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WEAK C_k^f-SPACES FOR MAPS AND THEIR DUALS

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Abstract. In this paper, we introduce and study the concepts of weak C_k^f-spaces for maps which are generalized concepts of C_k^f-spaces for maps, and introduce the dual concepts of weak C_k^f-spaces for maps and obtain some dual results.

1. Introduction

Throughout this paper, a space means a space of the homotopy type of a locally finite connected CW complex. All maps shall mean continuous functions. It is known that any space X is filtered by the projective spaces of ΩX by a result of Milnor [9] and Stasheff [11];

$$\Sigma\Omega X = P^1(\Omega X) \hookrightarrow P^2(\Omega X) \hookrightarrow \cdots \hookrightarrow P^\infty(\Omega X) \simeq X.$$

For each $1 \leq m \leq n$, let $j_{m,n}^X : P^m(\Omega X) \to P^n(\Omega X)$ and $e_n^X : P^n(\Omega X) \to P^\infty(\Omega X) \simeq X$ be the natural inclusions. Let $f : A \to X$ be a map. A space X is called [6] a C_k^f*-space* if the inclusion $e_k^X : P^k(\Omega X) \to X$ is f-cyclic. It is known [6] that a space X is a C_k^f-space for a map $f : A \to X$ if and only if $G^f(Z, X) = [Z, X]$ for any space Z with $cat\ Z \leq k$.

In this paper, we introduce the concepts of weak C_k^f-spaces for maps which are generalizations of C_k^f-spaces for maps [6] and study some properties of weak C_k^f-spaces for maps. We show that a space X is a weak C_k^f-space for a map $f : A \to X$ if and only if $WG^f(Z, X) = [Z, X]$ for any space Z with $cat\ Z \leq k$. Let $f : A \to X$ and $g : B \to Y$ be any maps. Then we show that the product space $X \times Y$ is a weak $C_k^{(f\times g)}$-space for a map $(f \times g) : A \times B \to X \times Y$ if and only if X is a weak C_k^f-space for a map $f : A \to X$ and Y is a weak C_k^g-space for a

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map $g : B \to Y$. We also introduce the dual concepts of weak C_k^f-spaces for maps and obtain some dual results.

2. Weak C_k^f-spaces for maps

Let $f : A \to X$ be a map. A based map $g : B \to X$ is called *f-cyclic* [10] if there is a map $\phi : B \times A \to X$ such that the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{\phi} & X \\ j\uparrow & & \nabla\uparrow \\ A \vee B & \xrightarrow{(f\vee g)} & X \vee X \end{array}$$

is homotopy commute, where $j : A \vee B \to A \times B$ is the inclusion and $\nabla : X \vee X \to X$ is the folding map. We call such a map ϕ an *associated map* of a f-cyclic map g. Clearly, g is f-cyclic iff f is g-cyclic. In the case $f = 1_X : X \to X$, a map $g : B \to X$ is called cyclic [12]. We denote the set of all homotopy classes of f-cyclic maps from B to X by $G^f(B, X)$ which is called the *Gottlieb set for a map* $f : A \to X$. In the case $f = 1_X : X \to X$, we called such a set $G^1(B, X)$ as the *Gottlieb set*, denoted by $G(B, X)$. In particular, $G^f(S^n, X)$ will be denoted by $G_n^f(X)$ which is called the *Gottlieb Group for a map* $f : A \to X$. Gottlieb [3] introduced and studied the *evaluation subgroups* $G_n(X) = G_n^1(X)$ of $\pi_n(X)$. In general, $G(B, X) \subset G^f(B, X) \subset [B, X]$ for any spaces A, B, X and any map $f : A \to X$.

It is shown [15] that $G_5(S^5 \times S^5) \cong 2\mathbb{Z} \oplus 2\mathbb{Z} \neq G_5^{i_1}(S^5 \times S^5) \cong 2\mathbb{Z} \oplus \mathbb{Z} \neq \pi_5(S^5 \times S^5) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Let $f : A \to X$ be a based map. A based map $g : B \to X$ is called a weakly f-cyclic [17] if $g_\#(\pi_n(B)) \subset G_n^f(X)$ for all n. In the case $f = 1_X : X \to X$, a map $g : B \to X$ is called weakly cyclic [13]. The set of all homotopy classes of weakly f-cyclic maps from B to X is denoted by $WG^f(B, X)$. In the case $f = 1_X : X \to X$, we called such a set $WG^1(B, X)$ as the *weak Gottlieb set*, denoted by $WG(B, X)$. In particular, $WG^f(S^n, X)$ will be denoted by $WG_n^f(X)$ which is called the *weak Gottlieb group for a map* $f : A \to X$. It is known [13] that any cyclic map is a weakly cyclic map, but the converse does not hold. That means, in general, $G(B, X) \subsetneq WG(B, X)$ and $G^f(B, X) \subset WG^f(B, X)$. A space X is called a G-space [3] if $G_n(X) = \pi_n(X)$ for all n. A space X is called a G^f-space for a map $f : A \to X$ [17] if $G_n^f(X) = \pi_n(X)$ for all n.

Lemma 2.1. *Let $f : A \to X$, $g : B \to Y$ be maps. If $\alpha : Z \to X$ is a weakly f-cyclic map and $\theta : C \to Z$ is an arbitrary map, then $\alpha \circ \theta : C \to X$ is a weakly f-cyclic map.*

Proof. Since $\alpha : Z \to X$ is a weakly f-cyclic map, $(\alpha \circ \theta)_\#(\pi_n(C)) = \alpha_\# \circ \theta_\#(\pi_n(C)) \subset \alpha_\#(\pi_n(Z)) \subset G_n^f(X)$ for all n. Thus we know $\alpha \circ \theta : C \to X$ is a weakly f-cyclic map. □

We can obtain, from the above definition, the following proposition.

Proposition 2.2. *The followings are equivalent;*
(1) *X is a G^f-space for a map $f : A \to X$*
(2) *$1_X : X \to X$ is weakly f-cyclic*
(3) *$WG^f(Z, X) = [Z, X]$ for any space Z.*

Proof. (1) $\Leftrightarrow$ (2). It follows from the definition of G^f-space. (2) implies (3). For any space Z, let $g : Z \to X$ be any map. Then we know, from Lemma 2.1, that $g = 1_X \circ g : Z \to X$ is a weakly f-cyclic. (3) implies (2). Taking $Z = X$, then $1_X : X \to X$ is a weakly f-cyclic map. □

It is known that any space X is filtered by the projective spaces of ΩX by a result of Milnor [9] and Stasheff [11];

$$\Sigma\Omega X = P^1(\Omega X) \hookrightarrow P^2(\Omega X) \hookrightarrow \cdots \hookrightarrow P^\infty(\Omega X) \simeq X.$$

For each $1 \le m \le n$, let $j_{m,n}^X : P^m(\Omega X) \to P^n(\Omega X)$ and $e_k^X : P^k(\Omega X) \to P^\infty(\Omega X) \simeq X$ be the natural inclusions.

Let $f : A \to X$ be a map. A space X is called [6] a *C_k^f-space for a map* $f : A \to X$ if the inclusion $e_k^X : P^k(\Omega X) \to X$ is f-cyclic.

Theorem 2.3. ([1],[2]) *The category cat $X \le k$ if and only if $e_k^X : P^k(\Omega X) \to X$ has a right homotopy inverse.*

It is known [6] that a space X is a C_k^f-space for a map $f : A \to X$ if and only if $G^f(Z, X) = [Z, X]$ for any space Z with *cat* $Z \le k$.

Definition 2.4. *Let $f : A \to X$ be a map. A space X is called a weak C_k^f-space for a map $f : A \to X$ if $e_k^X : P^k(\Omega X) \to X$ is a weakly f-cyclic map.*

ΣX denote the reduced suspension of X and ΩX denote the based loop space of X. The adjoint functor from the group $[\Sigma X, Y]$ to the group $[X, \Omega Y]$ will be denoted by τ. The symbols e and e' denote $\tau^{-1}(1_{\Omega X})$ and $\tau(1_{\Sigma X})$ respectively.

Proposition 2.5.

(1) *Any weak C_n^f-space for a map $f : A \to X$ is a weak C_m^f-space for a map $f : A \to X$ for $1 \leq m \leq n$.*

(2) *X is a weak C_1^f-space for a map $f : A \to X$ if and only if $WG^f(\Sigma B, X) = [\Sigma B, X]$ for any space B.*

Proof. (1) Since $e_m^X \sim e_n^X \circ j_{m,n}^X : P^m(\Omega X) \to X$ and $e_n^X : P^n(\Omega X) \to X$ is a weakly f-cyclic map, we know,from Lemma 2.1, that any weak C_n^f-space for a map $f : A \to X$ is a weak C_m^f-space for a map $f : A \to X$ for $1 \leq m \leq n$. (2) Suppose X is a weak C_1^f-space for a map $f : A \to X$. Thus $e : \Sigma\Omega X = P^1(\Omega X) \to X$ is a weakly f-cyclic map. Let B be any space and $g : \Sigma B \to X$ a map. Then we know, from Lemma 2.1, that $g = e \circ \Sigma\tau(g) : \Sigma B \to X$ is weakly f-cyclic. On the other hand, taking $B = \Omega X$, $e : \Sigma\Omega X \to X$ is a weakly f-cyclic map. Thus X is a weak C_1^f-space for a map $f : A \to X$. □

Theorem 2.6. *Let $f : A \to X$ be a map. Then X is a weak C_k^f-space for a map $f : A \to X$ if and only if $WG^f(Z, X) = [Z, X]$ for any space Z with cat $Z \leq k$.*

Proof. Suppose that X is a weak C_k^f-space. Then $e_k^X : P^k(\Omega X) \to X$ is weakly f-cyclic. Let Z be a space with *cat* $Z \leq k$ and $g : Z \to X$ any map. Since *cat* $Z \leq k$, there exists a map $s_k^Z : Z \to P^k(\Omega Z)$ such that $e_k^Z \circ s_k^Z \sim 1_Z$. We see $g \circ e_k^Z \sim e_k^X \circ P^k(\Omega g)$ by the naturality of the construction of $P^k(\Omega Z)$. Thus we know, from Lemma 2.1, that $g \sim g \circ e_k^Z \circ s_k^Z \sim e_k^X \circ (P^k(\Omega g) \circ s_k^Z) : Z \to X$ is weakly f-cyclic. It follows that $WG^f(Z, X) = [Z, X]$. Conversely, assume that $WG^f(Z, X) = [Z, X]$ for any space Z with *cat* $Z \leq k$. It is known that *cat* $C_\theta \leq$ *cat* $Y + 1$ for any map $\theta : X \to Y$. Thus *cat* $P^k(\Omega X) =$ *cat* $C_\theta \leq$ *cat* $P^{k-1}(\Omega X) + 1$, where $\theta : (\Omega X) * \cdots * (\Omega X) \to P^{k-1}(\Omega X)$ is the map. By induction, we have *cat* $P^k(\Omega X) \leq k$. Thus we know that $e_k : P^k(\Omega X) \to X$ is weakly f-cyclic by our assumption, and hence X is a weak C_k^f-space for a map $f : A \to X$. □

We have the following corollary from Theorem 2.6, Proposition 2.2 and Proposition 2.5.

Corollary 2.7. *Any G^f-space is a weak C_k^f-space and any weak C_k^f-space is a weak C_1^f-space.*

We can easily obtain the following proposition.

Proposition 2.8.

(1) *For any map* $f : A \to X,\ i : C \to A$ *and any space* Z, $G^f(Z, X) \subset G^{f\circ i}(Z, X)$.
(2) *If* $r : X \to Y$ *is a map, then* $r_\# : G^f(Z, X) \to G^{r\circ f}(Z, Y)$ *for any space* Z.

PROPOSITION 2.9. [6] *Let* $f : A \to X$ *and* $g : B \to Y$ *be any maps. The relation*

$$G^{(f\times g)}(Z, X \times Y) \cong G^f(Z, X) \times G^g(Z, Y)$$

holds for any space Z *(under the identification* $[Z, X \times Y] \cong [Z, X] \times [Z, Y]$*).*

LEMMA 2.10. *Let* $f : A \to X,\ g : B \to Y$ *be maps.*
(1) *If* $\alpha : Z \to X$ *is weakly* f*-cyclic and* $\beta : Z \to Y$ *is weakly* g*-cyclic, then* $(\alpha \times \beta)\Delta : Z \to X \times Y$ *is weakly* $(f \times g)$*-cyclic.*
(2) *If* $r : X \to Y$ *is a map, then* $r_\# : WG^f(Z, X) \to WG^{r\circ f}(Z, Y)$ *for any space* Z.

Proof. (1) Since $\alpha : Z \to X$ is weakly f-cyclic and $\beta : Z \to Y$ is weakly g-cyclic, we have, from Proposition 2.9, that $((\alpha\times\beta)\Delta)_\#(\pi_n(Z)) \cong \alpha_\#(\pi_n(Z)) \times \beta_\#(\pi_n(Z)) \subset G_n^f(X) \times G_n^g(X) \cong G_n^{f\times g}(X \times X)$. Thus $(\alpha \times \beta)\Delta : Z \to X \times Y$ is weakly $(f \times g)$-cyclic. (2) Let $g : Z \to X$ be a weakly f-cyclic map. Then $g_\#(\pi_n(Z)) \subset G_n^f(X)$ for all n. Thus we know, from Proposition 2.8(2), that $(r \circ g)_\#(\pi_n(Z)) = r_\# \circ g_\#(\pi_n(Z)) \subset r_\#(G_n^f(X)) \subset G_n^{r\circ f}(Y)$. Thus we have $r_\# : WG^f(Z, X) \to WG^{r\circ f}(Z, Y)$ for any space Z. □

PROPOSITION 2.11. *Let* $f : A \to X,\ g : B \to Y$ *be maps. Then* $WG^{(f\times g)}(Z, X \times Y) \cong WG^f(Z, X) \times WG^g(Z, Y)$ *for any space* Z.

Proof. Let $\alpha : Z \to X$ and $\beta : Z \to Y$ be maps. Suppose that $(\alpha, \beta) \in WG^f(Z, X)\times WG^g(Z, Y)$. We know, from Lemma 2.10(1), that $(\alpha\times\beta)\circ\Delta_Z$ is weakly $f\times g$-cyclic and $(\alpha\times\beta)\circ\Delta_Z \in WG^{f\times g}(Z, X\times Y)$, where $\Delta_Z : Z \to Z \times Z$ is the diagonal map. Conversely, suppose that $(\alpha\times\beta)\Delta \in WG^{f\times g}(Z, X\times Y)$. Let $p_1 : X\times Y \to X$ and $p_2 : X\times Y \to Y$ be the projections and $i_1 : X \to X \times Y$ and $i_2 : Y \to X \times Y$ be the inclusions defined by $i_1(x) = (x, y_0)$ and $i_2(y) = (x_0, y)$ for any $x \in X$ and $y \in Y$, where $x_0 \in X$ and $y_0 \in Y$ are base points. Then we have, from Lemma 2.10(2), that $\alpha \sim p_1 \circ (\alpha \times \beta)\Delta$ is weakly $p_1 \circ (f \times g)$-cyclic and α is weakly $f \sim p_1 \circ (f \times g) \circ i_1$-cyclic. Similarly $\beta \sim p_2 \circ (\alpha \times \beta)\Delta$ is weakly $g \sim p_2 \circ (f \times g) \circ i_2$-cyclic. It follows that $\alpha \in WG^f(Z, X)$ and $\beta \in WG^g(Z, Y)$. Thus $(\alpha, \beta) \in WG^f(Z, X) \times WG^g(Z, Y)$. □

THEOREM 2.12. *Let $f : A \to X,\ g : B \to Y$ be maps. Then $X \times Y$ is a weak $C_k^{(f\times g)}$-space for a map $(f \times g) : A \times B \to X \times Y$ if and only if X is a weak C_k^f-space for a map $f : A \to X$ and Y is a weak C_k^g-space for a map $g : B \to Y$.*

Proof. If $X \times Y$ is a weak $C_k^{(f\times g)}$-space for a map $(f \times g) : A \times B \to X \times Y$, then for any space Z with $cat\ Z \leq k$ we see, from Theorem 2.6 and Proposition 2.11, that $WG^f(Z, X) \times WG^g(Z, Y) \cong WG^{(f\times g)}(Z, X \times Y) = [Z, X \times Y] \cong [Z, X] \times [Z, Y]$, and hence $WG^f(Z, X) = [Z, X]$ and $WG^g(Z, Y) = [Z, Y]$. Thus X is a weak C_k^f-space for a map $f : A \to X$ and Y is a weak C_k^g-space for a map $g : B \to Y$.

Conversely, suppose that X is a weak C_k^f-space for a map $f : A \to X$ and Y is a weak C_k^g-space for a map $g : B \to Y$. Then $WG^f(Z, X) = [Z, X]$ and $WG^g(Z, Y) = [Z, Y]$ for any space Z with $cat\ Z \leq k$. It follows that $WG^{(f\times g)}(Z, X \times Y) \cong WG^f(Z, X) \times WG^g(Z, Y) = [Z, X] \times [Z, Y] \cong [Z, X \times Y]$ for any space Z with $cat\ Z \leq k$. Thus $X \times Y$ is a weak $C_k^{(f\times g)}$-space for a map $(f \times g) : A \times B \to X \times Y$. □

3. Weak DC_k^p-spaces for maps

In [2], Ganea introduced the concept of cocategory of a space as follows; Let X be any space. Define a sequence of cofibrations

$$\mathcal{C}_k : X \xrightarrow{e'_k} F_k \xrightarrow{s'_k} B_k \ (k \geq 0)$$

as follows, let $\mathcal{C}_0 : X \xrightarrow{e'_0} cX \xrightarrow{s'_0} \Sigma X$ be the standard cofibration. Assuming $\mathcal{C}_k$ to be defined, let F'_{k+1} be the fibre of s'_k and $e''_{k+1} : X \to F'_{k+1}$ lift e'_k. Define F_{k+1} as the reduced mapping cylinder of e''_{k+1}, let $e'_{k+1} : X \to F_{k+1}$ be the obvious inclusion map, and let $B_{k+1} = F_{k+1}/e'_{k+1}(X)$ and $s'_{k+1} : F_{k+1} \to F_{k+1}/e_{k+1}(X)$ the quotient map.

DEFINITION 3.1. [2] *The cocategory of X, cocat X, is the least integer $k \geq 0$ for which there is a map $r : F_k \to X$ such that $r \circ e'_k \sim 1$. If there is no such integer, cocat $X = \infty$.*

The following remark can easily obtained from the above definition.

REMARK 3.2. *cocat $X \leq k$ if and only if $e'_k : X \to F_k$ has a left homotopy inverse.*

For a map $p : X \to A$, a based map $g : X \to B$ is *p-cocyclic* [10] if there is a map $\theta : X \to A \vee B$ such that $j\theta \sim (p \times g)\Delta$, where $j : A \vee B \to A \times B$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. *The dual Gottlieb set for a map $p : X \to A$, $DG^p(X, B)$,* is the set of all homotopy classes of p-cocyclic maps from X to B. In the case $p = 1_X : X \to X$, we call a 1-cocyclic map is just a cocyclic map, and denoted by, $DG(X, B)$, which is the set of all homotopy classes of cocyclic maps from X to B. We can identify $H^n(X; \pi)$ with $[X, K(\pi, n)]$, and defined the coevaluation subgroup $G^n(X; \pi)$ of $H^n(X; \pi)$ to be the set of all homotopy classes of cocyclic maps from X to $K(\pi, n)$. The group $G^n(X) = G^n(X; \mathbb{Z})$ is the dual of Gottlieb group $G_n(X)$ of $\pi_n(X)$ for all n. A space X is called a G'-space [4] if $G^n(X) = H^n(X)$ for all n. In particular, $DG^p(X, K(\mathbb{Z}, n))$ will be denoted by $G^n_p(X)$ which is called the *dual Gottlieb group for a map $p : X \to A$.*

In general, $DG(X, B) \subset DG^p(X, B) \subset [X, B]$ for any map $p : X \to A$ and any space B. However, there is an example in [14] such that $DG(X, B) \neq DG^p(X, B) \neq [X, B]$.

It is introduced [19] that a space X is called *DC^p_k-space for a map* $p : X \to A$ if $e'^X_k : X \to F^X_k$ is p-cocyclic. It is known [18] that for a map $p : X \to A$, $g : X \to B$ is p-cocyclic if and only if $g^*([B, Z]) \subset DG^p(X, Z)$ for any space Z.

Definition 3.3. *Let $p : X \to A$ be a map, a map $g : X \to B$ is called a weakly p-cocyclic map if $g^*(H^n(B)) \subset G^n_p(X)$ for all n. The set of all homotopy classes of weakly p-cocyclic maps from X to B is denoted by $WDG^p(X, B)$. In particular, $WDG^p(X, K(\mathbb{Z}, n))$ will be denoted by $WG^n_p(X)$ which is called the weak Gottlieb group for a map $p : X \to A$.*

Clearly any p-cocyclic $g : X \to B$ is a weakly p-cocyclic, but the converse does not hold.It is well known [4] that $\mathbb{R}P^2$ is a G'-space, but not co-H-space. Thus we easily know that $1 : \mathbb{R}P^2 \to \mathbb{R}P^2$ is weakly cocyclic, but not cocyclic.

We showed [19] that a space X is a DC^p_k-space for a map $p : X \to A$ if and only if $DG^p(X, Z) = [X, Z]$ for any space Z with *cocat* $Z \leq k$.

Definition 3.4. *Let $p : X \to A$ be a map. A space X is called a weak DC^p_k−space for a map $p : X \to A$ if $e'^X_k : X \to F^X_k$ is a weakly p-cocyclic, that is, $(e'^X_k)^*(H^n(F^X_k)) \subset G^n_p(X)$ for all n.*

Theorem 3.5. *Let $p : X \to A$ be a map. Then a space X is a weak DC^p_k−space for a map $p : X \to A$ if and only if $WDG^p(X, Z) = [X, Z]$ for any space Z with cocat $Z \leq k$.*

Proof. Suppose X is a weak DC_k^p-space for a map $p : X \to A$. Let Z be a space with *cocat* $Z \leq k$ and $g : X \to Z$ any map. For any n, let $\alpha : Z \to K(\mathbb{Z}, n)$ be any map. Since *cocat* $Z \leq k$, there is a map $s : F_k \to Z$ such that $s \circ e'_k \sim 1_Z$. Since $e'_k : X \to F_k$ is weakly p-cocyclic, $\alpha \circ s \circ F_k(g) \circ e'_k : X \to K(\mathbb{Z}, n)$ is p-cocyclic. Thus we have a map $\theta : X \to A \vee K(\mathbb{Z}, n)$ such that $j\theta \sim (p \times (\alpha \circ s \circ F_k(g) \circ e'_k))\Delta$, where $j : A \vee K(\mathbb{Z}, n) \to A \times K(\mathbb{Z}, n)$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. Interpreting F_k as a functor, we have the following homotopy commutative diagram;

$$\begin{array}{ccccccc}
X & \xrightarrow{g} & Z & & & & \\
\downarrow{\scriptstyle e'_k} & & \downarrow{\scriptstyle e'_k} & \searrow{\scriptstyle 1} & & & \\
F_k(X) & \xrightarrow{F_k(g)} & F_k(Z) & \xrightarrow{s} & Z & \xrightarrow{\alpha} & K(\mathbb{Z}, n).
\end{array}$$

Thus we have that $\alpha \circ s \circ F_k(g) \circ e'_k \sim \alpha \circ g : X \to K(\mathbb{Z}, n)$ and $\alpha \circ g : X \to K(\mathbb{Z}, n)$ is p-cocyclic. Thus we know that g is a weakly p-cocyclic map and $WDG^p(X, Z) = [X, Z]$ for any space Z with *cocat* $Z \leq k$. On the other hand, we assume that for any space Z with *cocat* $Z \leq k$, $WDG^p(X, Z) = [X, Z]$. It is well known [1] that if $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration, then *cocat* $F \leq$ *cocat* $E + 1$. From the fact that $F_k \simeq F'_k \to F_{k-1} \overset{s'_{k-1}}{\to} B_{k-1}$ is a fibration, we know that *cocat* $F_k \leq$ *cocat* $F_{k-1} + 1$. Then we have, by induction, *cocat* $F_k \leq k$. Thus we know, by our assumption, that $e'_k : X \to F_k$ is weakly p-cocyclic and X is a weak DC_k^p-space for a map $p : X \to A$.

□

Proposition 3.6. *Let $p : X \to A$ and $q : Y \to A$ be any maps. Then the relation*

$$WDG^{\nabla(p \vee q)}(X \vee Y, B) \equiv WDG^p(X, B) \times WDG^q(Y, B)$$

holds for any space B.

Proof. Let $g : X \vee Y \to B$ be a weakly $\nabla(p \vee q)$-cocyclic map.For any n, let $\alpha : B \to K(\mathbb{Z}, n)$ be any map. Since $\alpha \circ g : X \vee Y \to K(\mathbb{Z}, n)$ is $\nabla(p \vee q)$-cocyclic, there is a map $G : X \vee Y \to A \vee K(\mathbb{Z}, n)$ such that $jG \sim ((\nabla(p \vee q)) \times (\alpha \circ g))\Delta$, where $j : A \vee K(\mathbb{Z}, n) \to A \times K(\mathbb{Z}, n)$ is the inclusion and $\Delta : X \vee Y \to (X \vee Y) \times (X \vee Y)$ is the diagonal. Consider the maps $G_1 = G \circ i_1 : X \to A \vee K(\mathbb{Z}, n)$ and $G_2 = G \circ i_2 : Y \to A \vee K(\mathbb{Z}, n)$, where $i_1 : X \to X \vee Y$, $i_2 : Y \to X \vee Y$ are natural inclusions. Then $j \circ G_1 \sim (p \times (\alpha \circ g \circ i_1))\Delta$, $j \circ G_2 \sim (q \times (\alpha \circ g \circ i_2))\Delta$, where $i_1 : X \to X \vee Y$, $i_2 : Y \to X \vee Y$ are natural inclusions. Thus we know

$(g \circ i_1, g \circ i_2) \in WDG^p(X, B) \times WDG^q(Y, B)$. On the other hand, let $(g_1, g_2) \in WDG^p(X, B) \times WDG^q(Y, B)$. For any n, let $\alpha : B \to K(\mathbb{Z}, n)$ be any map. Since $g_1 : X \to B$ is weakly p-cocyclic and $g_2 : Y \to B$ is weakly q-cocyclic, there are maps $G_1 : X \to A \vee K(\mathbb{Z}, n)$ and $G_2 : Y \to A \vee K(\mathbb{Z}, n)$ such that $j \circ G_1 \sim (p \times (\alpha \circ g_1))\Delta$, $j \circ G_2 \sim (q \times (\alpha \circ g_2))\Delta$ respectively. Let $T : A \vee K(\mathbb{Z}, n) \vee A \vee K(\mathbb{Z}, n) \to A \vee A \vee K(\mathbb{Z}, n) \vee K(\mathbb{Z}, n)$ be the switching map. Then consider the map $G = (\nabla \vee \nabla) \circ T \circ (G_1 \vee G_2) : X \vee Y \to A \vee K(\mathbb{Z}, n)$. Then $j \circ G \sim ((\nabla(p \vee q) \times \alpha \circ \nabla(g_1 \vee g_2))\Delta$, where $\Delta : X \vee Y \to (X \vee Y) \times (X \vee Y)$ is the diagonal map. Thus we know $\nabla(g_1 \vee g_2) \in WDG^{\nabla(p \vee q)}(X \vee Y, B)$. □

THEOREM 3.7. *Let $p : X \to A$ and $q : Y \to A$ be any maps. Then the wedge space $X \vee Y$ is a weak $DC_k^{\nabla(p \vee q)}$-space for a map $\nabla(p \vee q) : X \vee Y \to A$ if and only if X is a weak DC_k^p-space for a map $p : X \to A$ and Y is a weak DC_k^q-space for a map $q : Y \to A$.*

Proof. If $X \vee Y$ is a weak $DC_k^{\nabla(p \vee q)}$-space for a map $\nabla(p \vee q) : X \vee Y \to A$, then we know, from Theorem 3.5 and Proposition 3.6, that $WDG^p(X, Z) \times WDG^q(Y, Z) \equiv WDG^{\nabla(p \vee q)}(X \vee Y, Z) = [X \vee Y, Z] \equiv [X, Z] \times [Y, Z]$ for any space Z with *cocat* $Z \leq k$. Thus $WDG^p(X, Z) = [X, Z]$, $WDG^q(Y, Z) = [Y, Z]$ and X is a weak DC_k^p-space for a map $p : X \to A$ and Y is a weak DC_k^q-space for a map $q : Y \to A$. On the other hand, suppose that X is a weak DC_k^p-space for a map $p : X \to A$ and Y is a weak DC_k^q-space for a map $q : Y \to A$. Then we know that $WDG^p(X, Z) = [X, Z]$, $WDG^q(Y, Z) = [Y, Z]$ for any space Z with *cocat* $Z \leq k$. Thus we have, from Proposition 3.6, that $WDG^{\nabla(p \vee q)}(X \vee Y, Z) \equiv WDG^p(X, Z) \times WDG^q(Y, Z) = [X, Z] \times [Y, Z] \equiv [X \vee Y, Z]$ for any space Z with *cocat* $Z \leq k$. Thus we know, from Theorem 3.5, that $X \vee Y$ is a weak $DC_k^{\nabla(p \vee q)}$-space for a map $\nabla(p \vee q) : X \vee Y \to A$.

□

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