# A NUMERICAL INVESTIGATION ON THE ZEROS OF THE TANGENT POLYNOMIALS 

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#### Abstract

In this paper, we observe the behavior of complex roots of the tangent polynomials $T_{n}(x)$, using numerical investigation. By means of numerical experiments, we demonstrate a remarkably regular structure of the complex roots of the tangent polynomials $T_{n}(x)$. Finally, we give a table for the solutions of the tangent polynomials $T_{n}(x)$.

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## 1. Introduction

In the 21st century, the computing environment would make more and more rapid progress. Numerical experiments of Bernoulli polynomials, Euler polynomials, Genocchi polynomials, and tangent polynomials have been the subject of extensive study in recent year and much progress have been made both mathematically and computationally(see [1-15]). Using computer, a realistic study for tangent polynomials $T_{n}(x)$ is very interesting. It is the aim of this paper to observe an interesting phenomenon of 'scattering' of the zeros of the tangent polynomials $T_{n}(x)$ in complex plane. Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. Tangent numbers was introduced in [6]. First, we introduce the tangent numbers and tangent polynomials. As well known definition, the tangent numbers $T_{n}$ (cf. [6]) are defined by

$$
T_{0}=1, \quad \tan (t)=\sum_{n=0}^{\infty}(-1)^{n+1} T_{2 n+1} \frac{t^{2 n+1}}{(2 n+1)!}, \quad T_{2 n}=0, \quad(n \in \mathbb{N})
$$

[^0]Here is the list of the first tangent's numbers:

$$
\begin{aligned}
& T_{0}=1 \\
& T_{1}=-1 \\
& T_{3}=2 \\
& T_{5}=-16 \\
& T_{7}=272 \\
& T_{9}=-7936 \\
& T_{11}=353792 \\
& T_{13}=-22368256 \\
& T_{15}=1903757312 \\
& T_{17}=-209865342976 \\
& T_{19}=29088885112832 \\
& T_{21}=-4951498053124096 \\
& T_{23}=1015423886506852352 \\
& T_{25}=-246921480190207983616 \\
& T_{27}=70251601603943959887872 \\
& T_{29}=-23119184187809597841473536 \\
& T_{31}=8713962757125169296170811392
\end{aligned}
$$

In [6], we introduced the tangent polynomials $T_{n}(x)$. The tangent polynomials $T_{n}(x)$ are defined by the generating function:

$$
\begin{equation*}
F(x, t)=\sum_{n=0}^{\infty} T_{n}(x) \frac{t^{n}}{n!}=\left(\frac{2}{e^{2 t}+1}\right) e^{x t} \tag{1.1}
\end{equation*}
$$

where we use the technique method notation by replacing $T(x)^{n}$ by $T_{n}(x)$ symbolically. Note that

$$
T_{n}(x)=\sum_{k=0}^{n}\binom{n}{k} T_{k} x^{n-k} .
$$

In the special case $x=0$, we define $T_{n}(0)=T_{n}$.
Because

$$
\frac{\partial F}{\partial x}(x, t)=t F(x, t)=\sum_{n=0}^{\infty} \frac{d T_{n}}{d x}(x) \frac{t^{n}}{n!},
$$

it follows the important relation

$$
\frac{d T_{k}}{d x}(x)=k T_{k-1}(x)
$$

Since

$$
\begin{aligned}
\int_{a}^{b} T_{n}(x) d x & =\sum_{l=0}^{n}\binom{n}{l} T_{l} \int_{a}^{b} x^{n-l} d x \\
& =\left.\sum_{l=0}^{n}\binom{n}{l} T_{l} \frac{x^{n-l+1}}{n-l+1}\right|_{a} ^{b} \\
& =\left.\frac{1}{n+1} \sum_{l=0}^{n+1}\binom{n+1}{l} T_{l} x^{n-l+1}\right|_{a} ^{b}
\end{aligned}
$$

we see that

$$
\begin{equation*}
\int_{a}^{b} T_{n}(x) d x=\frac{T_{n+1}(b)-T_{n+1}(a)}{n+1} . \tag{1.2}
\end{equation*}
$$

Since $T_{n}(0)=T_{n}$, by (1.2), we have the following theorem.
Theorem 1.1. For $n \in \mathbb{N}$, we have

$$
T_{n}(x)=T_{n}+n \int_{0}^{x} T_{n-1, q}(t) d t .
$$

Then, it is easy to deduce that $T_{k}(x)$ are polynomials of degree $k$. Here is the list of the first tangent's polynomials:

$$
\begin{aligned}
& T_{0}(x)=1, \\
& T_{1}(x)=x-1, \\
& T_{2}(x)=x^{2}-2 x, \\
& T_{3}(x)=x^{3}-3 x^{2}+2, \\
& T_{4}(x)=x^{4}-4 x^{3}+8 x, \\
& T_{5}(x)=x^{5}-5 x^{4}+20 x^{2}-16, \\
& T_{6}(x)=x^{6}-6 x^{5}+40 x^{3}-96 x, \\
& T_{7}(x)=x^{7}-7 x^{6}+70 x^{4}-336 x^{2}+272, \\
& T_{8}(x)=x^{8}-8 x^{7}+112 x^{5}-896 x^{3}+2176 x, \\
& T_{9}(x)=x^{9}-9 x^{8}+168 x^{6}-2016 x^{4}+9792 x^{2}-7936, \\
& T_{10}(x)=x^{10}-10 x^{9}+240 x^{7}-4032 x^{5}+32640 x^{3}-79360 x .
\end{aligned}
$$

## 2. Beautiful zeros of the tangent polynomials

In this section, we display the shapes of the tangent polynomials $T_{n}(x)$ and we investigate the zeros of the tangent polynomials $T_{n}(x)$. For $n=1, \cdots, 10$, we can draw a plot of $T_{n}(x)$, respectively. This shows the ten plots combined into one. We display the shape of $T_{n}(x),-7 \leq x \leq 7$ (Figure 1). Next, we investigate the beautiful zeros of the $T_{n}(x)$ by using a computer. We plot the zeros of $T_{n}(x)$ for $n=20,30,40,60$, and $x \in \mathbb{C}$ (Figure 2). Stacks of zeros of $T_{n}(x)$ for $1 \leq n \leq 50$ from a 3-D structure are presented(Figure 3). In Figure 2(top-left), we choose


Figure 1. Curve of tangent polynomials $T_{n}(x)$
$n=20$. In Figure 2(top-right), we choose $n=30$. In Figure 2(bottom-left), we choose $n=40$. In Figure 2(bottom-right), we choose $n=50$.

Our numerical results for approximate solutions of real zeros of $T_{n}(x)$ are displayed in Table 1. The results are obtained by Mathematica software.

Table 1. Numbers of real and complex zeros of $T_{n}(x)$

| degree $n$ | real zeros | complex zeros |
| :---: | :---: | :---: |
| 1 | 1 | 0 |
| 2 | 2 | 0 |
| 3 | 3 | 0 |
| 4 | 4 | 0 |
| 5 | 5 | 0 |
| 6 | 2 | 4 |
| 7 | 3 | 4 |
| 8 | 4 | 4 |
| 9 | 5 | 4 |
| 10 | 6 | 4 |
| 11 | 3 | 8 |
| 12 | 4 | 8 |
| 13 | 5 | 8 |
| 14 | 6 | 8 |

We observe a remarkably regular structure of the complex roots of tangent polynomials. We hope to verify a remarkably regular structure of the complex roots of tangent polynomials(Table 1).


Figure 2. Zeros of $T_{n}(x)$ for $n=20,30,40,50$
Next, we calculated an approximate solution satisfying $T_{n}(x)$ and $x \in \mathbb{R}$. The results are given in Table 2.

Table 2. Approximate solutions of $T_{n}(x)=0, x \in \mathbb{R}$

| degree $n$ | 1.0000 |  |  |
| :---: | :---: | :---: | :---: |
| 1 | $0, \quad 2.0000$ |  |  |
| 2 | $-0.73205, \quad 1.0000, \quad 2.7321$ |  |  |
| 3 | $-1.2361, \quad 2.0000, \quad 3.2361$ |  |  |
| 4 | $-1.2361, \quad 1.0000, \quad 3.2361, \quad 3.2361$ |  |  |
| 5 | $0, \quad 2.0000$ |  |  |
| 6 | $-1.2361, \quad 1.99546, \quad 1.0000, \quad 2.9955$ |  |  |
| 7 | $0, \quad 2.0000, \quad 3.8647$ |  |  |
| 8 | $-1.8647, \quad 1.0002, \quad 1.000, \quad 3.0002, \quad 4.4395$ |  |  |
| 9 | $-2.4395, \quad-1.0, \quad 2.0000, \quad 4.0300, \quad 4.7304$ |  |  |
| 10 | $-2.7304, \quad-2.0300, \quad 0, \quad 1.000, \quad 3.0000$ |  |  |
| 11 | $-1.0000, \quad 1.0$ |  |  |



Figure 3. Stacks of zeros of $T_{n}(x), 1 \leq n \leq 50$

We plot the real zeros of the tangent polynomials $T_{n}(x)$ for $x \in \mathbb{C}$ (Figure 4).


Figure 4. Real zeros of $T_{n}(x), 1 \leq n \leq 50$

## 3. Observations

Since

$$
\begin{aligned}
\sum_{n=0}^{\infty} T_{n}(2-x) \frac{(-t)^{n}}{n!} & =F(2-x,-t)=\frac{2}{e^{-2 t}+1} e^{(2-x)(-t)} \\
& =\frac{2}{e^{2 t}+1} e^{x t}=F(x, t)=\sum_{n=0}^{\infty} T_{n}(x) \frac{t^{n}}{n!}
\end{aligned}
$$

we have the following theorem.
Theorem 3.1. For any positive integer n, we have

$$
\begin{equation*}
T_{n}(x)=(-1)^{n} T_{n}(2-x) . \tag{3.1}
\end{equation*}
$$

From (1.1), we have

$$
\begin{align*}
\sum_{n=0}^{\infty}\left(T_{n}(x+2)+T_{n}(x)\right) \frac{t^{n}}{n!} & =2 \sum_{n=0}^{\infty}(-1)^{n} e^{(2 n+x+2) t}+2 \sum_{n=0}^{\infty}(-1)^{n} e^{(2 n+x) t} \\
& =2 e^{x t} \\
& =\sum_{n=0}^{\infty} 2 x^{n} \frac{t^{n}}{n!} \tag{3.2}
\end{align*}
$$

Comparing the coefficient of $\frac{t^{n}}{n!}$ on both sides of (3.2), we get the following theorem.

Theorem 3.2. For any positive integer n, we have

$$
\begin{equation*}
T_{n}(x+2)+T_{n}(x)=2 x^{n} . \tag{3.3}
\end{equation*}
$$

By (3.3), we have the following corollary.
Corollary 3.3. For $n \in \mathbb{N}$, we have

$$
T_{n}=-T_{n}(2)
$$

The question is: what happens with the reflexive symmetry (3.1), when one considers tangent polynomials? Prove that $T_{n}(x), x \in \mathbb{C}$, has $\operatorname{Re}(x)=1$ reflection symmetry in addition to the usual $\operatorname{Im}(x)=0$ reflection symmetry analytic complex functions(Figures 2-4). Prove that $T_{n}(x)=0$ has $n$ distinct solutions. Find the numbers of complex zeros $C_{T_{n}(x)}$ of $T_{n}(x), \operatorname{Im}(x) \neq 0$. Since $n$ is the degree of the polynomial $T_{n}(x)$, the number of real zeros $R_{T_{n}(x)}$ lying on the real plane $\operatorname{Im}(x)=0$ is then $R_{T_{n}(x)}=n-C_{T_{n}(x)}$, where $C_{T_{n}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{T_{n}(x)}$ and $C_{T_{n}(x)}$. More studies and results in this subject we may see references [8], [9], [13], [14].

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