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LORENTZIAN ALMOST r-PARA-CONTACT STRUCTURE IN TANGENT BUNDLE

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Abstract. Almost contact and almost complex structures in the tangent bundle have been studied by K. Yano and S. Ishihara[5] and others. In the present paper, we have studied Lorentzian almost r-para-contact structure in the tangent bundle. Some results related to Lie-derivative have been studied.

1. Introduction

Let M be a n-dimensional differentiable manifold of C^∞ class and $T_p(M)$ the tangent space of M at a point p of M. Then the set $T(M) = \bigcup_{p\in M} T_p(M)$ is called the *tangent bundle* over the manifold M. For any point $\tilde{p}$ of $T(M)$, the correspondence $\tilde{p} \to p$ determines the bundle projection $\pi : T(M) \to M$. Thus $\pi(\tilde{p}) = p$, where $\pi : T(M) \to M$ defines the bundle projection of $T(M)$ over M. The set $\pi^{-1}(p)$ is called the *fibre* over $p \in M$ and M the *base space*.

Vertical lifts:

If f is a function in M, then we write f^V for the function in $T(M)$ obtained by forming the composition of $\pi : T(M) \to M$ and $f : M \to R$ so that $f^V = f \circ \pi$. Thus, if a point $\tilde{p} \in \pi^{-1}(U)$ has induced coordinates (x^h, y^h), then

$$f^V(\tilde{p}) = f^V(x, y) = f \circ \pi(\tilde{p}) = f(p) = f(x).$$

Thus the value of $f^V(\tilde{p})$ is constant along each fibre $T_p(M)$ and equal to the value $f(p)$. We call f^V the *vertical lift* of the function f.

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Vertical lifts to a unique algebraic isomorphism of the tensor algebra $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients by the conditions

$$(P \otimes Q)^V = P^V \otimes Q^V, \ (P+R)^V = P^V + R^V,$$

where P, Q and R are arbitrary elements of $\Im(M)$.

Complete lifts:

If f is a function in M, then we write f^C for the function in $T(M)$ defined by $f^C = i(df)$[4] and call f^C the *complete lift* of the function f. The complete lift f^C of a function f has the local expression $f^C = y^i \partial_i f = \partial f$ with respect to the induced coordinates in $T(M)$, where ∂f denotes $y^i \partial_i f$.

Suppose that $X \in \Im_0^1(M)$. We define a vector field X^C in $T(M)$ by $X^C f^C = (Xf)^C$, where f is an arbitrary function in M and call X^C the *complete lift* of X in $T(M)$. The complete lift X^C of X with components x^h in M has components

$$X^C = \begin{pmatrix} x^h \\ \partial x^h \end{pmatrix}$$

with respect to the induced coordinates in $T(M)$.

Suppose that $\omega \in \Im_1^0(M)$. Then a 1-form ω^C in $T(M)$ is defined by $\omega^C(X^C) = (\omega(X))^C$, where X is an arbitrary vector field in M. We call ω^C the *complete lift* of ω.

The complete lifts to a unique algebra isomorphism of the tensor algebra $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients is given by the conditions

$$(P \otimes Q)^C = P^C \otimes Q^V + P^V \otimes Q^C, \ (P+R)^C = P^C + R^C,$$

where P, Q and R are arbitrary elements of $\Im(M)$.

Horizontal lifts:

The *horizontal lift* f^H of $f \in \Im_0^0(M)$ to the tangent bundle $T(M)$ is defined by $f^H = f^C - \nabla_\gamma f$, where $\nabla_\gamma f = \gamma(\nabla f)$.

Let $X \in \Im_0^1(M)$. Then the horizontal lift X^H of X is defined in $T(M)$ by

$$X^H = X^C - \nabla_\gamma X,$$

where $\nabla_\gamma X = \gamma(\nabla X)$. The horizontal lift X^H of X has the components

$$X^H = \begin{pmatrix} x^h \\ -\Gamma_i^h x^i \end{pmatrix}$$

with respect to the induced coordinates in $T(M)$, where $\Gamma_i^h = y^j \Gamma_{ji}^h$.

The horizontal lift S^H of a tensor field S of arbitrary type in M to $T(M)$ is defined by

$$S^H = S^C - \nabla_\gamma S.$$

For any $P, Q \in \Im(M)$, we have

$$\nabla_\gamma (P \otimes Q) = (\nabla_\gamma P) \otimes Q^V + P^V \otimes (\nabla_\gamma Q)$$

or

$$(P \otimes Q)^H = P^H \otimes Q^V + P^V \otimes Q^H.$$

2. Almost product structure

Let M be a n-dimensional differentiable manifold of C^∞ class. If there exists a tensor field F of type (1,1) and of C^∞ class on M such that

$$F^2 = I,$$

where I denotes the unit tensor field. Then we say that F gives to M an *almost product structure.*

3. Complete lifts of almost product structure and Lorentzian almost r-para-contact structure in the tangent bundle

Let $\bar{M}$ be a $(2n+r)$-dimensional differentiable manifold of C^∞ class and $T(\bar{M})$ denotes the tangent bundle of $\bar{M}$. Suppose that there are given a tensor field F of type (1,1), a vector field U_α and a 1-form ω^α on $T(\bar{M})$ satisfying

$$F^2 = I + \sum_{\alpha=1}^{r} U_\alpha \otimes \omega^\alpha, \tag{3.1}$$

where

$$\begin{aligned} &FU_\alpha = 0, \\ &\omega^\alpha \circ F = 0, \\ &\omega^\alpha (U_\beta) = \delta_\beta^\alpha. \end{aligned} \tag{3.2}$$

Thus the manifold $\bar{M}$ satisfying conditions (3.1) and (3.2) will be said to possess *Lorentzian almost r-para-contact structure*([1], [3]).

THEOREM 3.1. *Let $\bar{M}$ be a differentiable manifold endowed with Lorentzian almost r-para-contact structure $(F, U_\alpha, \omega^\alpha)$. Then*

$$\tilde{J} = F^C + (U_\alpha^V \otimes \omega^{\alpha V} - U_\alpha^C \otimes \omega^{\alpha C})$$

is an almost product structure on $T(\bar{M})$.

Proof. From (3.1) and (3.2), we have [2]

$$(F^C)^2 = I + (U_\alpha^V \otimes \omega^{\alpha C} - U_\alpha^C \otimes \omega^{\alpha V}) \tag{3.3}$$

and

$$\begin{aligned} &F^C U_\alpha^V = 0,\ F^C U_\alpha^C = 0, \\ &\omega^{\alpha V} \circ F^C = 0,\ \omega^{\alpha C} \circ F^V = 0,\ \omega^{\alpha C} \circ F^C = 0, \\ &\omega^{\alpha V}(U_\alpha^V) = 0,\ \omega^{\alpha C}(U_\alpha^C) = 1,\ \omega^{\alpha C}(U_\alpha^V) = 1,\ \omega^{\alpha C}(U_\alpha^C) = 0. \end{aligned} \tag{3.4}$$

Let us define an element $\tilde{J}$ of $J(T(\bar{M}))$ by

$$\tilde{J} = F^C + (U_\alpha^V \otimes \omega^{\alpha V} - U_\alpha^C \otimes \omega^{\alpha C}). \tag{3.5}$$

Then we find by (3.3), (3.4) and (3.5),

$$\tilde{J}^2 = I.$$

Thus $\tilde{J}$ is an almost product structure in $T(\bar{M})$. □

In view of equation (3.5), we have

$$\tilde{J}X^V = -(FX)^V + (\omega^\alpha(X))^V U_\alpha^C,$$

$$\tilde{J}X^C = -(FX)^C - (\omega^\alpha(X))^V U_\alpha^C - (\omega^\alpha(X))^C U_\alpha^C. \tag{3.6}$$

In particular, we have

$$\tilde{J}X^V = -(FX)^V,\ \tilde{J}X^C = -(FX)^C, \tag{3.7}$$

$$\tilde{J}U_\alpha^V = U_\alpha^C,\ \tilde{J}U_\alpha^C = U_\alpha^C, \tag{3.8}$$

where X is an arbitrary vector field in M such that $\omega^\alpha(X) = 0$.

THEOREM 3.2. *Let the tangent bundle $T(M)$ of the manifold M admits $\tilde{J}$ defined in (3.5). Then for any vector fields X, Y such that $\omega^\alpha(Y) = 0$, we have*

(i) $(L_X V\tilde{J})Y^V = 0,$
(ii) $(L_X V\tilde{J})Y^C = -((L_X F)Y)^V + ((L_X\omega^\alpha)Y)^V U_\alpha^C,$
(iii) $(L_X V\tilde{J})U_\alpha^V = (L_X U_\alpha)^V,$
(iv) $(L_X V\tilde{J})U_\alpha^C = -((L_X F)U_\alpha)^V + (L_X\omega^\alpha(U_\alpha))^V U_\alpha^C$

and

$$
\begin{aligned}
&\text{(i)' } (L_X C\tilde{J})Y^V = -((L_X F)Y)^V + (L_X\omega^\alpha(Y))^V U_\alpha^C,\\
&\text{(ii)' } (L_X C\tilde{J})Y^C = -((L_X F)Y)^C - ((L_X\omega^\alpha)Y)U_\alpha + ((L_X\omega^\alpha)Y)^C U_\alpha^C,\\
&\text{(iii)' } (L_X C\tilde{J})U_\alpha^V = ((L_X F)U_\alpha)^C + [X,U_\alpha]^C + ((L_X\omega^\alpha)U_\alpha)^V U_\alpha^C,\\
&\text{(iv)' } (L_X C\tilde{J})U_\alpha^C = ((L_X F)U_\alpha)^C - [X,U_\alpha]^V - (L_X\omega^\alpha(U_\alpha))^V U_\alpha\\
&\qquad\qquad + ((L_X\omega^\alpha)U_\alpha)^C U_\alpha^C,
\end{aligned}
$$

where L is the Lie-derivative and [,] is the Lie-Bracket.

Proof. The proof follows easily from (3.4), (3.6), (3.8) and [5]. □

4. Horizontal lifts of Lorentzian almost r-para-contact structure

Let $(F, U_\alpha, \omega^\alpha)$ be Lorentzian almost r-para-contact structure in $\bar{M}$ with an affine connection. Then we have form (3.1) and (3.2) and [5],

$$(F^H)^2 = (I + U_\alpha \otimes \omega^\alpha)^H,$$

$$(F^H)^2 = I + (U_\alpha \otimes \omega^\alpha)^H, \tag{4.1}$$

$$(F^H)^2 = I + {U_\alpha}^H \otimes \omega^{\alpha V} + {U_\alpha}^V \otimes \omega^{\alpha H}.$$

Also,

$$F^H U_\alpha^H = 0,\ F^H U_\alpha^V = 0,$$

$$\omega^{\alpha H}(U_\alpha^H) = 0,\ \omega^{\alpha H}(U_\alpha^V) = 1,\ \omega^{\alpha V}(U_\alpha^H) = 1, \tag{4.2}$$

$$\omega^{\alpha H} \circ f^H = 0,\ \omega^{\alpha V} \circ f^H = 0.$$

Let us define a tenser field $\tilde{J}^*$ of type (1,1) in $T(\bar{M})$ by

$$\tilde{J}^* = F^H + U_\alpha^V \otimes \omega^{\alpha V} - U_\alpha^H \otimes \omega^{\alpha H}.$$

Then it is easy to show from (4.1) and (4.2) that

$$\tilde{J}^{*2} = I,$$

which means that $\tilde{J}^*$ is an almost product structure in $T(\bar{M})$. Thus we have

THEOREM 4.1. *Let $(F, U_\alpha, \omega^\alpha)$ be Lorentzian almost r-para-contact structure in $\bar{M}$ with an affine connection ∇. Then $\tilde{J}^*$ is a Lorentzian almost product structure in $T(\bar{M})$.*

References

[1] T. Adati, *An almost r-para contact Riemanian manifold and an almost product Riemannian Manifold*, Tensor, N. S. **43** (1986), 239-247.
[2] Lovejoy S. Das and Mohd. Nazrul Islam Khan, *Almost r-contact structures on the tangent bundle*, Differential Geometry-Dynamical Systems **7** (2005), 34-41.
[3] K. Matsumoto, *On Lorentzian paracontact manifolds*, Bull. of Yamagata Univ. Nat. Sci. **12** (1989), no. 2, 151-156.
[4] K. Yano and S. Ishihara, *Almost complex structures induced in tangent bundles*, Kodai Math. Sem. Rep. **19** (1967), 1-27.
[5] K. Yano and S. Ishihara, *Tangent and cotangent Bundles*, Marcel Dekker, Inc. New York, 1973.

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