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EXAMPLES OF MEASURE EXPANSIVE SYSTEMS

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Abstract. In this paper, we introduce some examples of measure expansive maps.

1. Introduction

In the middle of the twenty century, the notion of expansiveness was introduced by Utz [5]. Roughly speaking a system is expansive if two orbits cannot remain close to each other under the action of the system. This notion is very important in the context of the theory of dynamical systems. For instance, it is responsible for many chaotic properties for homeomorphisms defined on compact spaces.

As pointed out by Morales [3], in light of the rich consequences of expansiveness in the dynamics of a system, it is natural to consider other notions of expansiveness. In this paper we introduce a notion generalizing the usual concept of expansiveness which is called *measure expansive.*

Let X be a compact metric space with a metric d, and let f be a homeomorphism from X to X. First of all, we recall the notion of expansiveness.

Definition 1.1. *A homeomorphism $f : X \to X$ is called expansive if there is $\delta > 0$ such that for any distinct parts $x, y \in X$ there exists $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) > \delta$.*

Given $x \in X$ and $\delta > 0$, we define the *dynamical δ-ball at x,*

$$\Gamma^f_\delta(x) = \{y \in X : d(f^i(x), f^i(y)) \leq \delta \text{ for all } i \in \mathbb{Z}\}.$$

(Denote $\Gamma_\delta(x)$ for simplicity if there is no confusion.) Then we see that f is expansive if there is $\delta > 0$ such that $\Gamma_\delta(x) = \{x\}$ for all $x \in X$.

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Let β be the Borel σ-algebra on X. Denote by $\mathcal{M}(X)$ the set of Borel probability measures on X endowed with weak* topology. We say that $\mu \in \mathcal{M}(X)$ is *atomic* if there exists a point $x \in X$ such that $\mu(\{x\}) > 0$. Let $\mathcal{M}^*(X) = \{\mu \in \mathcal{M}(X) : \mu \text{ is nonatomic}\}$, and let $\mathcal{M}^*_f(X) = \{\mu \in \mathcal{M}^*(X) : \mu \text{ is } f\text{-invariant}\}$.

Definition 1.2. *Let $\mu \in \mathcal{M}(X)$. A homeomorphism $f : X \to X$ is said to be μ-expansive if there is $\delta > 0$ (called the expansive constant of μ with respect to f) such that $\mu(\Gamma_\delta(x)) = 0$ for all $x \in X$.*

Definition 1.3. *Let $\mu \in \mathcal{M}(X)$. A continuous map $f : X \to X$ is said to be positively μ-expansive if there is $\delta > 0$ (called the expansive constant of μ with respect to f) such that $\mu(\Gamma_\delta(x)) = 0$ for all $x \in X$. where,*

$$\Gamma^f_\delta(x) = \{y \in X : d(f^i(x), f^i(y)) \leq \delta \text{ for all } i \in \mathbb{Z}_+ = \{0\} \cup \mathbb{N}\}.$$

Notice that this definition does not assume that the map f (resp. the measure μ) is *measure preserving* (resp. *invariant*), i.e., $\mu = \mu \circ f^{-1}$.

Definition 1.4. *A homeomorphism $f : X \to X$ is said to be (positively) measure expansive if there is $\delta > 0$ such that f is (positively) μ-expansive for all $\mu \in \mathcal{M}^*(X)$.*

2. Examples of measure expansive maps

Example 2.1. *Let I be a unit interval, and $F_\alpha(x) = \alpha x(1 - x)$ with $\alpha > 2 + \sqrt{5}$. Then F_α is positively leb-expansive, where leb is the Lebesgue measure on I.*

Proof. Since $\alpha > 2 + \sqrt{5} > 4$, $\frac{\alpha}{4}$ is larger than one. So certain points leave I after one iteration of F_α. Denote the set of such points by A_0. Clearly, A_0 is an open interval centered at $1/2$, and has the property that if $x \in A_0$, then $F_\alpha(x) > 1$, so ${F_\alpha}^2(x) < 0$ and ${F_\alpha}^n(x) \to -\infty$. This property means that A_0 is the set of points which immediately escape from I. All other points in I remain in I after one iteration of F_α. Let

$$A_1 = \{x \in I : F_\alpha(x) \in A_0\}.$$

If $x \in A_1$, then ${F_\alpha}^2(x) > 1, {F_\alpha}^3(x) < 0$, and so ${F_\alpha}^n(x) \to -\infty$. Inductively, let

$$\begin{aligned} A_n &= \{x \in I : {F_\alpha}^n(x) \in A_0\} \\ &= \{x \in I : {F_\alpha}^i(x) \in I \quad \text{for} \quad i \leq n, \quad \text{but} \quad {F_\alpha}^{n+1}(x) \notin I\} \end{aligned}$$

That is, A_n consists of all points which escape from I at the $(n+1)$-th iteration. As above, if x lies in A_n, it follows that the orbit of x tends eventually to $-\infty$. Since we know the ultimate fate of any point which lies in A_n, it remains only to analyze the behavior of those points which never escape from I. And denote

$$\Lambda = I - (\cup_{n=0}^{\infty} A_n)$$

by the set of points which never escape from I.

We know that if $\alpha > 2 + \sqrt{5}$, then Λ is a Cantor set. So $leb(\Lambda) = 0$. Therefore F_α is positively leb-expansive. □

Let $\{(X_i, \beta_i, \mu_i) : i \in I \subset \mathbb{Z}\}$ be a countable family of probability spaces. Then we give a product measure on $X = \prod_{i \in I} X_i$ as follows. First, we give the product σ-algebra of $X = \prod_{i \in I} X_i$. We say that $R \subset X$ is a *measurable rectangle* if there exist a finite set $F \subset I$ and $B_i \in \beta_i (i \in F)$ such that $R = \cap_{i \in F} \pi_i^{-1}(B_i)$, where $\pi_i : X \to X_i$ is an i-th projection. Let $\mathcal{A}$ be the set of finite union of measurable rectangles. Then $\mathcal{A}$ becomes an algebra on X. Put $\beta = \beta(\mathcal{A})$. This β is the smallest σ-algebra generated by $\mathcal{A}$, and it is called the *product σ-algebra*. Second, we will give a product measure μ on β. For all $R = \cap_{i \in F} \pi^{-1}(B_i) \in \mathcal{A}$, we define $\tilde{\mu} : \mathcal{A} \to [0, \infty]$ by $\tilde{\mu}(R) = \prod_{i \in F} \mu_i(B_i)$. Then it is easy to check

· $\tilde{\mu}(\emptyset) = 0$, and
· $\tilde{\mu}(\cup_{i=1}^{\infty} R_{n_i}) = \sum_{i=1}^{\infty} \tilde{\mu}(R_{n_i})$, for all pairwise disjoint measurable rectangles $\{R_{n_i}\}_{i=1}^{\infty}$ with $\cup_{i=i}^{\infty} R_{n_i} \in \mathcal{A}$.

So by the Kolmogorov Extension theorem, there exists the unique probability measure $\mu : \beta \to [0, \infty)$ which is an extension of $\tilde{\mu}$. Thus (X, β, μ) becomes a probability space and we know that $\mu = \prod_{i \in I} \mu_i$ as a product measure. That is, the Borel σ-algebra of X coincide with the product σ-algebra defined above.

EXAMPLE 2.2. *Let $X = \{0,1\}^{\mathbb{N}} = \{(x_1, x_2, \ldots) : x_i = 0 \quad or \quad 1\}$ and define a map $\sigma : X \to X$ by $\sigma(x) = \sigma((x_1, x_2, x_3, \ldots)) = (x_2, x_3, \ldots)$ as a shift map on X. Then the shift map σ is μ-expansive, where μ is the product measure on X.*

Proof. Under the above notation, $X_i = \{0,1\}, I = \mathbb{N}$ and $\beta_i = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$. Let μ_i be a measure on β_i given by $\mu_i(\{0\}) = \mu_i(\{1\}) = 1/2$. Then each (X_i, β_i, μ_i) is a probability space.

Now, we give a product measure μ on X. For every finite sequence $u = (u_1, u_2, \ldots, u_n)$, where $u_i = 0$ or 1, we define

$$\begin{aligned}[u] &= \cap_{i=1}^{n} \pi_i^{-1}(u_i) \\ &= \{\omega \in X : \omega_i = u_i \quad \text{for} \quad i = 1, 2, \ldots, n\} \\ &= \{(u_1, u_2, \ldots, u_n, \ldots)\}\end{aligned}$$

which is called a *cylinder set.* Let $\mathcal{A}$ be the collection of finite union of *cylinder sets.*

Then $\mathcal{A}$ is an algebra on X. So we can find the smallest σ-algebra β containing $\mathcal{A}$. Then $\tilde{\mu}([u]) = \prod_{i=1}^{n} \mu_i(\{u_i\}) = (1/2)^n$. So there exists the unique probability measure μ which is an extension of $\tilde{\mu}$ on X.

For this measure μ, the shift map σ is μ-expansive. Indeed, let $x = (x_i)$ and $y = (y_i)$ be elements of X. If $x \neq y$, then there is $k \in \mathbb{N}$ such that $x_k \neq y_k$. From this fact, we know that

$$\begin{aligned}d(\sigma^{k-1}(x), \sigma^{k-1}(y)) &= d((x_k, x_{k+1}, \ldots), (y_k, y_{k+1}, \ldots)) \\ &= \frac{|x_k - y_k|}{2} + \frac{|x_{k+1} - y_{k+1}|}{2^2} + \cdots \\ &\geq \frac{1}{2}.\end{aligned}$$

Take $\delta = 1/2$. Then we have $\Gamma_\delta(x) = \{x\}$ for all $x \in X$.

To show that σ is μ-expansive, we only check that the measure of every one point set in X is zero. Assume $\mu(\{y\}) = c > 0$ for $y = (y_1, y_2, \ldots) \in X$. Then there is $n \in \mathbb{N}$ such that $\frac{1}{2^n} \leq c$, and consider the collection of disjoint measurable rectangles

$$\{[1 - y_1], [y_1, 1 - y_2], \ldots, [y_1, y_2, \ldots, 1 - y_n], \{y\}\}$$

which is a subset of X. Then

$$\begin{aligned}\mu(X) &> \mu([1 - y_1]) + \mu([y_1, 1 - y_2]) + \cdots + \mu([y_1, y_2, \ldots, 1 - y_n]) + \mu(\{y\}) \\ &= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + c \\ &\geq \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \frac{1}{2^n} = 1.\end{aligned}$$

This is a contradiction, because X is a probability space. Therefore the measure of every one point set in X is zero. This shows that σ is μ-expansive. $\square$

Definition 2.3. *A map* $h : S^1 \to S^1$ *is called a Denjoy map if* h *is nontransitive homeomorphism with irrational rotation number* $\rho(h)$*. Here,* $\rho(h,x) = \lim_{n\to\infty} \frac{1}{n}(H^n(x) - x)$*. Where,* $H : \mathbb{R} \to \mathbb{R}$ *: lifting.*

Example 2.4. *Denjoy map is measure expansive but not expansive.*

Proof. Let f be a *Denjoy* homeomorphism of S^1. As is well known f has no periodic points and exhibits a unique minimal set $\triangle$ which is a Cantor set [2]. In particular, $\triangle$ is compact without isolated points thus it exhibits a non-atomic Borel probability measure ν (for more details, see [4]). On the other hand, $f|_{\triangle}$ is expansive, so ν is an expansive measure of $f|_{\triangle}$. Then, we obtained that the measure μ defined by $\mu(A) = \nu(A \cap \triangle)$ for any Borelian subset A of X is an expansive measure of f. It is well known that there are no expansive homeomorphisms of S^1. $\square$

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