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ON HERMITE-HADAMARD-TYPE INEQUALITIES FOR DIFFERENTIABLE QUASI-CONVEX FUNCTIONS ON THE CO-ORDINATES†

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Abstract. In this paper, a new lemma is established and several new inequalities for differentiable co-ordinated quasi-convex functions in two variables which are related to the left-hand side of Hermite-Hadamard type inequality for co-ordinated quasi-convex functions in two variables are obtained.

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1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, we have the following double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}. \tag{1}$$

This remarkable result is well known in the literature as the Hermite-Hadamard inequality for convex mapping.

Definition 1.1. A function $f : [a, b] \to R$ is said to be quasi-convex on $[a, b]$, if

$$f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\}$$

holds for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$.

Clearly, any convex function is a quasi-convex function, but the converse is not generally true.

In [4], S. S. Dragomir defined convex functions on the co-ordinates as following:

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Let us consider the bidimensional interval $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$, a mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda) f(z,w)$$

holds for all (x,y), $(z,w) \in \Delta$ and $\lambda \in [0,1]$.

A function $f : \Delta \to \mathbb{R}$ is said to be co-ordinated convex on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex for all $y \in [c,d]$ and $x \in [a,b]$.

A formal definition for co-ordinated convex functions may be stated as follows:

Definition 1.2. A function $f : \Delta \to \mathbb{R}$ is said to be convex on co-ordinates on Δ if the inequality

$$\begin{aligned} f(\lambda x + (1-\lambda)z, ty + (1-t)w) &\leq \lambda t f(x,y) + \lambda(1-t) f(x,w) \\ &\quad + (1-\lambda) t f(z,y) + (1-t)(1-\lambda) f(z,w) \end{aligned}$$

holds for all (x,y), (z,y), (x,w), $(z,w) \in \Delta$ and $t, \lambda \in [0,1]$.

S. S. Dragomir in [4] established the following Hadamard-type inequalities for co-ordinated convex functions in a rectangle from the plane $\mathbb{R}^2$.

Theorem 1.3. *Suppose that $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx \\ &\leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}. \end{aligned} \tag{2}$$

The concept of quasi-convex function on the co-ordinates was introduced by Özdemir et al. in ([9], 2012).

Let us consider the bidimensional interval $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$, a mapping $f : \Delta \to \mathbb{R}$ is said to be a quasi-convex function on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \max\{f(x,y), f(z,w)\}$$

holds for all (x,y), $(z,w) \in \Delta$ and $\lambda \in [0,1]$.

A function $f : \Delta \to \mathbb{R}$ is said to be quasi-convex functions on the co-ordinates if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are quasi-convex for all $y \in [c,d]$ and $x \in [a,b]$.

A formal definition of quasi-convex functions on the co-ordinates as follows:

Definition 1.4. A function $f : \Delta \to \mathbb{R}$ is said to be a quasi-convex function on the co-ordinates on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, ty + (1-t)w) \leq \max\{f(x,y), f(x,w), f(z,y), f(z,w)\}$$

holds for all (x,y), (z,y), (x,w), $(z,w) \in \Delta$ with $t, \lambda \in [0,1]$.

In ([10], 2012), M. Z. Sarıkaya et al. established some inequalities for co-ordinated convex functions based on the following lemma.

Lemma 1.5. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\frac{\partial^2 f}{\partial u \partial v} \in L(\Delta)$, then the following equality holds:*

$$\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx$$
$$- \frac{1}{2}\left[\frac{1}{b-a}\int_a^b [f(x,c)+f(x,d)]dx + \frac{1}{d-c}\int_c^d [f(a,y)+f(b,y)]dy\right]$$
$$= \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 (1-2u)(1-2v)\frac{\partial^2 f}{\partial u \partial v}(ua+(1-u)b, vc+(1-v)d)dudv.$$

In ([7], 2012), M. E. Özdemir et al. established the following inequalities for quasi-convex functions on the co-ordinates based on Lemma 1.5.

Theorem 1.6. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\left|\frac{\partial^2 f}{\partial u \partial v}\right|$ is a quasi-convex function on the co-ordinates on Δ, then the following inequality holds:*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx - A\right|$$
$$\leq \frac{(b-a)(d-c)}{16}\max\left\{\left|\frac{\partial^2 f}{\partial u \partial v}(a,c)\right|, \left|\frac{\partial^2 f}{\partial u \partial v}(a,d)\right|, \left|\frac{\partial^2 f}{\partial u \partial v}(b,c)\right|, \left|\frac{\partial^2 f}{\partial u \partial v}(b,d)\right|\right\},$$

where

$$A = \frac{1}{2}\Big[\frac{1}{b-a}\int_a^b [f(x,c)+f(x,d)]dx + \frac{1}{d-c}\int_c^d [f(a,y)+f(b,y)]dy\Big].$$

Theorem 1.7. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\left|\frac{\partial^2 f}{\partial u \partial v}\right|^q$ is a quasi-convex function on the co-ordinates on Δ and $q > 1$, then:*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx - A\right|$$
$$\leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \times \left\{\max\left(\left|\frac{\partial^2 f}{\partial u \partial v}(a,c)\right|^q, \left|\frac{\partial^2 f}{\partial u \partial v}(a,d)\right|^q, \left|\frac{\partial^2 f}{\partial u \partial v}(b,c)\right|^q, \left|\frac{\partial^2 f}{\partial u \partial v}(b,d)\right|^q\right)\right\}^{\frac{1}{q}},$$

where A is defined in Theorem 1.6 and $\frac{1}{p} + \frac{1}{q} = 1$.

Some new integral inequalities that are related to the Hermite-Hadamard type for co-ordinated convex functions are also established by many authors.

In ([1], [2], 2008), M. Alomari and M. Darus defined co-ordinated s-convex functions and proved some inequalities based on this definition. In ([5], 2009), M. A. Latif and M. Alomari defined co-ordinated h-convex functions and proved some inequalities based on this definition. In ([3], 2009), Alomari et al. established some Hadamard-type inequalities for coordinated log-convex functions.

In ([6], 2012), M. A. Latif and S. S. Dragomir obtained some new Hadamard type inequalities for differentiable co-ordinated convex and concave functions in two variables which are related to the left-hand side of Hermite-Hadamard type inequality for co-ordinated convex functions in two variables based on the following lemma:

Lemma 1.8. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\frac{\partial^2 f}{\partial u \partial v} \in L(\Delta)$, then the following equality holds:*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx + f\Big(\frac{a+b}{2}, \frac{c+d}{2}\Big) \\
&- \frac{1}{b-a} \int_a^b f\Big(x, \frac{c+d}{2}\Big)dx - \frac{1}{d-c} \int_c^d f\Big(\frac{a+b}{2}, y\Big)dy \\
&= (b-a)(d-c) \int_0^1 \int_0^1 K(u,v) \frac{\partial^2 f}{\partial u \partial v}(ua + (1-u)b, vc + (1-v)d)dudv,
\end{aligned}$$

where

$$K(u,v) = \begin{cases} uv, & (u,v) \in \Big[0, \frac{1}{2}\Big] \times \Big[0, \frac{1}{2}\Big] \\ u(v-1), & (u,v) \in \Big[0, \frac{1}{2}\Big] \times \Big(\frac{1}{2}, 1\Big] \\ (u-1)v, & (u,v) \in \Big(\frac{1}{2}, 1\Big] \times \Big[0, \frac{1}{2}\Big] \\ (u-1)(v-1), & (u,v) \in \Big(\frac{1}{2}, 1\Big] \times \Big(\frac{1}{2}, 1\Big]. \end{cases}$$

Theorem 1.9 ([6]). *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\Big|\frac{\partial^2 f}{\partial u \partial v}\Big|$ is convex on the co-ordinates on Δ, then the following inequality holds:*

$$\begin{aligned}
&\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx + f\Big(\frac{a+b}{2}, \frac{c+d}{2}\Big) - A \right| \\
&\le \frac{(b-a)(d-c)}{16} \left(\frac{\Big|\frac{\partial^2 f}{\partial u \partial v}(a,c)\Big| + \Big|\frac{\partial^2 f}{\partial u \partial v}(a,d)\Big| + \Big|\frac{\partial^2 f}{\partial u \partial v}(b,c)\Big| + \Big|\frac{\partial^2 f}{\partial u \partial v}(b,d)\Big|}{4} \right),
\end{aligned}$$

where

$$A = \frac{1}{b-a} \int_a^b f\Big(x, \frac{c+d}{2}\Big)dx + \frac{1}{d-c} \int_c^d f\Big(\frac{a+b}{2}, y\Big)dy.$$

Theorem 1.10 ([6]). *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\Big|\frac{\partial^2 f}{\partial u \partial v}\Big|^q$ is convex on the* *co-ordinates on* Δ *and* $p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1$, *then the following inequality holds:*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx + f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big) - A\right|$$
$$\leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left(\frac{\left|\frac{\partial^2 f}{\partial u\partial v}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial u\partial v}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial u\partial v}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial u\partial v}(b,d)\right|^q}{4}\right)^{\frac{1}{q}},$$

where A *is as given in Theorem 1.9.*

For recent results and generalizations concerning Hermite-Hadamard type inequality for differentiable co-ordinated convex functions see ([8], 2012) and the references given therein.

In this paper, we establish several new inequalities for differentiable co-ordinated quasi-convex functions in two variables which are related to the left-hand side of Hermite-Hadamard type inequality for co-ordinated quasi-convex functions in two variables.

2. Main results

To establishing our results, we need the following lemma.

Lemma 2.1. *Let* $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be a partial differentiable mapping on* $\Delta := [a,b]\times[c,d]$ *in* $\mathbb{R}^2$ *with* $a<b$ *and* $c<d$. *If* $\frac{\partial^4 f}{\partial u^2\partial v^2} \in L(\Delta)$, *then the following equality holds:*

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx + f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)$$
$$-\frac{1}{b-a}\int_a^b f\Big(x,\frac{c+d}{2}\Big)dx - \frac{1}{d-c}\int_c^d f\Big(\frac{a+b}{2},y\Big)dy$$
$$= \frac{(b-a)^2(d-c)^2}{4}\int_0^1\int_0^1 M(u,v)\frac{\partial^4 f}{\partial u^2\partial v^2}(ua+(1-u)b, vc+(1-v)d)dudv,$$

where

$$M(u,v) = \begin{cases} u^2v^2, & (u,v)\in\left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{2}\right]\\ u^2(v-1)^2, & (u,v)\in\left[0,\frac{1}{2}\right]\times\left(\frac{1}{2},1\right]\\ (u-1)^2v^2, & (u,v)\in\left(\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right]\\ (u-1)^2(v-1)^2, & (u,v)\in\left(\frac{1}{2},1\right]\times\left(\frac{1}{2},1\right]. \end{cases}$$

Proof. Since

$$\int_0^1\int_0^1 M(u,v)\frac{\partial^4 f}{\partial u^2\partial v^2}(ua+(1-u)b, vc+(1-v)d)dudv$$

$$
\begin{aligned}
&= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} u^2 v^2 \frac{\partial^4 f}{\partial u^2 \partial v^2}(ua+(1-u)b, vc+(1-v)d)dudv \\
&\quad + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 u^2 (v-1)^2 \frac{\partial^4 f}{\partial u^2 \partial v^2}(ua+(1-u)b, vc+(1-v)d)dudv \\
&\quad + \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} (u-1)^2 v^2 \frac{\partial^4 f}{\partial u^2 \partial v^2}(ua+(1-u)b, vc+(1-v)d)dudv \\
&\quad + \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 (u-1)^2 (v-1)^2 \frac{\partial^4 f}{\partial u^2 \partial v^2}(ua+(1-u)b, vc+(1-v)d)dudv.
\end{aligned}
$$

Thus, by integration by parts, it follows that

$$
\begin{aligned}
&\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} u^2 v^2 \frac{\partial^4 f}{\partial u^2 \partial v^2}(ua+(1-u)b, vc+(1-v)d)dudv \\
&= \int_0^{\frac{1}{2}} u^2 \left\{ v^2 \frac{1}{c-d} \frac{\partial^3 f}{\partial u^2 \partial v}(ua+(1-u)b, vc+(1-v)d) \Big|_0^{\frac{1}{2}} \right. \\
&\quad \left. - \frac{1}{c-d} \int_0^{\frac{1}{2}} 2v \frac{\partial^3 f}{\partial u^2 \partial v}(ua+(1-u)b, vc+(1-v)d)dv \right\} du \\
&= \frac{1}{4} \frac{1}{c-d} \int_0^{\frac{1}{2}} u^2 \frac{\partial^3 f}{\partial u^2 \partial v}(ua+(1-u)b, \frac{c+d}{2})du \\
&\quad - \frac{1}{c-d} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} u^2 2v \frac{\partial^3 f}{\partial u^2 \partial v}(ua+(1-u)b, vc+(1-v)d)dvdu \\
&= \frac{1}{4} \frac{1}{c-d} \int_0^{\frac{1}{2}} u^2 \frac{\partial^3 f}{\partial u^2 \partial v}(ua+(1-u)b, \frac{c+d}{2})du \\
&\quad - \frac{1}{c-d} \int_0^{\frac{1}{2}} 2v \left\{ u^2 \frac{1}{a-b} \frac{\partial^2 f}{\partial u \partial v}(ua+(1-u)b, vc+(1-v)d) \Big|_0^{\frac{1}{2}} \right. \\
&\quad \left. - \int_0^{\frac{1}{2}} 2v 2u \frac{1}{a-b} \frac{\partial^2 f}{\partial u \partial v}(ua+(1-u)b, vc+(1-v)d)du \right\} dv \\
&= \frac{1}{4} \frac{1}{c-d} \int_0^{\frac{1}{2}} u^2 \frac{\partial^3 f}{\partial u^2 \partial v}(ua+(1-u)b, \frac{c+d}{2})du \\
&\quad - \frac{1}{2} \frac{1}{c-d} \frac{1}{a-b} \int_0^{\frac{1}{2}} v \frac{\partial^2 f}{\partial u \partial v}(\frac{a+b}{2}, vc+(1-v)d)dv \\
&\quad + \frac{4}{(a-b)(c-d)} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} uv \frac{\partial^2 f}{\partial u \partial v}(ua+(1-u)b, vc+(1-v)d)dudv.
\end{aligned}
$$

Similarly, we can get

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 u^2(v-1)^2\frac{\partial^4 f}{\partial u^2\partial v^2}(ua+(1-u)b,vc+(1-v)d)dudv\\
&=-\frac{1}{4}\frac{1}{c-d}\int_0^{\frac{1}{2}}u^2\frac{\partial^3 f}{\partial u^2\partial v}(ua+(1-u)b,\frac{c+d}{2})du\\
&\quad-\frac{1}{2}\frac{1}{c-d}\frac{1}{a-b}\int_{\frac{1}{2}}^1(v-1)\frac{\partial^2 f}{\partial u\partial v}(\frac{a+b}{2},vc+(1-v)d)dv\\
&\quad+\frac{4}{(a-b)(c-d)}\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 u(v-1)\frac{\partial^2 f}{\partial u\partial v}(ua+(1-u)b,vd+(1-v)c)dudv,
\end{aligned}$$

$$\begin{aligned}
&\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}(u-1)^2v^2\frac{\partial^4 f}{\partial u^2\partial v^2}(ua+(1-u)b,vc+(1-v)d)dudv\\
&=\frac{1}{4}\frac{1}{c-d}\int_{\frac{1}{2}}^1(u-1)^2\frac{\partial^3 f}{\partial u^2\partial v}(ua+(1-u)b,\frac{c+d}{2})du\\
&\quad+\frac{1}{2}\frac{1}{c-d}\frac{1}{a-b}\int_0^{\frac{1}{2}}v\frac{\partial^2 f}{\partial u\partial v}(\frac{a+b}{2},vd+(1-v)c)dv\\
&\quad+\frac{4}{(a-b)(c-d)}\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}(u-1)v\frac{\partial^2 f}{\partial u\partial v}(ua+(1-u)b,vd+(1-v)c)dudv,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1(u-1)^2(v-1)^2\frac{\partial^4 f}{\partial u^2\partial v^2}(ua+(1-u)b,vc+(1-v)d)dudv\\
&=-\frac{1}{4}\frac{1}{c-d}\int_{\frac{1}{2}}^1(u-1)^2\frac{\partial^3 f}{\partial u^2\partial v}(ua+(1-u)b,\frac{c+d}{2})du\\
&\quad+\frac{1}{2}\frac{1}{c-d}\frac{1}{a-b}\int_{\frac{1}{2}}^1(v-1)\frac{\partial^2 f}{\partial u\partial v}(\frac{a+b}{2},vd+(1-v)c)dv\\
&\quad+\frac{4}{(a-b)(c-d)}\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1(u-1)(v-1)\frac{\partial^2 f}{\partial u\partial v}(ua+(1-u)b,vd+(1-v)c)dudv.
\end{aligned}$$

Now

$$\begin{aligned}
&\int_0^1\int_0^1 M(u,v)\frac{\partial^4 f}{\partial u^2\partial v^2}(ua+(1-u)b,vc+(1-v)d)dudv\\
&=\frac{4}{(a-b)(c-d)}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}uv\frac{\partial^2 f}{\partial u\partial v}(ua+(1-u)b,vc+(1-v)d)dudv\\
&\quad+\frac{4}{(a-b)(c-d)}\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 u(v-1)\frac{\partial^2 f}{\partial u\partial v}(ua+(1-u)b,vd+(1-v)c)dudv
\end{aligned}$$

$$+\frac{4}{(a-b)(c-d)}\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}}(u-1)v\frac{\partial^2 f}{\partial u\partial v}(ua+(1-u)b,vd+(1-v)c)dudv$$
$$+\frac{4}{(a-b)(c-d)}\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}(u-1)(v-1)\frac{\partial^2 f}{\partial u\partial v}(ua+(1-u)b,vd+(1-v)c)dudv.$$

Multiplying the both sides by $\frac{(b-a)^2(d-c)^2}{4}$ and using Lemma 1.8, which completes the proof. □

Theorem 2.2. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\left|\frac{\partial^4 f}{\partial u^2\partial v^2}\right|$ is a quasi-convex function on the co-ordinates on Δ, then the following inequality holds:*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx+f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)-A\right|$$
$$\le\frac{(b-a)^2(d-c)^2}{24^2}\max\left\{\left|\frac{\partial^4 f}{\partial u^2\partial v^2}(a,c)\right|,\left|\frac{\partial^4 f}{\partial u^2\partial v^2}(a,d)\right|,\left|\frac{\partial^4 f}{\partial u^2\partial v^2}(b,c)\right|,\left|\frac{\partial^4 f}{\partial u^2\partial v^2}(b,d)\right|\right\},$$

where

$$A=\frac{1}{b-a}\int_a^b f\Big(x,\frac{c+d}{2}\Big)dx+\frac{1}{d-c}\int_c^d f\Big(\frac{a+b}{2},y\Big)dy.$$

Proof. From Lemma 2.1, we obtain

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx+f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)-A\right|$$
$$\le\frac{(b-a)^2(d-c)^2}{4}\int_0^1\int_0^1 M(u,v)\left|\frac{\partial^4 f}{\partial u^2\partial v^2}(ua+(1-u)b,vc+(1-v)d)\right|dudv.$$

Because $\left|\frac{\partial^4 f}{\partial u^2\partial v^2}\right|$ is quasi-convex on the co-ordinates on Δ, then one has

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx+f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)-A\right|$$
$$\le\frac{(b-a)^2(d-c)^2}{4}\int_0^1\int_0^1 M(u,v)\times\max\left\{\left|\frac{\partial^4 f}{\partial u^2\partial v^2}(a,c)\right|,\left|\frac{\partial^4 f}{\partial u^2\partial v^2}(a,d)\right|,\right.$$
$$\left.\left|\frac{\partial^4 f}{\partial u^2\partial v^2}(b,c)\right|,\left|\frac{\partial^4 f}{\partial u^2\partial v^2}(b,d)\right|\right\}dudv$$
$$=\frac{(b-a)^2(d-c)^2}{4}\max\left\{\left|\frac{\partial^4 f}{\partial u^2\partial v^2}(a,c)\right|,\left|\frac{\partial^4 f}{\partial u^2\partial v^2}(a,d)\right|,\left|\frac{\partial^4 f}{\partial u^2\partial v^2}(b,c)\right|,\left|\frac{\partial^4 f}{\partial u^2\partial v^2}(b,d)\right|\right\}$$

$$\int_0^1\int_0^1 M(u,v)dudv$$
$$= \frac{(b-a)^2(d-c)^2}{24^2}\max\left\{\left|\frac{\partial^4 f}{\partial u^2\partial v^2}(a,c)\right|, \left|\frac{\partial^4 f}{\partial u^2\partial v^2}(a,d)\right|, \left|\frac{\partial^4 f}{\partial u^2\partial v^2}(b,c)\right|, \left|\frac{\partial^4 f}{\partial u^2\partial v^2}(b,d)\right|\right\}.$$

On the other hand, we have

$$\int_0^1\int_0^1 M(u,v)dudv = \frac{1}{144}.$$

The proof is completed. □

The corresponding version for powers of the absolute value of the fourth partial derivative is incorporated in the following theorems.

Theorem 2.3. *Let $f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\left|\frac{\partial^4 f}{\partial u^2\partial v^2}\right|^q$ is a quasi-convex function on the co-ordinates on Δ and $q > 1$, then:*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx + f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big) - A\right|$$
$$\le \frac{(b-a)^2(d-c)^2}{64}\Big(\frac{1}{2p+1}\Big)^{\frac{2}{p}}$$
$$\times\left(\max\left\{\left|\frac{\partial^4 f}{\partial u^2\partial v^2}(a,c)\right|^q + \left|\frac{\partial^4 f}{\partial u^2\partial v^2}(a,d)\right|^q + \left|\frac{\partial^4 f}{\partial u^2\partial v^2}(b,c)\right|^q + \left|\frac{\partial^4 f}{\partial u^2\partial v^2}(b,d)\right|^q\right\}\right)^{\frac{1}{q}},$$

where A is defined in Theorem 3.1 and $\frac{1}{p}+\frac{1}{q} = 1$.

Proof. From Lemma 2.1, we obtain

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx + f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big) - A\right|$$
$$\le \frac{(b-a)^2(d-c)^2}{4}\int_0^1\int_0^1 M(u,v)\left|\frac{\partial^4 f}{\partial u^2\partial v^2}(ua+(1-u)b, vc+(1-v)d)\right|dudv.$$

By using the well known Hölder's inequality for double integrals, then one has

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx + f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big) - A\right|$$
$$\le \frac{(b-a)^2(d-c)^2}{4}\left(\int_0^1\int_0^1 [M(u,v)]^p dudv\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^4 f}{\partial u^2\partial v^2}(ua+(1-u)b, vc+(1-v)d)\right|^q dudv\right)^{\frac{1}{q}}.$$

Because $\left|\dfrac{\partial^4 f}{\partial u^2 \partial v^2}\right|^q$ is quasi-convex on the co-ordinates on Δ, then one has

$$\begin{aligned}&\int_0^1\int_0^1 \left|\frac{\partial^4 f}{\partial u^2 \partial v^2}(ua+(1-u)b, vc+(1-v)d)\right|^q dudv\\ &\leq \max\left\{\left|\frac{\partial^4 f}{\partial u^2 \partial v^2}(a,c)\right|^q+\left|\frac{\partial^4 f}{\partial u^2 \partial v^2}(a,d)\right|^q+\left|\frac{\partial^4 f}{\partial u^2 \partial v^2}(b,c)\right|^q+\left|\frac{\partial^4 f}{\partial u^2 \partial v^2}(b,d)\right|^q\right\}.\end{aligned}$$

We note that

$$\begin{aligned}\int_0^1\int_0^1 [M(u,v)]^p dudv &= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} u^{2p}v^{2p}dudv+\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 u^{2p}(1-v)^{2p}dudv\\ &\quad+\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}(1-u)^{2p}v^{2p}dudv+\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1(1-u)^{2p}(1-v)^{2p}dudv\\ &=\frac{4}{(2p+1)^2}\Big(\frac{1}{2}\Big)^{4p+2}.\end{aligned}$$

Hence, it follows that

$$\begin{aligned}&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx+f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)-A\right|\\ &\leq\frac{(b-a)^2(d-c)^2}{64}\Big(\frac{1}{2p+1}\Big)^{\frac{2}{p}}\\ &\quad\times\left(\max\left\{\left|\frac{\partial^4 f}{\partial u^2 \partial v^2}(a,c)\right|^q+\left|\frac{\partial^4 f}{\partial u^2 \partial v^2}(a,d)\right|^q+\left|\frac{\partial^4 f}{\partial u^2 \partial v^2}(b,c)\right|^q+\left|\frac{\partial^4 f}{\partial u^2 \partial v^2}(b,d)\right|^q\right\}\right)^{\frac{1}{q}}.\end{aligned}$$

So, the proof is completed. □

Theorem 2.4. *Let $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\left|\dfrac{\partial^4 f}{\partial u^2 \partial v^2}\right|^q$ is a quasi-convex function on the co-ordinates on Δ and $q\geq 1$, then:*

$$\begin{aligned}&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx+f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)-A\right|\\ &\leq\frac{(b-a)^2(d-c)^2}{24^2}\\ &\quad\times\max\left\{\left|\frac{\partial^4 f}{\partial u^2 \partial v^2}(a,c)\right|^q+\left|\frac{\partial^4 f}{\partial u^2 \partial v^2}(a,d)\right|^q+\left|\frac{\partial^4 f}{\partial u^2 \partial v^2}(b,c)\right|^q+\left|\frac{\partial^4 f}{\partial u^2 \partial v^2}(b,d)\right|^q\right\},\end{aligned}$$

where A is defined in Theorem 3.1.

Proof. From Lemma 2.1, we obtain

$$\begin{aligned}&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx+f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)-A\right|\\ &\leq\frac{(b-a)^2(d-c)^2}{4}\int_0^1\int_0^1 M(u,v)\left|\frac{\partial^4 f}{\partial u^2 \partial v^2}(ua+(1-u)b, vc+(1-v)d)\right|dudv.\end{aligned}$$

By using the well known power mean inequality for double integrals, then one has

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx+f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)-A\right|$$
$$\leq \frac{(b-a)^2(d-c)^2}{4}\left(\int_0^1\int_0^1 M(u,v)dudv\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1 M(u,v)\Big|\frac{\partial^4 f}{\partial u^2\partial v^2}(ua+(1-u)b,vc+(1-v)d)\Big|^q dudv\right)^{\frac{1}{q}}.$$

Because $\left|\frac{\partial^4 f}{\partial u^2\partial v^2}\right|^q$ is quasi-convex on the co-ordinates on Δ, then one has

$$\int_0^1\int_0^1\Big|\frac{\partial^4 f}{\partial u^2\partial v^2}(ua+(1-u)b,vc+(1-v)d)\Big|^q dudv$$
$$\leq \max\left\{\Big|\frac{\partial^4 f}{\partial u^2\partial v^2}(a,c)\Big|^q+\Big|\frac{\partial^4 f}{\partial u^2\partial v^2}(a,d)\Big|^q+\Big|\frac{\partial^4 f}{\partial u^2\partial v^2}(b,c)\Big|^q+\Big|\frac{\partial^4 f}{\partial u^2\partial v^2}(b,d)\Big|^q\right\}.$$

Thus, it follows that

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx+f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)-A\right|$$
$$\leq \frac{(b-a)^2(d-c)^2}{4}\left(\int_0^1\int_0^1 M(u,v)dudv\right)^{1-\frac{1}{q}}\left(\int_0^1\int_0^1 M(u,v)dudv\right)^{\frac{1}{q}}$$
$$\times\max\left\{\Big|\frac{\partial^4 f}{\partial u^2\partial v^2}(a,c)\Big|^q+\Big|\frac{\partial^4 f}{\partial u^2\partial v^2}(a,d)\Big|^q+\Big|\frac{\partial^4 f}{\partial u^2\partial v^2}(b,c)\Big|^q+\Big|\frac{\partial^4 f}{\partial u^2\partial v^2}(b,d)\Big|^q\right\}.$$

Thus, we get the following inequality

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx+f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)-A\right|$$
$$\leq \frac{(b-a)^2(d-c)^2}{24^2}$$
$$\times\max\left\{\Big|\frac{\partial^4 f}{\partial u^2\partial v^2}(a,c)\Big|^q+\Big|\frac{\partial^4 f}{\partial u^2\partial v^2}(a,d)\Big|^q+\Big|\frac{\partial^4 f}{\partial u^2\partial v^2}(b,c)\Big|^q+\Big|\frac{\partial^4 f}{\partial u^2\partial v^2}(b,d)\Big|^q\right\},$$

which complete the proof. □

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