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THE RELATION BETWEEN MCSHANE INTEGRAL AND MCSHANE DELTA INTEGRAL

Jae Myung Park*, Deok Ho Lee**, Ju Han Yoon***, Young Kuk Kim****, and Jong Tae Lim*****

Abstract. In this paper, we define an extension $f^* : [a, b] \to \mathbb{R}$ of a function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ for a time scale $\mathbb{T}$ and show that f is McShane delta integrable on $[a, b]_{\mathbb{T}}$ if and only if f^* is McShane integrable on $[a, b]$.

1. Introduction and preliminaries

First, we introduce some concepts related to the notion of time scales. A time scale $\mathbb{T}$ is any closed nonempty subset of $\mathbb{R}$. For each $t \in \mathbb{T}$, we define the forward jump operator $\sigma(t)$ by

$$\sigma(t) = \inf\{z > t : z \in \mathbb{T}\}$$

and the backward jump operator $\rho(t)$ by

$$\rho(t) = \sup\{z < t : z \in \mathbb{T}\}$$

where $\inf \phi = \sup \mathbb{T}$ and $\sup \phi = \inf \mathbb{T}$.

If $\sigma(t) > t$, we say the t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense,while if $\rho(t) = t$, we say that t is left-dense. The forward graininess function $\mu(t)$ is defined by $\mu(t) = \sigma(t) - t$, and the backward graininess function $\nu(t)$ is defined by $\nu(t) = t - \rho(t)$.

For $a, b \in \mathbb{T}$, we define the time scale interval in $\mathbb{T}$ by

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}.$$

A pair $\delta = (\delta_L, \delta_R)$ of two real-valued functions on $[a, b]_{\mathbb{T}}$ is a $\triangle$-gauge on $[a, b]_{\mathbb{T}}$ by $\delta_L(t) > 0$ on $(a, b]_{\mathbb{T}}$, $\delta_R(t) > 0$ on $[a, b)_{\mathbb{T}}$, $\delta_L(a) \ge 0, \delta_R(b) \ge 0$, and $\delta_R(t) \ge \mu(t)$ for each $t \in [a, b)_{\mathbb{T}}$.

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Correspondence should be addressed to Ju Han Yoon, yoonjh@cbnu.ac.kr.

A collection $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i]_{\mathbb{T}})\}_{i=1}^n$ of tagged intervals is a $\delta-$fine McShane partition of $[a,b]_{\mathbb{T}}$ if $\bigcup_{i=1}^n [t_{i-1}, t_i]_{\mathbb{T}} = [a,b]_{\mathbb{T}}$, $[t_{i-1}, t_i]_{\mathbb{T}} \subset [\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)]$ and $\xi_i \in [a,b]_{\mathbb{T}}$ for each $i = 1, 2, \cdots, n$.

For a partition $\mathcal{P} = \{(\xi_i, [t_{i-1}, t_i])\}_{i=1}^n$, we write

$$f(\mathcal{P}) = \sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}),$$

whenever $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$.

2. The McShane and McShane delta integrals

DEFINITION 2.1. A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is McShane delta integrable (or $M_\triangle$-integrable) on $[a,b]_{\mathbb{T}}$ if there exists a number A such that for each $\epsilon > 0$ there exists a $\triangle$-gauge δ on $[a,b]_{\mathbb{T}}$ such that

$$\left| f(\mathcal{P}) - A \right| < \epsilon$$

for every δ-fine McShane partition $\mathcal{P}$ of $[a,b]_{\mathbb{T}}$. The number A is called the $M_\triangle$-integral of f on $[a,b]_{\mathbb{T}}$, and we write $A = (M_\triangle) \int_a^b f$.

Recall that $f : [a,b] \to \mathbb{R}$ is McShane integrable (or M-integrable) on $[a,b]$ if there exists a number A such that for each $\epsilon > 0$ there exists a gauge $\delta : [a,b] \to \mathbb{R}^+$ on $[a,b]$ such that

$$\left| f(\mathcal{P}) - A \right| < \epsilon$$

for every δ-fine McShane partition $\mathcal{P}$ of $[a,b]$.

Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a function on $[a,b]_{\mathbb{T}}$, and let $\{(a_k, b_k)\}_{k=1}^\infty$ be the sequence of intervals contiguous to $[a,b]_{\mathbb{T}}$ in $[a,b]$.

Define a function $f^* : [a,b] \to \mathbb{R}$ on $[a,b]$ by

$$f^*(t) = \begin{cases} f(a_k) & \text{if} \quad t \in (a_k, b_k) \quad \text{for some } k. \\ f(t) & \text{if} \quad t \in [a,b]_{\mathbb{T}}. \end{cases}$$

It is well-known [6] that $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is Henstock delta(or $H_\triangle$)$-$integrable on $[a,b]_{\mathbb{T}}$ if and only if $f^* : [a,b] \to \mathbb{R}$ is Henstock(or H)$-$integrable, and $(H_\triangle) \int_a^b f = (H) \int_a^b f^*$.

THEOREM 2.2. *If $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is $M_\triangle$-integrable on $[a,b]_{\mathbb{T}}$, then $f^* : [a,b] \to \mathbb{R}$ is M-integrable on $[a,b]$. In this case, $(M_\triangle) \int_a^b f = (M) \int_a^b f^*$.*

Proof. Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be $M_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and let $\epsilon > 0$. Then there exists a $\Delta-$gauge $\delta = (\delta_L, \delta_R)$ on $[a,b]_{\mathbb{T}}$ such that

(1) $\left|f(\mathcal{P}) - (M_\Delta)\int_a^b f\right| < \epsilon$ for each $\delta-$fine McShane partition $\mathcal{P} = \{(\xi_i, [a_i, b_i])\}$ of $[a,b]_{\mathbb{T}}$,
(2) $\delta_R(t) = \sigma(t) - t$ if $t \in [a,b]_{\mathbb{T}}^{rs}$, where $[a,b]_{\mathbb{T}}^{rs}$ is the set of all right scattered points of $[a,b]_{\mathbb{T}}$,
(3) $t + \delta_R(t) \in [a,b]_{\mathbb{T}}$ if $t \in [a,b]_{\mathbb{T}}^{rd}$, where $[a,b]_{\mathbb{T}}^{rd}$ is the set of all right dense points of $[a,b]_{\mathbb{T}}$,
(4) $\delta_L(t) < \min\left\{\frac{s_k - \rho(s_k)}{3}, \frac{\epsilon}{2^k(|f(s_k)|+|f(\rho(s_k))|+1)}\right\}$ if $t = s_k$, where $[a,b]_{\mathbb{T}}^{ls} = \{s_1, s_2, s_k, \cdots\}$, and
(5) $t - \delta_L(t) \in [a,b]_{\mathbb{T}}$ if $t \in [a,b]_{\mathbb{T}}^{ld}$, where $[a,b]_{\mathbb{T}}^{ld}$ is the set of all left dense points of $[a,b]_{\mathbb{T}}$.

Define a gauge $\delta^1 = (\delta_L^1, \delta_R^1)$ on $[a,b]$ by

$$\delta^1(t) = \begin{cases} (\delta_L(t), \delta_R(t)) & \text{if} \quad t \in [a,b]_{\mathbb{T}} \\ (t - \rho(s_k), s_k - t) & \text{if} \quad t \in (\rho(s_k), s_k), \quad k = 1, 2, 3, \cdots . \end{cases}$$

Assume that $D = \{(\zeta_i, [u_i, v_i])\}_{i=1}^n$ is a δ^1-fine McShane partition of $[a,b]$. Let $i \leq n$ and $k \in \mathbb{N}$ be given. Suppose That $(\rho(s_k), s_k) \not\subset (u_i, v_i)$, $(\rho(s_k), s_k) \not\supset (u_i, v_i)$ and $(\rho(s_k), s_k) \cap (u_i, v_i) \neq \emptyset$. Then we note that the substituting $(\zeta_i, [\rho(s_k), s_k] \cap [u_i, v_i])$ and $(\zeta_i, (\rho(s_k), s_k)^c \cap [u_i, v_i])$ instead of $(\zeta_i, [u_i, v_i])$ in D does not alter the value of $f^*(D)$.

Hence we may assume that

$(\rho(s_k), s_k) \subset (u_i, v_i)$ or $(\rho(s_k), s_k) \supset (u_i, v_i)$ or $(\rho(s_k), s_k) \cap (u_i, v_i) = \emptyset$ for each $i \leq n$ and $k \in \mathbb{N}$.

Let K be the set of all $k \in \mathbb{N}$ such that $(u_i, v_i) \subset (\rho(s_k), s_k)$ for some $i \leq n$. Then K is clearly a finite set. Denote $K = \{k_1, k_2, , \cdots, k_q\}$.

Let $A_j = \{i \leq n | (u_i, v_i) \subset (\rho(s_{k_j}), s_{k_j})\}$ for $j = 1, 2, \cdots, q$,

$$A = \bigcup_{j=1}^{q} A_j \quad \text{and} \quad B = \{1, 2, \cdots, n\} - A.$$

If $i \in B$, then $\zeta_i \in [a,b]_{\mathbb{T}}$ and $u_i, v_i \in [a,b]_{\mathbb{T}}$.

Let $A_j^1 = \{i \in A_j | \rho(s_{k_j}) \leq \zeta_i \leq s_{k_j}\}$ and $A_j^2 = A_j - A_j^1$.

Define

$$\zeta_i^1 = \begin{cases} \zeta_i & \text{if} \quad i \in B \\ \rho(s_k) & \text{if} \quad i \in A_j^1 \quad (j = 1, 2, \cdots, q) \\ \zeta_i & \text{if} \quad i \in A_j^2 \quad (j = 1, 2, \cdots, q). \end{cases}$$

Let $D^1 = \{(\zeta_i^1, [u_i, v_i])\}_{i=1}^n$. Then clearly D^1 is a δ^1-fine McShane partition on $[a, b]$ and $\zeta_i^1 \in [a, b]_{\mathbb{T}}$ for each $i \le n$, and

$$\begin{aligned}
&|f^*(D^1) - f^*(D)| \\
&= \Big| \sum_{i \in A} [f^*(\zeta_i^1) - f^*(\zeta_i)](v_i - u_i) \Big| \\
&= \Big| \sum_{j=1}^{q} \sum_{i \in A_j^1} [f^*(\zeta_i^1) - f^*(\zeta_i)](v_i - u_i) \Big| \\
&\le \sum_{j=1}^{q} \sum_{i \in A_j^1} |f^*(\zeta_i^1) - f^*(\zeta_i)|(v_i - u_i) \\
&\le \sum_{j=1}^{q} \Big[(|f^*(\rho(s_{k_j}))| + |f^*(s_{k_j})|) \sum_{i \in A_j^1} (v_i - u_i) \Big] \\
&< \sum_{j=1}^{q} \frac{\epsilon\big(|f(\rho(s_{k_j}))| + |f(s_{k_j})|\big)}{2^{k_j}\big(|f(\rho(s_k))| + |f(s_{k_j})| + 1\big)} < \epsilon.
\end{aligned}$$

For each j, choose $\lambda_j, \mu_j \in A_j$ such that

$$f(\zeta_{\lambda_j}^1) = \min_{i \in A_j} f(\zeta_i^1) \quad \text{and} \quad f(\zeta_{\mu_j}^1) = \max_{i \in A_j} f(\zeta_i^1).$$

Then the partitions

$$D_1^1 = \{(\zeta_i^1, [u_i, v_i]) | i \in B\} \cup \{(\zeta_{\lambda_j}^1, [\rho(s_{k_j}), s_{k_j}]) | j = 1, 2, \cdots, q\}$$

and

$$D_2^1 = \{(\zeta_i^1, [u_i, v_i]) | i \in B\} \cup \{(\zeta_{\mu_j}^1, [\rho(s_{k_j}), s_{k_j}]) | j = 1, 2, \cdots, q\}$$

are both $\delta-$fine McShane partitions on $[a, b]_{\mathbb{T}}$ and

$$\begin{aligned}
f^*(D^1) &= \sum_{i \in B} f(\zeta_i^1)(v_i - u_i) + \sum_{j=1}^{q} \sum_{i \in A_j} f(\zeta_i^1)(v_i - u_i) \\
&\le \sum_{i \in B} f(\zeta_i^1)(v_i - u_i) + \sum_{j=1}^{q} f(\zeta_{\mu_j}^1)(s_{k_j} - \rho(s_{k_j})) \\
&= f(D_2^1) < (M_\Delta) \int_a^b f + \epsilon.
\end{aligned}$$

Similarly, we have

$$f^*(D^1) \geq f(D_1^1) > (M_\Delta)\int_a^b f - \epsilon,$$

so that $|f^*(D^1) - (M_\Delta)\int_a^b f| < \epsilon$.

Hence,

$$\begin{aligned}\left|f^*(D) - (M_\Delta)\int_a^b f\right| &\leq |f^*(D) - f^*(D^1)| + \left|f^*(D^1) - (M_\Delta)\int_a^b f\right| \\ &< 2\epsilon.\end{aligned}$$

Thus, f^* is McShane integrable and

$$(M)\int_a^b f^* = (M_\Delta)\int_a^b f.$$

□

THEOREM 2.3. *If $f^* : [a,b] \to \mathbb{R}$ is M-integrable on $[a,b]$, then $f : [a,b]_\mathbb{T} \to \mathbb{R}$ is $M_\Delta-$integrable on $[a,b]_\mathbb{T}$.*

Proof. Suppose that f^* is M-integrable on $[a,b]$. Define $g : [a,b] \to \mathbb{R}$ by

$$g(t) = \begin{cases} |f(\rho(t))| + |f(t)| & \text{if} \quad t \in [a,b]_\mathbb{T}^{ls} \\ 0 & \text{if} \quad t \in [a,b] - [a,b]_\mathbb{T}^{ls}. \end{cases}$$

Then since $[a,b]_\mathbb{T}^{ls}$ is a countable set, $g = 0$ a.e. on $[a,b]$. Hence, g is M-integrable on $[a,b]$ and $(M)\int_a^b g = 0$. Let $\epsilon > 0$. Then there exists a gauge $\delta = (\delta_L, \delta_R)$ on $[a,b]$ such that

$$\left|f^*(D) - (M)\int_a^b f^*\right| < \epsilon \qquad \text{and} \qquad |\text{g(D)}| < \epsilon$$

for each $\delta-$fine McShane partition D of $[a,b]$.

Define a $\Delta-$gauage $\delta^1 = (\delta_L^1, \delta_R^1)$ on $[a,b]_\mathbb{T}$ by

$$\delta_L^1(t) = \delta_L(t) \qquad \text{if} \quad t \in [a,b]_\mathbb{T}$$

$$\delta_R(t) = \begin{cases} \delta_R(t) & \text{if} \quad t \in [a,b]_\mathbb{T}^{rd} \\ \\ \sigma(t) - t & \text{if} \quad t \in [a,b]_\mathbb{T}^{rs}. \end{cases}$$

Assume that $D = \{(\xi_i, [u_i, v_i])\}_{i=1}^n$ is a δ^1-fine McShane partition of $[a,b]_\mathbb{T}$. Let $A = \{i \leq n | \xi_i \in [a,b]_\mathbb{T}^{rs}, [\xi_i, \sigma(\xi_i)] \subset [u_i, v_i]\}$,

$$B = \{1, 2, \cdots, n\} - A.$$

For each $i \in A$, choose a $\delta-$fine McShane partition $D_i = \{(\xi_{ij}, [u_{ij}, v_{ij}])\}_{j=1}^{p_i}$ of $[\xi_i, \sigma(\xi_i)]$.

Let $D^* = \{(\xi_i, [u_i, v_i])\}_{i \in B} \cup \{(\xi_i, [u_i, \xi_i]) | i \in A, u_i < \xi_i\} \cup \Big[\bigcup_{i \in A} D_i\Big]$. Then D^* is a $\delta-$fine McShane partition of $[a, b]$, and

$$\begin{aligned} |f(D) - f^*(D^*)| &= \Big| \sum_{i \in A} f(\xi_i)(\sigma(\xi_i) - \xi_i) - \sum_{i \in A} \sum_{j \le p_i} f^*(\xi_{ij})(v_{ij} - u_{ij}) \Big| \\ &= \Big| \sum_{i \in A} \Big[f(\xi_i)(\sigma(\xi_i) - \xi_i) - \sum_{\substack{j \le p_i \\ \xi_{ij} < \sigma(\xi_i)}} f(\xi_i)(v_{ij} - u_{ij}) \\ &\quad - \sum_{\substack{i \le p_i \\ \xi_{ij} = \sigma(\xi_i)}} f(\sigma(\xi_i))(v_{ij} - u_{ij}) \Big] \Big| \\ &= \Big| \sum_{i \in A} \sum_{\substack{j \le p_i \\ \xi_{ij} = \sigma(\xi_i)}} \Big[f(\xi_i) - f(\sigma(\xi_i)) \Big] (v_{ij} - u_{ij}) \Big| \\ &\le \sum_{i \in A} \sum_{\substack{j \le p_i \\ \xi_{ij} = \sigma(\xi_i)}} \Big[|f(\xi_i| + |f(\sigma(\xi_i))| \Big] (v_{ij} - u_{ij}) < \epsilon. \end{aligned}$$

Hence,

$$\begin{aligned} &\Big| f(D) - (M) \int_a^b f^* \Big| \\ &\le |f(D) - f^*(D^*)| + \Big| f^*(D^*) - (M) \int_a^b f^* \Big| < 2\epsilon. \end{aligned}$$

Thus, f is $M_\Delta-$integrable on $[a, b]_{\mathbb{T}}$ and

$$(M_\Delta) \int_a^b f = (M) \int_a^b f^*.$$

□

From Theorem 2.2 and 2.3, we get the following theorem.

THEOREM 2.4. *A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is $M_\Delta-$ integrable $[a, b]_{\mathbb{T}}$ if and only if $f^* : [a, b] \to \mathbb{R}$ is M-integrable on $[a, b]$. In this case,*

$$(M_\Delta) \int_a^b f = (M) \int_a^b f^*.$$

We next verify the basic properties of the McShane delta integral.

THEOREM 2.5. *Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ and let $c \in [a,b]$.*

(1) *If f is $M_\Delta-$integrable on $[a,b]_{\mathbb{T}}$, then f is $M_\Delta-$integrable on every subinterval of $[a,b]_{\mathbb{T}}$.*
(2) *If f is $M_\Delta-$integrable on each of the intervals $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$, then f is $M_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and*

$$(M_\Delta)\int_a^b f = (M_\Delta)\int_a^c f + (M_\Delta)\int_c^b f.$$

Proof. (1) If f is $M_\Delta-$integrable on $[a,b]_{\mathbb{T}}$, then f^* is M-integrable on $[a,b]$. Since f^* is M-integrable on every subinterval of $[a,b]$, f is $M_\Delta-$integrable on every subinterval of $[a,b]_{\mathbb{T}}$ by Theorem 2.4.

(2) If f is $M_\Delta-$integrable on each of $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$, then f^* is M-integrable on each of $[a,c]$ and $[c,b]$. By the property of the McShane integral, f^* is M-integrable on $[a,b]$ and

$$(M)\int_a^b f^* = (M)\int_a^c f^* + (M)\int_c^b f^*.$$

By Theorem 2.4, f is $M_\Delta-$integrable on $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$ and

$$(M_\Delta)\int_a^b f = (M_\Delta)\int_a^c f + (M_\Delta)\int_c^b f.$$

□

THEOREM 2.6. *Let $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be $M_\Delta-$ integrable on $[a,b]_{\mathbb{T}}$. Then $\alpha f + \beta g$ is $M_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ for every $\alpha, \beta \in \mathbb{R}$ and*

$$(M_\Delta)\int_a^b (\alpha f + \beta g) = \alpha(M_\Delta)\int_a^b f + \beta(M_\Delta)\int_a^b g.$$

Proof. If f and g are $M_\Delta-$integrable on $[a,b]_{\mathbb{T}}$, then f^* and g^* are M-integrable on $[a,b]$. Hence $\alpha f^* + \beta g^*$ is M-integrable on $[a,b]$, and $(M)\int_a^b(\alpha f^* + \beta g^*) = \alpha(M)\int_a^b f^* + \beta(M)\int_a^b g^*$. By Theorem 2.4, $\alpha f + \beta g$ is $M_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and

$$(M_\Delta)\int_a^b (\alpha f + \beta g) = \alpha(M_\Delta)\int_a^b f + \beta(M_\Delta)\int_a^b g.$$

□

THEOREM 2.7. *(Monotone Convergence Theorem) Let $\{f_n\}$ be a monotone sequence of $M_\Delta-$integrable functions defined on $[a,b]_{\mathbb{T}}$ and* *suppose that* $\{f_n\}$ *converges pointwise to* f *on* $[a,b]_{\mathbb{T}}$*. If* $\lim_{n\to\infty}(M_\Delta)\int_a^b f_n$ *is finite, then* f *is* $M_\Delta-$*integrable on* $[a,b]_{\mathbb{T}}$ *and*

$$(M_\Delta)\int_a^b f = \lim_{n\to\infty}(M_\Delta)\int_a^b f_n.$$

Proof. Let $\{f_n\}$ be a monotone sequence of $M_\Delta-$integrable functions on $[a,b]_{\mathbb{T}}$. Then by Theorem 2.4, $\{f_n^*\}$ is a monotone sequence of M-integrable functions on $[a,b]$, and $\{f_n^*\}$ converges pointwise to f^* on $[a,b]$. Since $(M_\Delta)\int_a^b f_n = (M)\int_a^b f_n^*$ for each n, $\lim_{n\to\infty}(M)\int_a^b f_n^*$ is finite. By the Monotone Convergence Theorem for the M-integral, f^* is M-integrable on $[a,b]$ and

$$(M)\int_a^b f^* = \lim_{n\to\infty}(M)\int_a^b f_n^*.$$

By Theorem 2.4, f is $M_\Delta-$integrable on $[a,b]_{\mathbb{T}}$ and

$$(M_\Delta)\int_a^b f = \lim_{n\to\infty}(M_\Delta)\int_a^b f_n.$$

□

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Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: parkjm@cnu.ac.kr

**
Department of Mathematics Education
KongJu National University
Kongju 314-701, Republic of Korea
E-mail: dhlee@kongju.ac.kr

***
Department of Mathematics Education
Chungbuk National University
Chungju 360-763, Republic of Korea
E-mail: yoonjh@cbnu.ac.kr

****
Department of Mathematics Education
Seowon University
Chungju 361-742, Republic of Korea
E-mail: ykkim@dragon.seowon.ac.kr

*****
Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: ssongtae@edurang.dje.go.kr