\in K} \inf \frac{p_{ij}}{q_{ij}} > 0$ and $(x_{ij}) \in {}_2m_0^I(f,q)$. Then there exists $\beta > 0$ such that $p_{ij} > \beta q_{ij}$, for all sufficiently large $(i,j) \in K$.
Since $(x_{ij}) \in {}_2m_0^I(f,q)$, for a given $\epsilon > 0$, we have

$$B_0 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|x_{ij}|^{q_{ij}}) \ge \epsilon\} \in I$$

Let $G_0 = K^c \cup B_0$ Then $G_0 \in I$. Then for all sufficiently large $(i,j) \in G_0$,

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|x_{ij}|^{p_{ij}}) \ge \epsilon\} \subseteq \{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|x_{ij}|^{\beta q_{ij}}) \ge \epsilon\} \in I.$$

Therefore $(x_{ij}) \in {}_2m_0^I(f,p)$. □

Corollary 2.6. *Let (p_{ij}) and (q_{ij}) be two sequences of positive real numbers. Then ${}_2m_0^I(f,q) \supseteq {}_2m_0^I(f,p)$ if and only if* $\lim_{(i,j)\in K} \inf \frac{q_{ij}}{p_{ij}} > 0$, *where $K^c \subseteq \mathbb{N}\times\mathbb{N}$ such that $K \in I$.*

Theorem 2.7. *Let (p_{ij}) and (q_{ij}) be two sequences of positive real numbers. Then ${}_2m_0^I(f,q) = {}_2m_0^I(f,p)$ if and only if* $\lim_{(i,j)\in K} \inf \frac{p_{ij}}{q_{ij}} > 0$, *and* $\lim_{(i,j)\in K} \inf \frac{q_{ij}}{p_{ij}} > 0$, *where $K \subseteq \mathbb{N}\times\mathbb{N}$ such that $K^c \in I$.*

Proof. On combining Theorem 2.5 and 2.6 we get the required result. □

Theorem 2.8. *Let $h = \inf_{(i,j)} p_{ij}$ and $H = \sup_{(i,j)} p_{ij}$. Then the following results are equivalent. (a) $H < \infty$ and $h > 0$. (b) ${}_2c_0^I(f,p) = {}_2c_0^I$.*

Proof. Suppose that $H < \infty$ and $h > 0$,then the inequalities $min\{1, s^h\} \leq s^{p_{ij}} \leq max\{1, s^H\}$ hold for any $s > 0$ and for all $(i,j) \in \mathbb{N}\times\mathbb{N}$. Therefore the equivalence of (a) and (b) is obvious. □

Theorem 2.9. *Let f be a modulus function. Then ${}_2c_0^I(f,p) \subset {}_2c^I(f,p) \subset {}_2l_\infty^I(f,p)$. The strict inclusions follow from the strict inclusion of the spaces ${}_2c_0^I(f) \subset {}_2c^I(f) \subset {}_2l_\infty^I(f)$ (see [13]).*

Proof. Let $(x_{ij}) \in {}_2c^I(f,p)$. Then there exists $L \in \mathbb{C}$ such that

$$I - \lim f(|x_{ij} - L|^{p_{ij}}) = 0.$$

We have

$$f(|x_{ij}|^{p_{ij}}) \leq \frac{1}{2} f(|x_{ij} - L|^{p_{ij}}) + \frac{1}{2} f(|L|^{p_{ij}}).$$

Taking supremum over (i,j) both sides we get $(x_{ij}) \in l_\infty^I(f,p)$ and the inclusion ${}_2c_0^I(f,p) \subset {}_2c^I(f,p)$ is obvious. Hence ${}_2c_0^I(f,p) \subset {}_2c^I(f,p) \subset {}_2l_\infty^I(f,p)$ and the inclusions are proper. □

Theorem 2.10. *If $H = \sup_{i,j} p_{ij} < \infty$,then for any modulus f, we have ${}_2l_\infty^I \subset M(m^I(f,p))$, where the inclusion may be proper.*

Proof. Let $a \in {}_2l_\infty^I$.This implies that $\sup_{i,j} |a_{ij}| < 1 + K$. for some $K > 0$ and all (i,j). Therefore $x = (x_{ij}) \in {}_2m^I(f,p)$ implies

$$\sup_{i,j} f(|a_{ij}x_{ij}|^{p_{ij}}) \leq (1+K)^H \sup_{i,j} f(|x_{ij}|^{p_{ij}}) < \infty.$$

which gives ${}_2l_\infty^I \subset M({}_2m^I(f,p))$. To show that the inclusion may be proper, consider the case when $p_{ij} = \frac{1}{(ij)}$ for all (i,j). Take $a_{ij} = (i \times j)$ for all (i,j). Therefore $x \in {}_2m^I(f,p)$ implies

$$\sup_{i,j} f(|a_{ij}x_{ij}|^{p_{ij}}) \leq \sup_{i,j} f(|i \times j|^{\frac{1}{(ij)}}) \sup_{i,j} f(|x_{ij}|^{\frac{1}{ij}}) < \infty.$$

Thus in this case $a = (a_{ij}) \in M({}_2m^I(f,p))$ while $a \notin {}_2l_\infty^I$. □

Theorem 2.11. *The function* $\hbar : {}_2m^I(f,p) \to \boldsymbol{R}$ *is the Lipschitz function,where* ${}_2m^I(f,p) = {}_2c^I(f,p) \cap {}_2l_\infty(f,p)$, *and hence uniformly continuous.*

Proof. Let $x, y \in {}_2m^I(f,p)$,$x \neq y$. Then the sets

$$A_x = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - \hbar(x)|^{p_{ij}} \geq ||x-y||\} \in I,$$

$$A_y = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - \hbar(y)|^{p_{ij}} \geq ||x-y||\} \in I.$$

Here $||x-y|| = \sup_{i,j} f(|x_{ij} - y_{ij}|^{\frac{p_{ij}}{M}})$ where $M = max\{1, \sup_{i,j} p_{ij}\}$
Thus the sets,

$$B_x = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - \hbar(x)|^{p_{ij}} < ||x-y||\} \in {}_2m^I(f,p),$$

$$B_y = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - \hbar(y)|^{p_{ij}} < ||x-y||\} \in {}_2m^I(f,p).$$

Hence also $B = B_x \cap B_y \in {}_2m^I(f,p)$, so that $B \neq \phi$. Now taking (i,j) in B such that

$$|\hbar(x) - \hbar(y)|^{p_{ij}} \leq |\hbar(x) - x_{ij}|^{p_{ij}} + |x_{ij} - y_{ij}|^{p_{ij}} + |y_{ij} - \hbar(y)|^{p_{ij}} \leq 3||x-y||.$$

Thus $\hbar$ is a Lipschitz function. For ${}_2m_0^I(f,p)$ the result can be proved similarly. □

Theorem 2.12. *If* $x, y \in {}_2m^I(f,p)$,*then* $(x.y) \in {}_2m^I(f,p)$ *and* $\hbar(xy) = \hbar(x)\hbar(y)$.

Proof. For $\epsilon > 0$

$$B_x = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - \hbar(x)|^{p_{ij}} < \epsilon\} \in {}_2m^I(f,p),$$

$$B_y = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - \hbar(y)|^{p_{ij}} < \epsilon\} \in {}_2m^I(f,p).$$

Now,

$$|x_{ij}y_{ij} - \hbar(x)\hbar(y)|^{p_{ij}} = |x_{ij}y_{ij} - x_{ij}\hbar(y) + x_{ij}\hbar(y) - \hbar(x)\hbar(y)|^{p_{ij}}$$
$$\leq |x_{ij}|^{p_{ij}}|y_{ij} - \hbar(y)|^{p_{ij}} + |\hbar(y)|^{p_{ij}}|x_{ij} - \hbar(x)|^{p_{ij}} \quad (2)$$

As ${}_2m^I(f,p) \subseteq l_\infty(f,p)$,there exists an $M \in \boldsymbol{R}$ such that $|x_{ij}|^{p_{ij}} < M$ and $|\hbar(y)|^{p_k} < M$. Using eqn(2) we get

$$|x_{ij}y_{ij} - \hbar(x)\hbar(y)|^{p_{ij}} \leq M\epsilon + M\epsilon = 2M\epsilon$$

For all $(i,j) \in B_x \cap B_y \in {}_2m^I(f.p)$. Hence $(x.y) \in m^I(f,p)$ and $\hbar(xy) = \hbar(x)\hbar(y)$. For ${}_2m_0^I(f,p)$ the result can be proved similarly. □

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