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A QUADRAPARAMETRIC FAMILY OF EIGHTH-ORDER ROOT-FINDING METHODS

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Abstract. A new three-step quadraparametric family of eighth-order iterative methods free from second derivatives are proposed in this paper to find a simple root of a nonlinear equation. Convergence analysis as well as numerical experiments confirms the eighth-order convergence and asymptotic error constants.

1. Introduction

High-order iterative methods have been investigated by many researchers such as Bi-Ren-Wu[1], Bi-Wu-Ren[2], Chun-Ham[3], Kou-Li-Wang[6], Ren-Wu-Bi[7], Wang-Kou-Li[10] and Wang-Liu[11]. These methods have convergence order of at least 6 and are 3-step second derivative-free methods, the 2nd-step of which frequently uses King's fourth-order method[5], Jarratt's fourth-order method[4] or their variants. Undoubtedly, special attention has been paid to high-order iterative methods free from second derivatives to find a numerical solution of a nonlinear equation $f(x) = 0$. In this paper, a three-step quadraparametric family of eighth-order methods free of second derivatives are proposed with their convergence results as well as numerical experiments for various test functions.

Let $f : \mathbb{C} \to \mathbb{C}$ have a simple root α and be analytic in a small region containing α. A parametric family of three-step iterative methods are considered below: for $n = 0, 1, \cdots$,

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$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - K_f(x_n)\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - W_f(x_n)\frac{f(z_n)}{f'(z_n)}, \end{cases} \tag{1.1}$$

where, denoting $u_n = f(y_n)/f(x_n)$ and $v_n = f(z_n)/f(x_n)$,

$$K_f(x_n) = \frac{1 + \beta u_n + \lambda u_n^2}{1 + (\beta - 2)u_n + \mu u_n^2}, \tag{1.2}$$

$$W_f(x_n) = \frac{1 + a u_n + b v_n}{1 + c u_n + d v_n}, \tag{1.3}$$

with $\beta, \lambda, \mu, a, b, c$ and d as constant parameters to be determined later. We immediately find that both K_f and W_f are extensions proposed from classical King's method[5]. Functions K_f and W_f can be viewed as *weighting functions* for error terms $z_n - y_n$ and $x_{n+1} - z_n$, respectively. With a proper selection of the constant parameters based on the analysis to be presented in Section 2, they play a crucial role of maximizing the order of convergence up to 4 and 8 in the second step and third step. Observe that (1.1) requires 5 new function evaluations for $f(x_n), f(y_n), f(z_n), f'(x_n)$ and $f'(z_n)$ per iteration. We wish to reduce the number of function evaluations by one. To this end, we approximate $f'(z_n)$ using $f(x_n), f(y_n), f(z_n), f'(x_n)$. Taylor series expansion of $f'(z_n)$ about y_n leads to an approximation:

$$f'(z_n) \approx f'(y_n) + f''(y_n)(z_n - y_n). \tag{1.4}$$

The fact that $y_n \approx z_n$ for sufficiently large values of n yields the following approximations:

$$f'(y_n) \approx \frac{f(z_n) - f(y_n)}{z_n - y_n}, \tag{1.5}$$

$$f'(z_n) \approx \frac{f(x_n) - f(z_n)}{x_n - z_n}, \tag{1.6}$$

$$f''(y_n) \approx f''(z_n) = \frac{f'(z_n) - f'(x_n)}{z_n - x_n} \approx \frac{\frac{f(z_n) - f(x_n)}{z_n - x_n} - f'(x_n)}{z_n - x_n}. \tag{1.7}$$

Hence, $f'(z_n)$ in (1.4) now approximates

$$f'(z_n) \approx \frac{f(z_n) - f(y_n)}{z_n - y_n} + \left(\frac{z_n - y_n}{z_n - x_n}\right)\left(\frac{f(z_n) - f(x_n)}{z_n - x_n} - f'(x_n)\right). \tag{1.8}$$

Consequently, (1.1) can be rewritten as follows: with K_f and W_f described as in (1.2) and (1.3), respectively

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - K_f(x_n)\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - W_f(x_n)\frac{f(z_n)}{F(x_n)}, \end{cases} \tag{1.9}$$

where

$$F(x_n) = \frac{f(z_n) - f(y_n)}{z_n - y_n} + \left(\frac{z_n - y_n}{z_n - x_n}\right)\left(\frac{f(z_n) - f(x_n)}{z_n - x_n} - f'(x_n)\right), \tag{1.10}$$

explicitly depending on $f(x_n), f(y_n), f(z_n), f'(x_n)$ not on $f'(y_n)$.

Note that (1.9) now requires only four function evaluations for $f(x_n)$, $f(y_n), f(z_n), f'(x_n)$ per iteration. The main objective of this paper is to show iteration scheme (1.9) has eighth-order convergence with relations

$$\beta = \frac{1}{2}(\lambda - \mu - 1),\ c = a,\ d = b - 2, \tag{1.11}$$

or equivalently with relations below:

$$K_f(x_n) = \frac{1 + \frac{1}{2}(\lambda - \mu - 1)u_n + \lambda u_n^2}{1 + \frac{1}{2}(\lambda - \mu - 5)u_n + \mu u_n^2}, \tag{1.12}$$

$$W_f(x_n) = \frac{1 + a u_n + b v_n}{1 + a u_n + (b-2)v_n}, \tag{1.13}$$

where λ, μ, a and b are four independent constant parameters to be freely chosen in $\mathbb{R}$. In addition, deriving the asymptotic error constant or error equation is another goal of this paper. To measure convergence behavior within a given error bound, the values of $|x_n - \alpha|$ of proposed scheme (1.9) will be compared made with those of the existing seventh- or eighth-order methods some of which are described as follows.

(1) **Kou-Li-Wang**[6]: seventh-order method

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \frac{1+H_2(x_n,y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \left[(1 + H_2(x_n, y_n))^2 + H_\theta(y_n, z_n)\right]\frac{f(z_n)}{f'(z_n)},\ \theta \in \mathbb{R}, \end{cases} \tag{1.14}$$

with $H_\theta(y,z) = \frac{f(z)}{f(y)-\theta f(z)}$. The error equation of this method is asserted in [6] to satisfy the relation with $e_n = x_n - \alpha$ and $c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}$ for $j = 2, 3, 4$:

$$e_{n+1} = -2({c_3}^2 - 2{c_2}^2 c_3 + c_2 c_4)({c_2}^2 - c_3)e_n^7 + O(e_n^8). \tag{1.14a}$$

REMARK 1.1. In fact, error equation (1.14a) asserted in the paper by Kou, Li and Wang[6] has been found to be incorrect and needs to be completely corrected as the following:

$$e_{n+1} = 4{c_2}^2({c_2}^2 - c_3)^2{e_n}^7 + O({e_n}^8). \tag{1.14b}$$

(2) **Bi-Ren-Wu**[1]: eighth-order method

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{2f(x_n)-f(y_n)}{2f(x_n)-5f(y_n)}\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(x_n)+(2+\theta)f(z_n)}{f(x_n)+\theta f(z_n)} \cdot \frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}, \quad \theta \in \mathbb{R}, \end{cases} \tag{1.15}$$

where $f[z,y] = \frac{f(z)-f(y)}{z-y}$ and $f[z,x,x] = \frac{f'(x)-f[z,x]}{x-z}$. Observe that this method is a special case when $\lambda = \mu = 0$ in (1.12) and $a = 0, b = 2 + \theta$ in (1.13). The error equation of this method satisfies

$$e_{n+1} = {c_2}^2c_3(3{c_2}^3 + 2c_2c_3 - c_4){e_n}^8 + O({e_n}^9). \tag{1.15a}$$

(3) **Bi-Wu-Ren**[2]: eighth-order method

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \left[\frac{f(x_n)}{f(x_n)-3f(y_n)}\right]^{2/3} \cdot \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(x_n)+(2+\theta)f(z_n)}{f(x_n)+\theta f(z_n)} \cdot \frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}, \quad \theta \in \mathbb{R}. \end{cases} \tag{1.16}$$

The error equation of this method satisfies

$$e_{n+1} = {c_2}^2c_3(\frac{4}{3}{c_2}^3 + 2c_2c_3 - c_4){e_n}^8 + O({e_n}^9). \tag{1.16a}$$

Some interesting choices of four parameters (λ, μ, a, b) in (1.12) and (1.13) will be further discussed in Section 2. Observe that (1.9) has four function evaluations per iteration and its efficiency index[9] is same as that of 3-step methods mentioned above. It's convergence order is optimal as well as consistent with the conjecture of Kung-Traub[8]. The advantage of proposed method (1.9) is that we can choose four free parameters as compared with methods (1.14), (1.15) and (1.16) having only one free parameter. Such a choice of four free parameters (λ, μ, a, b) gives a wide range of iterative methods which can be conveniently chosen from, depending on problems, to find a numerical root of the given nonlinear equation $f(x)$. Section 3 will discuss numerical experiments for various test functions.

2. Main results

Development of proposed scheme (1.9) and its convergent analysis will be described in Theorem 2.1 below:

THEOREM 2.1. *Let f and α be described as in Section 1. Let $c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}$ for $j = 2, 3, \cdots$. Assume that all three values c_2, c_3 and c_4 are not vanishing simultaneously. Let x_0 be an initial guess chosen in a sufficiently small neighborhood of α. Let $\beta = \frac{1}{2}(\lambda - \mu - 1)$, $c = a$ and $d = b-2$ with λ, μ, a and b as independent parameters freely chosen in $\mathbb{R}$. Then with $K_f(x_n)$ in (1.2) and $W_f(x_n)$ in (1.3), iteration scheme (1.9) defines a quadraparametric family of eighth-order methods and gives its error equation by*

$$e_{n+1} = {c_2}^2 c_3\{2(a+1)c_2c_3 - c_4 + {c_2}^3(5\lambda - \mu + 3)\}e_n^8 + O(e_n^9), \quad (2.1)$$

where $e_n = x_n - \alpha$ for $n = 0, 1, 2, \cdots$.

Proof. Taylor series expansion of $f(x_n)$ about α up to ninth-order terms yields with $f(\alpha) = 0$:

$$f(x_n) = f'(\alpha)(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8 + c_9e_n^9 + O(e_n^{10})). \quad (2.2)$$

For ease of notation, e_n will be denoted by e (not to be confused with Napier's base for natural logarithms) for the time being. A lengthy algebraic computation induces relations (2.3)-(2.7) below:

$$f'(x_n) = f'(\alpha)(1 + 2c_2e + 3c_3e^2 + 4c_4e^3 + 5c_5e^4 + 6c_6e^5 + 7c_7e^6 + 8c_8e^7 + 9c_9e^8 + O(e^9)), \quad (2.3)$$

$$\frac{f(x_n)}{f'(x_n)} = e - c_2e^2 + 2(c_2^2 - c_3)e^3 + (-4c_2^3 + 7c_2c_3 - 3c4)e^4 + (8c_2^4 - 20c_2^2c_3 + 6c_3^2 + 10c_2c_4 - 4c_5)e^5 + (-16c_2^5 + 52c_2^3c_3 - 33c_2c_3^2 - 28c_2^2c_4 + 17c_3c_4 + 13c_2c_5 - 5c_6)e^6 + 2(16c_2^6 - 64c_2^4c_3 - 9c_3^3 + 36c_2^3c_4 + 6c_4^2 + 9c_2^2(7c_3^2 - 2c_5) + 11c_3c_5 + c_2(-46c_3c_4 + 8c_6) - 3c_7)e^7 + (-64c_2^7 + 304c_2^5c_3 - 176c_2^4c_4 - 75c_3^2c_4 + 31c_4c_5 + c_2^3(-408c_3^2 + 92c_5) + 4c_2^2(87c_3c_4 - 11c_6) + 27c_3c_6 + c_2(135c_3^3 - 64c_4^2 - 118c_3c_5 + 19c_7) - 7c_8)e^8 + O(e^9), \quad (2.4)$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2e^2 - 2(c_2^2 - c_3)e^3 + (4c_2^3 - 7c_2c_3 + 3c4)e^4 -$$

$$2(4c_2^4-10c_2^2c_3+3c_3^2+5c_2c_4-2c_5)e^5+$$
$$(16c_2^5-52c_2^3c_3+33c_2c_3^2+28c_2^2c_4-17c_3c_4-13c_2c_5+5c_6)e^6-$$
$$2(16c_2^6-64c_2^4c_3-9c_3^3+36c_2^3c_4+6c_4^2+9c_2^2(7c_3^2-2c_5)+$$
$$11c_3c_5+c_2(-46c_3c_4+8c_6)-3c_7)e^7+(64c_2^7-304c_2^5c_3+176c_2^4c_4+$$
$$75c_3^2c_4-31c_4c_5+c_2^3(408c_3^2-92c_5)-4c_2^2(87c_3c_4-11c_6)-27c_3c_6-$$
$$c_2(135c_3^3-64c_4^2-118c_3c_5+19c_7)+7c_8)e^8+O(e^9), \qquad (2.5)$$

$$f(y_n)=f'(\alpha)(c_2^2e^2-2(c_2^2-c_3)e^3+(5c_2^3-7c_2c_3+3c_4)e^4-$$
$$2(6c_2^4-12c_2^2c_3+3c_3^2+5c_2c_4-2c_5)e^5+$$
$$(28c_2^5-73c_2^3c_3+37c_2c_3^2+34c_2^2c_4-17c_3c_4-13c_2c_5+5c_6)e^6-$$
$$2(32c_2^6-103c_2^4c_3-9c_3^3+52c_2^3c_4+6c_4^2+c_2^2(80c_3^2-22c_5)+$$
$$11c_3c_5+c_2(-52c_3c_4+8_6)-3c_7)e^7+(144c_2^7-552c_2^5c_3+297c_2^4c_4+$$
$$75c_3^2c_4+2c_2^3(291c_3^2-67c_5)-31c_4c_5-27c_3c_6+c_2^2(-455c_3c_4+54c_6)+$$
$$c_2(-147c_3^3+73c_4^2+134c_3c_5-19c_7)+7c_8)e^8+O(e^9)). \qquad (2.6)$$

$$u_n=\frac{f(y_n)}{f(x_n)}=c_2e+(-3c_2^2+2c_3)e^2+(8c_2^3-10c_2c_3+3c_4)e^3+$$
$$(-20c_2^4+37c_2^2c_3-8c_3^2-14c_2c_4+4c_5)e^4+$$
$$(48c_2^5-118c_2^3c_3+55c_2c_3^2+51c_2^2c_4-22c_3c_4-18c_2c_5+5c_6)e^5+$$
$$(-112c_2^6+344c_2^4c_3-252c_2^2c_3^2+26c_3^3-163c_2^3c_4+$$
$$150c_2c_3c_4-15c_4^2+65c_2^2c_5-28c_3c_5-22c_2c_6+6c_7)e^6+$$
$$(256c_2^7-944c_2^5c_3+480c_2^4c_4+105c_3^2c_4+c_2^3(952c_3^2-207c_5)-38c_4c_5-$$
$$34c_3c_6+c_2^2(-693c_3c_4+79c_6)-2c_2(114c_3^3-51c_4^2-95c_3c_5+13c_7)+7c_8)e^7+$$
$$(-576c_2^8+2480c_2^6c_3-72c_3^4-1336c_2^5c_4+141c_3c_4^2+$$
$$132c_3^2c_5-24c_5^2+c_2^4(-3200c_3^2+607c_5)+c_2^3(2660c_3c_4-251c_6)-$$
$$46c_4c_6-40c_3c_7+3c_2^2(418c_3^3-159c_4^2-292c_3c_5+31c_7)+$$
$$c_2(-936c_3^2c_4+258c_4c_5+230c_3c_6-30c_8)+8c_9)e^8+O(e^7). \qquad (2.7)$$

Substituting relations (2.2)-(2.7) into (1.9), we get x_{n+1} as follows by the aid of symbolic computation of Mathematica:

$$x_{n+1}=\alpha+A_5e^5+A_6e^6+A_7e^7++A_8e^8+O(e^9), \qquad (2.8)$$

where $A_i=A_i(a,b,c,d,\beta,\lambda,\mu)(i=5,6,7,8,9)$ are multivariate polynomials in a,b,c,d,β,λ and μ; for instance,

$$A_5=(c-a)c_2(-c_2c_3+{c_2}^3(1+2\beta-\lambda+\mu)). \qquad (2.9)$$

We impose conditions $A_5 = A_6 = A_7 = 0$ and $A_8 \neq 0$ independently of c_j's so that iteration scheme (1.9) has eighth-order convergence. Solving $A_5 = 0$ for c yields

$$c = a. \tag{2.10}$$

Substituting this c into $A_6 = 0$ and $A_7 = 0$ after simplification yields

$$c_3(-2 + b - d) + {c_2}^2(-b + d)(1 + 2\beta - \lambda + \mu) = 0,$$

from which two relations follow independently of c_2 and c_3:

$$d = b - 2,\ \beta = \frac{1}{2}(\lambda - \mu - 1). \tag{2.11}$$

Substituting these c, d, β into A_8 after simplification yields

$$c_2^2 c_3(2(1 + a)c_2c_3 - c_4 + c_2^3(3 + 5\lambda - \mu)). \tag{2.12}$$

Now restoring notation e back to e_n in (2.11) yields the error equation and the asymptotic error constant η with convergence order 8, respectively, as follows:

$$e_{n+1} = {c_2}^2 c_3\{2(a + 1)c_2c_3 - c_4 + {c_2}^3(5\lambda - \mu + 3)\}{e_n}^8 + O({e_n}^9), \tag{2.13}$$

$$\eta = \lim_{n\to\infty} \left|\frac{e_{n+1}}{{e_n}^8}\right| = \left|{c_2}^2 c_3\{2(a + 1)c_2c_3 - c_4 + {c_2}^3(5\lambda - \mu + 3)\}\right|, \tag{2.14}$$

yielding desired (2.1). Substituting $d = b-2$ and $\beta = \frac{1}{2}(\lambda-\mu-1)$ found by (2.11) into (1.2) and (1.3) gives desired $K_f(x_n)$ in (1.12) and $W_f(x_n)$ in (1.13). This completes the proof. □

Although iteration scheme (1.9) defines a quadraparametric family of eighth-order methods, it is interesting to observe that error equation (2.13) depends only on three parameters λ, μ, a and independent of parameter b. This kind of parameter-independency for the error equation occurs frequently as can be seen in many iterative methods such as (1.14), (1.15) and (1.16). Given choices of parameters, with the introduction of two normalized variables $u_n = f(y_n)/f(x_n)$ and $v_n = f(z_n)/f(x_n)$, it is convenient to display a variety of $K_f(x_n), W_f(x_n)$ and η. It also simplifies coding of numerical Algorithm 3.1. Table 1 lists a number of such choices of λ, μ, a and b, being accompanied with $K_f(x_n), W_f(x_n)$ and η.

TABLE 1. Typical choices of a, b, λ and μ for $K_f(x_n)$, $W_f(x_n)$ and η

Case	(λ, μ, a, b)	$K_f(x_n)$	$W_f(x_n)$	η
0	$(0,0,0,3)$	$\frac{2-u_n}{2-5u_n}$	$\frac{1+3v_n}{1+v_n}$	$\lvert c_2{}^2 c_3(3c_2{}^3 + 2c_2c_3 - c_4)\rvert$
1	$(-1,-2,-1,0)$	$\frac{1-u_n^2}{1-2u_n-2u_n^2}$	$\frac{1-u_n}{1-u_n-2v_n}$	$\lvert c_2{}^2 c_3 c_4\rvert$
2	$(-2,-7,-1,0)$	$\frac{1+2u_n-2u_n^2}{1-7u_n^2}$	$\frac{1-u_n}{1-u_n-2v_n}$	$\lvert c_2{}^2 c_3 c_4\rvert$
3	$(0,-2,-1,0)$	$\frac{2+u_n}{2-3u_n-4u_n^2}$	$\frac{1-u_n}{1-u_n-2v_n}$	$\lvert c_2{}^2 c_3(5c_2{}^3 - c_4)\rvert$
4	$(\frac{9}{16}, -\frac{87}{16}, -1, 0)$	$\frac{(4+9u_n)(4+u_n)}{(16+8u_n-87u_n^2)}$	$\frac{1-u_n}{1-u_n-2v_n}$	$\lvert c_2{}^2 c_3(\frac{45}{4}c_2{}^3 - c_4)\rvert$
5	$(\frac{9}{16}, \frac{73}{16}, -1, 0)$	$\frac{(4-9u_n)(4-u_n)}{(16-72u_n+73u_n^2)}$	$\frac{1-u_n}{1-u_n-2v_n}$	$\lvert c_2{}^2 c_3(\frac{5}{4}c_2{}^3 - c_4)\rvert$
6	$(-\frac{9}{16}, \frac{39}{16}, -1, 0)$	$\frac{(4-9u_n)(4+u_n)}{(4-13u_n)(4-3u_n)}$	$\frac{1-u_n}{1-u_n-2v_n}$	$\lvert c_2{}^2 c_3(\frac{9}{4}c_2{}^3 + c_4)\rvert$
7	$(-\frac{9}{16}, -\frac{89}{16}, -1, 0)$	$\frac{(4+9u_n)(4-u_n)}{16-89u_n^2}$	$\frac{1-u_n}{1-u_n-2v_n}$	$\lvert c_2{}^2 c_3(\frac{23}{4}c_2{}^3 - c_4)\rvert$
8	$(1,4,-1,2)$	$(\frac{1-u_n}{1-2u_n})^2$	$\frac{1-u_n+2v_n}{1-u_n}$	$\lvert c_2{}^2 c_3(4c_2{}^3 - c_4)\rvert$
9	$(0,-1,-1,2)$	$\frac{1}{1-2u_n-u_n^2}$	$\frac{1-u_n+2v_n}{1-u_n}$	$\lvert c_2{}^2 c_3(4c_2{}^3 - c_4)\rvert$
10	$(1,0,-1,1)$	$\frac{1+u_n^2}{1-2u_n}$	$\frac{1-u_n+v_n}{1-u_n-v_n}$	$\lvert c_2{}^2 c_3(8c_2{}^3 - c_4)\rvert$
11	$(1,-4,-1,2)$	$\frac{(1+u_n)^2}{1-4u_n^2}$	$\frac{1-u_n+2v_n}{1-u_n}$	$\lvert c_2{}^2 c_3(12c_2{}^3 - c_4)\rvert$
12	$(2,1,-1,2)$	$\frac{1+2u_n{}^2}{(1-u_n)^2}$	$\frac{1-u_n+2v_n}{1-u_n}$	$\lvert c_2{}^2 c_3(12c_2{}^3 - c_4)\rvert$
13	$(1,-5,-1,2)$	$\frac{(2+u_n)(1+2u_n)}{(2+5u_n)(1-2u_n)}$	$\frac{1-u_n+2v_n}{1-u_n}$	$\lvert c_2{}^2 c_3(13c_2{}^3 - c_4)\rvert$
14	$(5,0,-1,1)$	$1 + 2u_n + 5u_n^2$	$\frac{1-u_n+v_n}{1-u_n-v_n}$	$\lvert c_2{}^2 c_3(28c_2{}^3 - c_4)\rvert$

3. Algorithm, numerical results and discussions

The analysis described in Section 2 allows us to develop a zero-finding algorithm to be implemented with *Mathematica*[12]:

Algorithm 3.1 (Zero-Finding Algorithm)

Step 1. Construct iteration scheme (1.1) with the given function f having a simple zero α for $n \in \mathbb{N} \cup \{0\}$ as mentioned in Section 1.

Step 2. Set the minimum number of precision digits. With exact or most accurate zero α, supply the theoretical asymptotic error constant η, order of convergence p as well as c_2, c_3, c_4, λ and μ, a, b stated in Section 2. Set the error bound ϵ, the maximum iteration number n_{max} and the initial guess x_0. Compute $|f(x_0)|$ and $|x_0 - \alpha|$.

Step 3. Tabulate the values of $n,\ x_n,\ |f(x_n)|,\ |e_n| = |x_n - \alpha|,\ |\frac{e_n}{e_{n-1}{}^p}|$ and η.

Throughout the numerical experiments, the minimum number of precision digits was chosen as 350, being large enough to minimize round-off errors as well as to clearly observe the computed asymptotic error constants requiring small-number divisions. The zero α, however, was separately computed with 700 digits of precision to have 400 significant digits, whenever its exact value is not known. The error bound $\epsilon = 10^{-300}$ was used for moderately accurate computation. The values of initial guess x_0 were selected closely to α to guarantee the convergence. The computed asymptotic error constant agrees up to 8 significant digits with the theoretical one. The computed zero is accurate up to 300 significant digits, although the first 15 digits are displayed.

Iterative method **YK1** with $(\lambda, \mu, a, b) = (-1, -2, -1, 0)$ applied to test functions $f(x) = 1 - \frac{x}{2} + \sin^{-1}(x^2 - 1)$, $f(x) = e^{-x^2}\frac{\sin x}{x^2-1} + \cos x \cdot \log(1+x-\pi)$ and $f(x) = e^{(x-1)^2+5} + (x-1)^4 + 5(x-1)^2 - 1$, clearly shows successful asymptotic error constants with eighth-order convergence for suitable initial values chosen near α. Tables 2, 3 and 4 list iteration indexes n, approximate zeros x_n, residual errors $|f(x_n)|$, errors $|e_n| = |x_n - \alpha|$ and computational asymptotic error constants $|\frac{e_n}{e_{n-1}{}^8}|$ as well as the theoretical asymptotic error constant η.

TABLE 2. Convergence for
$f(x) = 1 - \frac{x}{2} + \sin^{-1}(x^2 - 1)$ with $\alpha \approx 0.594810968398$

n	x_n	$\lvert f(x_n)\rvert$	$\lvert e_n\rvert = \lvert x_n - \alpha\rvert$	$\lvert \frac{e_n}{e_{n-1}{}^8}\rvert$	η
0	0.7	0.114815	0.105189		
1	0.594810968314147	8.91737×10^{-11}	8.42220×10^{-11}	0.0056190363	0.0028544234
2	0.594810968398369	7.65128×10^{-84}	7.22641×10^{-84}	0.0028544234	
3	0.594810968398369	$0.\times10^{-350}$	$0.\times10^{-350}$		

TABLE 3. Convergence for
$f(x) = e^{-x^2}\frac{\sin x}{x^2-1} + \cos x \cdot \log(1 + x - \pi)$ with $\alpha = \pi$

n	x_n	$\lvert f(x_n)\rvert$	$\lvert e_n\rvert = \lvert x_n - \alpha\rvert$	$\lvert \frac{e_n}{e_{n-1}{}^8}\rvert$	η
0	2.965	0.191286	0.176593		
1	3.14159265248208	1.10772×10^{-9}	1.10771×10^{-9}	0.001171241754	0.000012094207
2	3.14159265358979	2.74158×10^{-77}	2.74156×10^{-77}	0.000012094207	
3	3.14159265358979	$0.\times10^{-349}$	$0.\times10^{-349}$		

TABLE 4. Convergence for
$f(x) = e^{(x-1)^2+5} + (x-1)^4 + 5(x-1)^2 - 1$ with $\alpha = 1 + i\sqrt{5}$

n	x_n	$\lvert f(x_n)\rvert$	$\lvert e_n\rvert = \lvert x_n - \alpha\rvert$	$\lvert \frac{e_n}{e_{n-1}{}^8}\rvert$	η
0	$0.96 + 2.3\ i$	1.51612	0.0754142		
1	$0.99999999909 + 2.23606797751i$	1.622×10^{-8}	9.071×10^{-10}	0.86709637	0.13743472
2	$1.00000000000 + 2.23606797749i$	1.127×10^{-72}	6.304×10^{-74}	0.13743472	
3	$1.00000000000 + 2.23606797749i$	$0.\times10^{-348}$	$0.\times10^{-349}$		

TABLE 5. Comparison of $|x_n - \alpha|$ for various iterative methods

$f(x)$	x_0	$\|x_n - \alpha\|$	$\binom{\text{KLW}}{\theta=0}$	$\binom{\text{BRW}}{\theta=1}$	$\binom{\text{BWR}}{\theta=1}$	**YK**1	**YK**5	**YK**8
f_1	-0.86	$\|x_1 - \alpha\|$	5.60e-07*	2.18e-07	1.02e-07	2.74e-08	5.82e-08	2.20e-07
		$\|x_2 - \alpha\|$	1.03e-44	2.38e-54	3.37e-57	2.31e-62	2.67e-59	2.71e-54
		$\|x_3 - \alpha\|$	7.47e-309	0.e-350	0.e-350	0.e-350	0.e-350	0.e-350
f_2	1.45	$\|x_1 - \alpha\|$	1.00e-07	2.41e-08	1.82e-08	6.14e-09	1.02e-08	2.21e-08
		$\|x_2 - \alpha\|$	1.15e-49	3.96e-61	2.40e-62	3.04e-67	1.10e-64	1.89e-61
		$\|x_3 - \alpha\|$	3.05e-343	0.e-349	0.e-349	0.e-349	0.e-349	0.e-349
f_3	-1.3	$\|x_1 - \alpha\|$	4.95e-08	3.29e-07	2.51e-07	6.22e-08	1.00e-07	2.71e-07
		$\|x_2 - \alpha\|$	4.27e-52	8.83e-51	5.95e-52	2.30e-57	9.13e-56	1.51e-51
		$\|x_3 - \alpha\|$	0.e-349	0.e-349	0.e-349	0.e-349	0.e-349	0.e-349
f_4	0.065	$\|x_1 - \alpha\|$	4.46e-07	3.03e-09	7.52e-11	4.97e-10	2.29e-10	6.62e-09
		$\|x_2 - \alpha\|$	7.60e-43	2.49e-67	1.70e-80	2.49e-75	1.08e-76	1.60e-64
		$\|x_3 - \alpha\|$	3.17e-293	0.e-482	0.e-508	0.e-498	0.e-501	0.e-476
		$\|x_4 - \alpha\|$	0.e-641					
f_5	-1.75	$\|x_1 - \alpha\|$	1.99e-07	3.71e-08	3.71e-08	3.05e-08	3.04e-08	3.09e-08
		$\|x_2 - \alpha\|$	3.25e-49	2.61e-62	2.51e-62	6.62e-63	6.75e-63	8.03e-63
		$\|x_3 - \alpha\|$	9.95e-342	0.e-349	0.e-349	0.e-349	0.e-349	0.e-349
f_6	$1.65i$	$\|x_1 - \alpha\|$	5.45e-08	1.74e-08	1.07e-08	5.03e-08	3.07e-08	7.23e-09
		$\|x_2 - \alpha\|$	3.17e-52	3.72e-63	2.28e-64	1.42e-58	2.23e-60	9.94e-66
		$\|x_3 - \alpha\|$	0.e-349	0.e-349	0.e-349	0.e-349	0.e-349	0.e-349
f_7	1.3	$\|x_1 - \alpha\|$	2.12e-07	5.85e-08	4.68e-08	8.68e-09	1.95e-08	3.51e-08
		$\|x_2 - \alpha\|$	3.55e-48	3.69e-58	4.64e-59	1.11e-65	1.78e-62	4.63e-60
		$\|x_3 - \alpha\|$	1.29e-333	0.e-349	0.e-349	0.e-349	0.e-349	0.e-349

* 5.60e-07 $\equiv$ 5.60 $\times 10^{-7}$

Convergence behavior was confirmed for further test functions below:

$$\begin{cases} f_1(x) = (1+x^2)\cos(\frac{\pi x}{2}) + \frac{\log(x^2+2x+2)}{1+x^2},\ \alpha = -1,\ x_0 = -0.86, \\ f_2(x) = x^5 + x^4 + 4x^2 - 15,\ \alpha \approx 1.347428098968304,\ x_0 = 1.45, \\ f_3(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5,\ \alpha \approx -1.207647827130918,\ x_0 = -1.3, \\ f_4(x) = e^x \sin x + \log(1+x^2),\ \alpha = 0,\ x_0 = 0.065, \\ f_5(x) = \sqrt{x^2+2}\sin(\frac{\pi}{x^2}) + \frac{1}{(x^4+1)} - \sqrt{3} - \frac{1}{17},\ \alpha = -2,\ x_0 = -1.75, \\ f_6(x) = x^2 + \pi - \sin x^2 + \log(x^2+\pi+1),\ \alpha = i\sqrt{\pi},\ x_0 = 1.65i,\ i = \sqrt{-1}, \\ f_7(x) = x^4 + \sin(\frac{\pi}{x^2}) - 5,\ \alpha = \sqrt{2},\ x_0 = 1.3, \end{cases}$$

Here $\log z$ $(z \in \mathbb{C})$ represents a principal analytic branch with $-\pi \leq Im(\log z) < \pi$.
Table 5 lists the values of $|x_n - \alpha|$ within the prescribed error bound for various seventh- or eighth-order methods **KLW** (1.14), **BRW** (1.15), **BWR** (1.16) and **YK**i (1.9) identified by case number i in Table 1.

As Table 5 suggests, proposed 3-step methods **YK**i show acceptable performance as compared with existing 3-step iterative methods **KLW**, **BRW** and **BWR**. Under the same order of convergence, one should note that the speed of local convergence of $|x_n - \alpha|$ is dependent on c_j, namely $f(x)$ and α. During the current numerical experiments for the chosen test functions, **YK**1 has shown best performance for f_1, f_2, f_3, f_5, f_7, **YK**8 for f_6, while **BWR** for f_4. The efficiency index defined by $EI = p^{1/d}$, with p as the order of convergence and d the number of new evaluations of $f(x)$ or its derivatives per iteration, is equally $8^{1/4} \approx 1.68179$ for **YK**i, **BRW** and **BWR**, and is better than $\sqrt{2}$, the efficiency index of Newton's method.

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