

GROWTH OF POLYNOMIALS HAVING ZEROS ON THE DISK

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ABSTRACT. A well known result due to Ankeny and Rivlin [1] states that if $p(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree n satisfying $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$

$$\max_{|z|=R} |p(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z)|.$$

It was proposed by Professor R.P. Boas Jr. to obtain an inequality analogous to this inequality for polynomials having no zeros in $|z| < k$, $k > 0$. In this paper, we obtain some results in this direction, by considering polynomials of degree $n \geq 2$, having all its zeros on the disk $|z| = k$, $k \leq 1$.

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1. Introduction

For an arbitrary entire function $f(z)$, let $M(f, r) = \max_{|z|=r} |f(z)|$. As a consequence of Maximum Modulus Principle [6, Vol. I, p. 137, Problem III, 269] it is known that if $p(z)$ is a polynomial of degree n , then

$$M(p, R) \leq R^n M(p, 1), \quad R \geq 1. \quad (1.1)$$

The result is best possible and equality holds for polynomials having zeros at the origin.

Ankeny and Rivlin [1] considered polynomials not vanishing in the interior of the unit circle and obtained refinement of inequality (1.1) by proving that if $p(z) \neq 0$ in $|z| < 1$, then

$$M(p, R) \leq \left(\frac{R^n + 1}{2} \right) M(p, 1), \quad R \geq 1. \quad (1.2)$$

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The inequality (1.2) is sharp and equality holds for $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.

While trying to obtain inequality analogous to (1.2) for polynomials not vanishing in $|z| < k$, $k \leq 1$, recently the authors [2] proved the following result.

Theorem 1.1 ([2]). *If $p(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then for every positive integer s*

$$\{M(p, R)\}^s \leq \left(\frac{k^{n-1}(1+k) + (R^{ns} - 1)}{k^{n-1} + k^n} \right) \{M(p, 1)\}^s, \quad R \geq 1. \quad (1.3)$$

By involving the coefficients of $p(z)$, Dewan and Ahuja [2] in the same paper obtained the following refinement of Theorem 1.1.

Theorem 1.2 ([2]). *If $p(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then for every positive integer s*

$$\{M(p, R)\}^s \leq \frac{1}{k^n} \frac{\left(n|a_n|\{k^n(1+k^2) + k^2(R^{ns} - 1)\} + |a_{n-1}|\{2k^n + R^{ns} - 1\} \right)}{2|a_{n-1}| + n|a_n|(1+k^2)} \{M(p, 1)\}^s, \quad R \geq 1. \quad (1.4)$$

In this paper, we restrict ourselves to the class of polynomials of degree $n \geq 2$ having all its zeros on $|z| = k$, $k \leq 1$ and obtain improvement and generalization of Theorems 1.1 & 1.2. More precisely, we prove

Theorem 1.3. *If $p(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree $n \geq 2$ having all its zeros on $|z| = k$, $k \leq 1$, then for every positive integer s and $R \geq 1$*

$$\begin{aligned} \{M(p, R)\}^s &\leq \frac{k^{n-1}(1+k) + (R^{ns} - 1)}{k^{n-1} + k^n} \{M(p, 1)\}^s \\ &\quad - s|a_1| \left(\frac{R^{ns} - 1}{ns} - \frac{R^{ns-2} - 1}{ns - 2} \right) \{M(p, 1)\}^{s-1}, \quad \text{if } n > 2 \end{aligned} \quad (1.5)$$

and

$$\begin{aligned} \{M(p, R)\}^s &\leq \frac{k^{n-1}(1+k) + (R^{ns} - 1)}{k^{n-1} + k^n} \{M(p, 1)\}^s \\ &\quad - s|a_1| \left(\frac{R^{ns} - 1}{ns} - \frac{R^{ns-1} - 1}{ns - 1} \right) \{M(p, 1)\}^{s-1}, \quad \text{if } n = 2. \end{aligned} \quad (1.6)$$

The following result immediately follows by choosing $s = 1$ in Theorem 1.3.

Corollary 1.4. *If $p(z) = \sum_{j=0}^n a_j z^j$ is polynomial of degree $n \geq 2$ having all its zeros on $|z| = k$, $k \leq 1$, then for $R \geq 1$*

$$\begin{aligned} M(p, R) &\leq \frac{k^{n-1}(1+k) + R^n - 1}{k^{n-1} + k^n} M(p, 1) \\ &\quad - |a_1| \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n - 2} \right), \quad \text{if } n > 2 \end{aligned} \quad (1.7)$$

and

$$M(p, R) \leq \frac{k^{n-1}(1+k) + R^n - 1}{k^{n-1} + k^n} M(p, 1) - |a_1| \left(\frac{R^n - 1}{n} - \frac{R^{n-1} - 1}{n-1} \right), \text{ if } n = 2. \quad (1.8)$$

Our next result is a refinement of Theorem 1.3, the proof of which is omitted as it follows on same lines as proved in [2] by the same authors.

Theorem 1.5. *If $p(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree $n \geq 2$ having all its zeros on $|z| = k$, $k \leq 1$, then for every positive integer s and $R \geq 1$*

$$\{M(p, R)\}^s \leq \frac{1}{k^n} \frac{\left(n|a_n| \{k^n(1+k^2) + k^2(R^{ns} - 1)\} + |a_{n-1}| \{2k^n + R^{ns} - 1\} \right)}{2|a_{n-1}| + n|a_n|(1+k^2)} \{M(p, 1)\}^s - s|a_1| \left(\frac{R^{ns} - 1}{ns} - \frac{R^{ns-2} - 1}{ns-2} \right) \{M(p, 1)\}^{s-1}, \text{ if } n > 2 \quad (1.9)$$

and

$$\{M(p, R)\}^s \leq \frac{1}{k^n} \frac{\left(n|a_n| \{k^n(1+k^2) + k^2(R^{ns} - 1)\} + |a_{n-1}| \{2k^n + R^{ns} - 1\} \right)}{2|a_{n-1}| + n|a_n|(1+k^2)} \{M(p, 1)\}^s - s|a_1| \left(\frac{R^{ns} - 1}{ns} - \frac{R^{ns-1} - 1}{ns-1} \right) \{M(p, 1)\}^{s-1}, \text{ if } n = 2. \quad (1.10)$$

Remark 1.1. The above theorem also generalize as well as improves upon Theorem 1.2 for $n \geq 2$.

If we choose $s = 1$ in Theorem 2, we get the following result.

Corollary 1.6. *If $p(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree $n \geq 2$ having all its zeros on $|z| = k$, $k \leq 1$, then for $R \geq 1$*

$$M(p, R) \leq \frac{1}{k^n} \frac{\left(n|a_n| \{k^n(1+k^2) + k^2(R^n - 1)\} + |a_{n-1}| \{2k^n + R^n - 1\} \right)}{2|a_{n-1}| + n|a_n|(1+k^2)} M(p, 1) - |a_1| \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right), \text{ if } n > 2 \quad (1.11)$$

and

$$M(p, R) \leq \frac{1}{k^n} \frac{\left(n|a_n| \{k^n(1+k^2) + k^2(R^n - 1)\} + |a_{n-1}| \{2k^n + R^n - 1\} \right)}{2|a_{n-1}| + n|a_n|(1+k^2)} M(p, 1) - |a_1| \left(\frac{R^n - 1}{n} - \frac{R^{n-1} - 1}{n-1} \right), \text{ if } n = 2. \quad (1.12)$$

2. Proof of Theorems

For the proof of these theorems, we need the following lemmas. The first lemma is due to Govil [5].

Lemma 2.1. *If $p(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{k^{n-1} + k^n} \max_{|z|=1} |p(z)|. \quad (2.13)$$

Lemma 2.2. *If $p(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{k^n} \left(\frac{n|a_n|k^2 + |a_{n-1}|}{2|a_{n-1}| + n|a_n|(1+k^2)} \right) \max_{|z|=1} |p(z)|. \quad (2.14)$$

The above lemma is due to Dewan and Mir [3].

Lemma 2.3. *If $p(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree n , then for all $R \geq 1$,*

$$\max_{|z|=R} |p(z)| \leq R^n M(p, 1) - (R^n - R^{n-2})|p(0)|, \text{ if } n > 1 \quad (2.15)$$

and

$$\max_{|z|=R} |p(z)| \leq R M(p, 1) - (R - 1)|p(0)|, \text{ if } n = 1. \quad (2.16)$$

The above result is due to Frappier, Rahman and Ruscheweyh [4].

Proof of Theorem 1.3. We first consider the case when $p(z)$ is of degree $n > 2$. Note that for every θ , $0 \leq \theta < 2\pi$ and $R \geq 1$, we have

$$\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s = \int_1^R \frac{d}{dt} \{p(te^{i\theta})\}^s dt = \int_1^R s \{p(te^{i\theta})\}^{s-1} p'(te^{i\theta}) e^{i\theta} dt.$$

Then

$$|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| \leq s \int_1^R |p(te^{i\theta})|^{s-1} |p'(te^{i\theta})| dt. \quad (2.17)$$

Since $p(z)$ is of degree $n > 2$, the polynomial $p'(z)$ is of degree $(n-1) \geq 2$, hence applying inequality (2.15) of Lemma 2.3 to $p'(z)$, we have for $r \geq 1$ and $0 \leq \theta < 2\pi$

$$|p'(re^{i\theta})| \leq r^{n-1} M(p', 1) - (r^{n-1} - r^{n-3})|p'(0)|. \quad (2.18)$$

Inequality (2.18) in conjunction with inequalities (2.17) and (1.1), yields for $n > 2$ and $R \geq 1$

$$\begin{aligned} |\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| &\leq s \int_1^R (t^n M(p, 1))^{s-1} [t^{n-1} M(p', 1) \\ &\quad - (t^{n-1} - t^{n-3})|p'(0)|] dt \\ &= s \int_1^R t^{ns-1} \{M(p, 1)\}^{s-1} M(p', 1) \end{aligned}$$

$$\begin{aligned}
& - (t^{ns-1} - t^{ns-3}) \{M(p, 1)\}^{s-1} |p'(0)| dt \\
& = s \left[\frac{R^{ns} - 1}{ns} \{M(p, 1)\}^{s-1} M(p', 1) \right. \\
& \quad \left. - \left(\frac{R^{ns} - 1}{ns} - \frac{R^{ns-2}}{ns-2} \right) \{M(p, 1)\}^{s-1} |p'(0)| \right].
\end{aligned}$$

On applying Lemma 2.1 to the above inequality, we get for $n > 2$

$$\begin{aligned}
|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| & \leq \frac{R^{ns} - 1}{k^{n-1} + k^n} \{M(p, 1)\}^s \\
& - s \left(\frac{R^{ns} - 1}{ns} - \frac{R^{ns-2} - 1}{ns-2} \right) \{M(p, 1)\}^{s-1} |p'(0)|.
\end{aligned}$$

This gives

$$\begin{aligned}
\{M(p, R)\}^s & \leq \frac{R^{ns} - 1 + k^{n-1} + k^n}{k^{n-1} + k^n} \{M(p, 1)\}^s \\
& - s \left(\frac{R^{ns} - 1}{ns} - \frac{R^{ns-2} - 1}{ns-2} \right) \{M(p, 1)\}^{s-1} |p'(0)|
\end{aligned}$$

from which proof of inequality (1.5) follows.

The proof of inequality (1.6) follows on the same lines as that of (1.5), but instead of using (2.15) of Lemma 2.3 we use (2.16) of the same lemma. \square

Proof of Theorem 1.5. We first consider the case when polynomial $p(z)$ is of degree $n > 2$, then the polynomial $p'(z)$ is of degree $(n-1) \geq 2$, hence applying inequality (2.15) of Lemma 2.3 to $p'(z)$, we have for $r \geq 1$ and $0 \leq \theta < 2\pi$

$$|p'(re^{i\theta})| \leq r^{n-1} M(p', 1) - (r^{n-1} - r^{n-3}) |p'(0)|. \quad (2.19)$$

Now for every θ , $0 \leq \theta < 2\pi$ and $R \geq 1$, we have

$$\begin{aligned}
\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s & = \int_1^R \frac{d}{dt} \{p(te^{i\theta})\}^s dt \\
& = \int_1^R s \{p(te^{i\theta})\}^{s-1} p'(te^{i\theta}) e^{i\theta} dt \\
|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| & \leq s \int_1^R |p(te^{i\theta})|^{s-1} |p'(te^{i\theta})| dt. \quad (2.20)
\end{aligned}$$

Inequality (2.20) in conjunction with inequality (1.1) and (2.19), gives for $n > 2$

$$\begin{aligned}
|\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| & \leq s \int_1^R (t^n M(p, 1))^{s-1} [t^{n-1} M(p', 1) - (t^{n-1} - t^{n-3}) |p'(0)|] dt \\
& = s \int_1^R t^{ns-1} \{M(p, 1)\}^{s-1} M(p', 1) \\
& \quad - (t^{ns-1} - t^{ns-3}) \{M(p, 1)\}^{s-1} |p'(0)| dt \\
& = s \left[\frac{R^{ns} - 1}{ns} \{M(p, 1)\}^{s-1} M(p', 1) \right.
\end{aligned}$$

$$- \left(\frac{R^{ns} - 1}{ns} - \frac{R^{ns-2} - 1}{ns - 2} \right) \{M(p, 1)\}^{s-1} |p'(0)| \Big],$$

which on combining with Lemma 2.2, yields for $n > 2$

$$\begin{aligned} |\{p(Re^{i\theta})\}^s - \{p(e^{i\theta})\}^s| &\leq \frac{R^{ns} - 1}{k^n} \left(\frac{n|a_n|k^2 + |a_{n-1}|}{2|a_{n-1}| + n|a_n|(1 + k^2)} \right) \{M(p, 1)\}^s \\ &\quad - s \left(\frac{R^{ns} - 1}{ns} - \frac{R^{ns-2} - 1}{ns - 2} \right) \{M(p, 1)\}^{s-1} |p'(0)|, \end{aligned}$$

from which we get the desired result.

The proof of inequality (1.10) follows on the same lines as that of inequality (1.9), but instead on using (2.15) of Lemma 2.3, we use inequality (2.16) of the same lemma. \square

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