A NOTE ON A QUESTION RELATED TO KOLMOGOROV FORWARD EQUATION

Joongul Lee*

ABSTRACT. We examine a question raised in [1] regarding existence and uniqueness of the steady-state solution of certain type of Kolmogorov forward equation in this paper, and prove that the question holds for S^1 by giving an explicit solution.

1. Introduction

We first review the question raised in [1].

Let \mathcal{M} be an orientable Riemannian manifold of dimension n, whose inner product at each point is denoted by \langle , \rangle . The inner product gives us a canonical isomorphism between the 1-forms on \mathcal{M} and the vector fields on \mathcal{M} . For convenience we denote the isomorphism and its inverse by a *tilde*. Then if f is a function on \mathcal{M} , its gradient is the vector field

$$\nabla f = \widetilde{df}.$$

And if E is a vector field on \mathcal{M} , its divergence is the function

$$\nabla \cdot E = *d * \widetilde{E}$$

where * is the Hodge star operator. (cf. [2])

QUESTION 1.1. Assume \mathcal{M} is compact. For a vector field X on \mathcal{M} , is it possible to fi is it possible to find unique vector fields E, F with the

- 1. X = E + F,
- 2. $F = -\nabla \phi$ for some function ϕ on \mathcal{M} ,
- 3. $\langle E, F \rangle + \nabla \cdot E = 0$ at every point of \mathcal{M} .

As explained in [1], one can derive the steady-state solution of certain type of Kolmogorov forward equation from such decomposition.

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2. The main result

THEOREM 2.1. For $\mathcal{M} = S^1$, the question has an affirmative answer for every vector field.

Proof. Let $p:\mathbb{R}\to S^1$ be the covering map $p(x)=(\cos x,\sin x)$. p is a local isometry and is periodic with period 2π . Therefore we may identify the vector fields on S^1 with periodic functions (with period 2π) on \mathbb{R} , and under such identification the gradient and divergence are just the derivative. A periodic function g on \mathbb{R} is a derivative of a periodic function if and only if $\int_0^{2\pi} g\,dx=0$.

Let f(x) be the given periodic function, and consider the differential equation

$$\frac{dy}{dx} + fy - y^2 = 0. ag{2.1}$$

This is a Bernoulli equation and it can be transformed into a first order linear differential equation. Let $\alpha = \int_0^{2\pi} f(t) dt$. Choose a primitive F(x) of f(x) (i.e. $\frac{dF}{dx} = f(x)$), and let $I(x) = e^{-F(x)}$. Multiplying I on the differential equation and integrating, we get the general solution y = 0 or

$$y(x) = \frac{I(x)}{c - \int_0^x I(t) \, dt}$$
 (2.2)

where c is a constant.

To find a periodic solution, first assume $\alpha \neq 0$. Setting $y(0) = y(2\pi)$ on Equation (2.2) gives

$$c = \frac{\int_0^{2\pi} I(t) \, dt}{1 - e^{-\alpha}},$$

and so

$$y(x) = \frac{(1 - e^{-\alpha})I(x)}{\int_0^{2\pi} I(t) dt - (1 - e^{-\alpha}) \int_0^x I(t) dt}.$$
 (2.3)

Equation (2.3) makes sense and gives the periodic solution y = 0 when $\alpha = 0$. Also note that the denominator of Equation (2.3) is never 0.

To prove that Equation (2.3) gives a periodic solution, let $z(x) = y(x+2\pi)$. z(x) is also a solution of the given differential equation (2.1) because f is periodic. As $z(0) = y(2\pi) = y(0)$, it follows from the uniqueness of the solution of differential equation that z = y.

f-y is a derivative of a periodic function because either f itself is a derivative (when $\alpha=0$) or

$$f - y = -\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} \{-\log|y|\},$$

together with the fact that $y(x) \neq 0$ for $x \in \mathbb{R}$ when $\alpha \neq 0$. Finally we note that y(x) in Equation (2.3) is independent of the choice of F. \square

References

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Department of Mathematics Education Hongik University Seoul 121-791, Republic of Korea *E-mail*: jglee@hongik.ac.kr