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TRANSITIVE AND ABSORBENT FILTERS OF LATTICE IMPLICATION ALGEBRAS

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Abstract. The notion of transitive filters is introduced in lattice implication algebras. A necessary and sufficient condition is derived for every filter to become a transitive filter. Some sufficient conditions are also derived for a filter to become a transitive filter. The concept of absorbent filters is introduced and their properties are studied. A set of equivalent conditions is obtained for a filter to become an absorbent filter.

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1. Introduction

In order to research the logical system whose propositional value is given in a lattice from the semantic viewpoint, Y. Xu [6] proposed the concept of lattice implication algebras and discussed some of their properties in [5, 6]. Y. Xu and K.Y. Qin [7] introduced the notion of filters in a lattice implication algebra and investigated some of their properties. Later, Y.B. Jun [1, 2, 3] introduced various types of filters like implicative filter, positive implicative filter, associative filters and fantastic filters e.t.c. in lattice implication algebras and studied their properties. In [2], Y.B. Jun, Y. Xu and K.Y. Qin introduced the notion of positive implicative filters and observed a relation between implicative filters and positive implicative filters.

In this paper, the concept of transitive filters is introduced in lattice implication algebras and their properties are studied. A necessary and sufficient condition is obtained for every filter of a lattice implication algebra to become a transitive filter. Some sufficient conditions are also derived for a filter to become a transitive filter. The notion of absorbent filters are introduced in lattice implication algebra and their properties are studied. In [2], Y.B. Jun studied

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the properties of positive implicative filters and proved that every positive implicative filter is an implicative filter and left to the readers an open problem on the converse of the result. In this paper, a set of equivalent conditions is derived for the existing of the converse which in turns lead to an equivalence among absorbent, implicative and positive implicative filters.

2. Preliminaries

In this section, we cite some elementary aspects and results which have taken mostly from [1], [2], [6] and [7] those will be used in the sequel of this paper as well as for a ready reference of the reader.

Definition 2.1 ([6])**.** By a lattice implication algebra we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution $'$ and a binary operation $\to$ satisfying the following axioms:

(1) $x \to (y \to z) = y \to (x \to z)$
(2) $x \to x = 1$
(3) $x \to y = y' \to x'$
(4) $x \to y = y \to x = 1 \Rightarrow x = y$
(5) $(x \to y) \to y = (y \to x) \to x$
(6) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$
(7) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$

for all $x, y, z \in L$.

Note that the conditions (6) and (7) are equivalent to the conditions

(6′) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$ and
(7′) $x \to (y \vee z) = (x \to y) \vee (x \to z)$, respectively.

We can define a partial ordering $\leq$ on a lattice implication algebra L by $x \leq y$ if and only if $x \to y = 1$.

Theorem 2.2 ([6])**.** *In a lattice implication algebra L, the following conditions hold for all $x, y, z \in L$.*

(1) $0 \to x = 1, 1 \to x = x$ *and* $x \to 1 = 1$
(2) $x' = x \to 0$
(3) $x \to y \leq (y \to z) \to (x \to z)$
(4) $x \leq y$ *implies* $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$
(5) $((x \to y) \to y) \to y = x \to y$
(6) $x \to ((x \to y) \to y) = 1$

In what follows, L would mean a lattice implication algebra unless otherwise specified. In [7], Y. Xu and K. Y. Qin defined the notions of a filter and an implicative filter in a lattice implication algebra.

Definition 2.3 ([7])**.** Let $(L, \vee, \wedge, \to, 0, 1)$ be a lattice implication algebra. A subset F of L is called a filter of L if it satisfies for all $x, y \in L$:

$(F1)$ $1 \in F$

$(F2)$ $x \in F$ and $x \to y \in F$ imply $y \in F$

Lemma 2.4 ([7]). *Every filter F of L has the following property:*

$$x \leq y \text{ and } x \in F \text{ imply } y \in F \text{ for all } x, y \in L$$

Definition 2.5 ([1]). A subset F of a lattice implication algebra L is called an implicative filter of L, if it satisfies $(F1)$ and

$(F3)$ $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$ for all $x, y, z \in L$

Definition 2.6 ([2]). A subset F of a lattice implication algebra L is called a positive implicative filter of L if it satisfies $(F1)$ and

$(F4)$ $x \to ((y \to z) \to y) \in F$ and $x \in F$ imply $y \in F$ for all $x, y, z \in L$

Definition 2.7 ([2]). Let x be a fixed element of L. A subset F of L is called an associative filter of L with respect to x if it satisfies the following:

$(A1)$ $1 \in F$

$(A1)$ $x \to (y \to z) \in F$ and $x \to y \in F$ imply $z \in F$

for all $y, z \in L$. An associative filter of L with respect to all $x \neq 0$ is called an associative filter of L.

Theorem 2.8 ([2]). *Every associative filter is a filter.*

Theorem 2.9 ([2]). *Let F be a filter of L. Then F is an associative filter if and only if it satisfies the following property:*

$$x \to (y \to z) \in F \text{ implies } (x \to y) \to z \in F \quad \text{for all } x, y, z \in L$$

Definition 2.10 ([3]). A non-empty subset F of L is called a fantastic filter of a lattice implication algebra L if it satisfies $(F1)$ and

$(F7)$ $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$ for all $x, y, z \in L$

Theorem 2.11 ([3]). *Every positive implicative filter is a fantastic filter.*

3. Transitive filters

In this section, the concept of transitive filters is introduced in a lattice implication algebra and the properties of these filters are studied. An equivalent condition is obtained for every lattice filter to become a transitive filter.

Definition 3.1. A non-empty subset F of a lattice implication algebra L is called a transitive filter if it satisfies the following properties, for all $x, y, z \in L$:

$(T1)$ $1 \in F$

$(T2)$ $x \to y \in F$, $y \to z \in F$ imply that $x \to z \in F$

It is evident that the principal filter $[1)$ of a lattice implication algebra is a transitive filter. For, consider $x, y, z \in L$ be such that $x \to y \in [1)$ and $y \to z \in [1)$. Then $x \leq y$ and $y \leq z$. Hence $x \leq z$, which implies $x \to z = 1 \in [1)$.

Example 3.2. Let $L = \{0, a, b, c, 1\}$ be a set. Define the partially ordered relation on L as $0 < a < b < c < 1$ and also define $x \wedge y = min\{x, y\}$, $x \vee y = max\{x, y\}$ for all $x, y \in L$. Define the operations "$'$" and $\to$ on L are as follows:

x	x'
0	1
a	c
b	b
c	a
1	0

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	b	c	1	1
1	0	a	b	c	1

Then clearly L is a lattice implication algebra. Now consider the set $F = \{a, b, c, 1\}$. It can be easily verified that F is a transitive filter of L.

Some properties of transitive filters can be observed in the following.

Lemma 3.3. *Let F be a transitive filter of L. Then we have the following:*

(1) $x \in F, x \leq y$ imply that $y \in F$

(2) $x \in F, x \to y \in F$ imply that $y \in F$

(3) Set intersection of transitive filters is again a transitive filter

Proof. (1). Suppose $x \in F$ and $x \leq y$. Then we get that $x \to y = 1 \in F$. Since $1 \to x \in F$ and $x \to y \in F$, we get that $y = 1 \to y \in F$.
(2). Let $x \in F$ and $x \to y \in F$. Then we get that $1 \to x \in F$ and $x \to y \in F$. Since F is transitive, it yields that $y \in F$.
(3). It is clear. □

Theorem 3.4. *Every transitive filter of L is a lattice filter.*

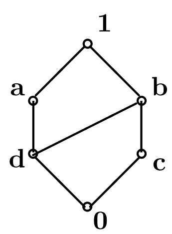
Proof. Let F be a transitive filter of a lattice implication algebra L. Let $x, y \in F$. We have $y \leq x \to y = 1 \wedge (x \to y) = (x \to x) \wedge (x \to y) = x \to (x \wedge y)$. By Lemma 3.3(1), we get that $x \to (x \wedge y) \in F$. Since $1 \to x \in F$ and F is transitive, it yields that $x \wedge y \in F$. Therefore F is a lattice filter of L. □

But the converse of the above Theorem 3.4 is not true. It can be observed in the following example.

Example 3.5. Let $L = \{0, a, b, c, d, 1\}$ be a set whose Hasse diagram is given in the following figure. Define a unary operation "$'$" and a binary operation $\to$ on L as in the following tables respectively:

x	x'
0	1
a	c
b	d
c	a
d	b
1	0

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1



Define the operations $\vee$ and $\wedge$ on L as follows:

$$x \vee y = (x \to y) \to y$$
$$x \wedge y = ((x' \to y') \to y')'$$

for all $x, y \in L$. Then L is a lattice implication algebra. It is easy to check that $F = \{b, 1\}$ is a lattice filter of L but not a transitive filter, since $a \to b = b \in F$ and $b \to c = b \in F$ but $a \to c = c \notin F$.

In the following, a sufficient condition is obtained for every lattice filter of a lattice implication algebra to become a transitive filter.

Theorem 3.6. *Let F be a lattice filter of L. If $x \wedge (x \to y) = x \wedge y$ for all $x, y \in L$, then F is a transitive filter.*

Proof. Let F be a filter of L. Let $x, y, z \in L$ be such that $x \to y \in F$ and $y \to z \in F$. Since $y \to z \leq x \to (y \to z)$, we get $x \to (y \to z) \in F$. Hence we get

$$\begin{aligned}(x \to y) \wedge (x \to z) &= x \to (y \wedge z)\\ &= x \to [y \wedge (y \to z)]\\ &= (x \to y) \wedge [x \to (y \to z)] \in F\end{aligned}$$

Thus $x \to z \in F$. Therefore F is a transitive filter. $\square$

From the Lemma 3.3(2), it can be easily observed that every transitive filter is a filter. However, in the following, a necessary and sufficient condition is derived for every filter of a lattice implication algebra to become a transitive filter.

Theorem 3.7. *Let F be a filter of a lattice implication algebra L. Then F is a transitive filter if and only if for all $x, y, z \in L$, it satisfies the following condition:*

$(T3)$ $\quad x \to y \in F,\ (x \to y) \to (y \to z) \in F$ *implies* $(x \to y) \to (x \to z) \in F$

Proof. Let F be a filter of L. Assume that F is a transitive filter of L. Let $x, y, z \in L$ be such that $x \to y \in F$ and $(x \to y) \to (y \to z) \in F$. Then we have the following consequence

$$\begin{aligned}(y \to z) \to [(x \to y) \to (x \to z)] &= (x \to y) \to [(y \to z) \to (x \to z)]\\ &= (x \to y) \to \{x \to [(y \to z) \to z]\}\\ &= (x \to y) \to \{x \to [(z \to y) \to y]\}\\ &= (x \to y) \to [(z \to y) \to (x \to y)]\\ &= (z \to y) \to [(x \to y) \to (x \to y)]\\ &= (z \to y) \to 1\\ &= 1 \in F\end{aligned}$$

Since $(x \to y) \to (y \to z) \in F$, $(y \to z) \to [(x \to y) \to (x \to z)] \in F$ and F is a transitive filter, we can conclude that $(x \to y) \to [(x \to y) \to (x \to z)] \in F$. Since $x \to y \in F$ and F is a filter, we get that $(x \to y) \to (x \to z) \in F$.

Conversely, assume the condition $(T3)$. Let $x \to y \in F$ and $y \to z \in F$. Then we get $y \to z \leq (x \to y) \to (y \to z) \in F$. By the assumed condition $(T3)$, we get $(x \to y) \to (x \to z) \in F$. Now $x \to y \in F, (x \to y) \to (x \to z) \in F$ and F is a filter, we get that $x \to z \in F$. Therefore F is a transitive filter. $\square$

In the following, some sufficient conditions are obtained for a filter of a lattice implication algebra to become a transitive filter.

Theorem 3.8. *Let F be a filter of L. Then F is a transitive filter if it satisfies the following condition:*

$$x \to (y \to z) \in F \textit{ implies } (x \to y) \to z \in F \textit{ for all } x, y, z \in L$$

Proof. Let F be a filter of L which is satisfying the given condition. Let $x, y, z \in L$ be such that $x \to y \in F$ and $y \to z \in F$. Since $y \to z \leq x \to (y \to z)$, we can get $x \to (y \to z) \in F$. By the assumed condition, we get $(x \to y) \to z \in F$. Since F is a filter and $x \to y \in F$, it yields that $z \in F$. Since $z \leq x \to z$, we get $x \to z \in F$. Therefore F is a transitive filter. $\square$

By the Theorem 2.9, the following corollary is a direct consequence.

Corollary 3.9. *Let F be a filter of L. If F is an associative filter, then it is a transitive filter.*

OPEN PROBLEM. Does the converse of the Corollary 3.9 hold?

4. Absorbent filters

In this section, the concept of absorbent filters is introduced in a lattice implication algebra and the properties of absorbent filters are studied. A set of equivalent conditions is obtained for every filter to become an absorbent filter.

Definition 4.1. A non-empty subset F of a lattice implication algebra L is called an absorbent filter if it satisfies the following properties, for all $x, y \in L$:

$(A1)$ $1 \in F$

$(A2)$ $(x \to y) \to x \in F$ imply that $x \in F$

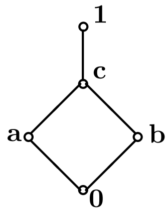
Theorem 4.2. *Every associative filter of L is an absorbent filter.*

Proof. Let F be an associative filter of L. Let $x, y \in L$ be such that $(x \to y) \to x \in F$. Then $(x \to y) \to (1 \to x) = 1 \to [(x \to y) \to x] = (x \to y) \to x \in F$. Since F is associative, it yields that $x = 1 \to x = [(x \to y) \to 1] \to x \in F$. Therefore F is an absorbent filter of L. $\square$

Example 4.3. Let $L = \{0, a, b, c, 1\}$ be a distributive lattice whose Hasse diagram is given in the following figure. Then clearly $(L, \vee, \wedge, \rightarrow, 0, 1)$ is a Heyting algebra where the binary operation $\rightarrow$ is defined as follows:

x	x'
0	1
a	b
b	a
c	c
1	0

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	c	1	1
b	a	c	1	1	1
c	c	c	c	1	1
1	0	a	b	c	1



Then $F = \{1, c, b\}$ is an absorbent filter of L. But F is not an associative filter, because of $b \rightarrow (c \rightarrow a) = b \rightarrow c = 1 \in F$ and $(b \rightarrow c) \rightarrow a = 1 \rightarrow a = a \notin F$.

In general, every filter of a lattice implication algebra need not be an absorbent filter. It can be seen from the Example 3.5. Consider the set $F = \{c, 1\}$. Then clearly F is a filter of L but not an absorbent filter. For consider $c, d \in L$. Then it is clear that $(d \rightarrow c) \rightarrow d = b \rightarrow d = c \in F$ and $d \notin F$.

In [2], Y.B. Jun proved that every positive implicative filter is an implicative filter and raised an open problem about the converse. However, in the following, we derive a set of equivalent conditions for every implicative filter to become a positive implicative filter which leads to a characterization of an absorbent filter.

Theorem 4.4. *Let F be a filter of L and $x, y, z \in L$. Then the following conditions are equivalent.*

(1) F is absorbent

(2) $a \in F$, $(x \rightarrow y) \rightarrow (a \rightarrow x) \in F$ imply $x \in F$

(3) F is implicative

(4) F is positive implicative

Proof. (1) $\Rightarrow$ (2): Assume that F is absorbent. Let $a \in F$, $(x \rightarrow y) \rightarrow (a \rightarrow x) \in F$. Then $a \rightarrow [(x \rightarrow y) \rightarrow x] \in F$. Since $a \in F$ and F is a filter, we get that $(x \rightarrow y) \rightarrow x \in F$. Since F is an absorbent filter, we get that $x \in F$.

(2) $\Rightarrow$ (3): Assume the condition (2). Suppose that $x \rightarrow (y \rightarrow z) \in F$ and $x \rightarrow y \in F$. We have $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) \leq (x \rightarrow y) \rightarrow [x \rightarrow (x \rightarrow z)]$. Since F is a filter and $x \rightarrow (y \rightarrow z) \in F$, we get $(x \rightarrow y) \rightarrow [x \rightarrow (x \rightarrow z)] \in F$. Since $x \rightarrow y \in F$ and F is a filter, we get $x \rightarrow (x \rightarrow z) \in F$. Put $x \rightarrow (x \rightarrow z) = a$. Then we have the following consequence

$$\begin{aligned}
[(x \rightarrow z) \rightarrow z] \rightarrow [a \rightarrow (x \rightarrow z)] &= a \rightarrow \{[(x \rightarrow z) \rightarrow z] \rightarrow (x \rightarrow z)\} \\
&= a \rightarrow \{x \rightarrow [((x \rightarrow z) \rightarrow z) \rightarrow z]\} \\
&= a \rightarrow [x \rightarrow (x \rightarrow z)] \\
&= a \rightarrow a
\end{aligned}$$

$$= 1 \in F$$

Then by the condition (2), it yields that $x \to z \in F$.

(3) $\Rightarrow$ (4): Assume that condition (3). Let $x, y, z \in L$ be such that $x \to [(y \to z) \to y] \in F$ and $x \in F$. Since F is a filter and $x \in F$, we get that $[(y \to z) \to y] \in F$. We have always $y' = y \to 0 \leq y \to z$. Hence

$$\begin{aligned}(y \to z) \to y &\leq y' \to y \\ &= y' \to y'' \\ &= y' \to (y' \to 0)\end{aligned}$$

Since F is a filter, we get $y' \to (y' \to 0) \in F$. Also $y' \to y' = 1 \in F$. Since F is an implicative filter, it yields that $y = y'' = y' \to 0 \in F$.

(4) $\Rightarrow$ (1): Assume that F is a positive implicative filter. Suppose $(x \to y) \to x \in F$. Then $1 \to [(x \to y) \to x] \in F$. Since F is positive implicative, we get $x = 1 \to x \in F$. Therefore F is an absorbent filter. □

Corollary 4.5. *Let F be a filter of L. If F is an absorbent filter of L, then it is a fantastic filter of L.*

Proof. Let F be an absorbent filter. Then by the main theorem, F is a positive implicative filter. From the Theorem 2.11, F is a fantastic filter. □

OPEN PROBLEM. Does the converse of the above Corollary hold?

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