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REMARK ON THE MEAN VALUE OF $L(\frac{1}{2},\chi)$ IN THE HYPERELLIPTIC ENSEMBLE

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Abstract. Let $\mathbb{A}=\mathbb{F}_q[T]$ be a polynomial ring over $\mathbb{F}_q$. In this paper we determine an asymptotic mean value of quadratic Dirichlet L-functions $L(s,\chi_{\gamma D})$ at the central point $s=\frac{1}{2}$, where D runs over all monic square-free polynomials of even degree in $\mathbb{A}$ and γ is a generator of $\mathbb{F}_q^*$.

1. Introduction and statement of results

Let $k=\mathbb{F}_q(T)$ be a rational function field over the finite field $\mathbb{F}_q$ and $\mathbb{A}=\mathbb{F}_q[T]$ be the polynomial ring. We assume that $q>3$ is odd. For a nonnegative integer n, let $\mathcal{H}_n$ be the set of monic square-free polynomials of degree n in $\mathbb{A}$. For any $D\in\mathcal{H}_n$, let χ_D be the quadratic Dirichlet character modulo D defined by the Jacobi symbol $\chi_D(N)=(\frac{D}{N})$ and $L(s,\chi_D)$ be the quadratic Dirichlet L-function associated to χ_D. As $g\to\infty$, Andrade and Keating determined the asymptotic mean value of $L(\frac{1}{2},\chi_D)$ over $D\in\mathcal{H}_{2g+1}$ ([1, Corollary 2.2]). Their result is the function field analogues of those obtained previously by Jutila [3] in the number-field setting and is consistent with recent general conjectures for the moments of L-functions by Keating and Snaith [4] motivated by Random Matrix Theory. Recently, Jung determined the asymptotic mean value of $L(\frac{1}{2},\chi_D)$ over $D\in\mathcal{H}_{2g+2}$ as $g\to\infty$ ([2, Corollary 1.3]). Let ∞_k be the infinite place of k associated to $1/T$. If $D\in\mathcal{H}_{2g+1}$, then ∞_k ramifies in $k(\sqrt{D})$, i.e., $k(\sqrt{D})$ is a ramified imaginary quadratic extension of k. If $D\in\mathcal{H}_{2g+2}$, then ∞_k splits in $k(\sqrt{D})$, i.e., $k(\sqrt{D})$ is a real quadratic

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extension of k. Let γ be a generator of $\mathbb{F}_q^*$. For any $D \in \mathcal{H}_{2g+2}$, ∞_k is inert in $k(\sqrt{\gamma D})$, i.e., $k(\sqrt{D})$ is an inert imaginary quadratic extension of k. Set $|N| = q^{\deg N}$ for $0 \neq N \in \mathbb{A}$, and let $\zeta_{\mathbb{A}}(s) = \frac{1}{1-q^{1-s}}$ be the zeta function of $\mathbb{A}$. Put $\mathrm{P}(s) = \prod_P (1-(1+|P|)^{-1}|P|^{-s})$, where P runs over all monic irreducible polynomials in $\mathbb{A}$. The aim of this paper is to determine the asymptotic mean value of $L(\frac{1}{2}, \chi_{\gamma D})$ over $D \in \mathcal{H}_{2g+2}$ as $g \to \infty$. Our main result is the following theorem and its corollary.

Theorem 1.1. *Let q be a power of an odd prime number which is greater than 3. Then we have*

$$\begin{aligned}&\sum_{D \in \mathcal{H}_{2g+2}} L(\tfrac{1}{2}, \chi_{\gamma D}) \\ &= \frac{\mathrm{P}(1)}{2\zeta_{\mathbb{A}}(2)}|D|\left\{\log_q |D| + \frac{4}{\log q}\frac{\mathrm{P}'}{\mathrm{P}}(1) + 2(\sqrt{q}+1)\right\} + O\left(|D|^{\frac{3}{4}+\frac{\log_q 2}{2}}\right).\end{aligned}$$

Comparing (2.1) with Theorem 1.1, we obtain the following asymptotic mean value.

Corollary 1.2. *Under the same assumption of Theorem 1.1, we have*

$$\frac{1}{\#\mathcal{H}_{2g+2}} \sum_{D \in \mathcal{H}_{2g+2}} L(\tfrac{1}{2}, \chi_{\gamma D}) \sim \mathrm{P}(1)(g+1)$$

as $g \to \infty$.

2. Preliminaries

2.1. Quadratic Dirichlet L-function

Let $\mathbb{A}^+$ be the set of all monic polynomials in $\mathbb{A}$ and $\mathbb{A}_n^+ = \{N \in \mathbb{A}^+ : \deg N = n\}$ $(n \geq 0)$. The zeta function $\zeta_{\mathbb{A}}(s)$ of $\mathbb{A}$ is defined by the infinite series

$$\zeta_{\mathbb{A}}(s) = \sum_{N \in \mathbb{A}^+} |N|^{-s}.$$

It is straightforward to see that $\zeta_{\mathbb{A}}(s) = \frac{1}{1-q^{1-s}}$. For any square-free $D \in \mathbb{A}$, the quadratic character χ_D is defined by the Jacobi symbol $\chi_D(N) = (\frac{D}{N})$ and the quadratic Dirichlet L-function $L(s, \chi_D)$ associated to χ_D is

$$L(s, \chi_D) = \sum_{N \in \mathbb{A}^+} \chi_D(N)|N|^{-s}.$$

We can write $L(s, \chi_D) = \sum_{n=0}^{\infty} \sigma_n(D) q^{-ns}$ with $\sigma_n(D) = \sum_{N \in \mathbb{A}_n^+} \chi_D(N)$. Since $\sigma_n(D) = 0$ for $n \geq \deg D$, $L(s, \chi_D)$ is a polynomial in q^{-s} of degree $\leq \deg D - 1$. Putting $u = q^{-s}$, write

$$\mathcal{L}(u, \chi_D) = \sum_{n=0}^{\deg D - 1} \sigma_n(D) u^n = L(s, \chi_D).$$

The cardinality of $\mathcal{H}_n$ is $\#\mathcal{H}_1 = q$ and $\#\mathcal{H}_n = (1 - q^{-1}) q^d$ $(n \geq 2)$. In particular, we have

$$\#\mathcal{H}_{2g+2} = (q-1) q^{2g+1} = \frac{q^{2g+2}}{\zeta_{\mathbb{A}}(2)}. \tag{2.1}$$

Fix a generator γ of $\mathbb{F}_q^*$. Write $\bar{D} = \gamma D$ for any $D \in H_{2g+2}$. Since $(\frac{\gamma}{N}) = (-1)^{\deg N}$, we have $(\frac{\bar{D}}{N}) = (-1)^{\deg N} (\frac{D}{N})$. Hence, $\sigma_n(\bar{D}) = (-1)^n \sigma_n(D)$. For $D \in \mathcal{H}_{2g+2}$, $\mathcal{L}(u, \chi_{\bar{D}})$ has a trivial zero at $u = -1$. The complete L-function $\tilde{\mathcal{L}}(u, \chi_{\bar{D}})$ is defined by

$$\tilde{\mathcal{L}}(u, \chi_{\bar{D}}) = (1+u)^{-1} \mathcal{L}(u, \chi_{\bar{D}}).$$

It is a polynomial of even degree $2g$ and satisfies the functional equation

$$\tilde{\mathcal{L}}(u, \chi_{\bar{D}}) = (qu^2)^g \tilde{\mathcal{L}}((qu)^{-1}, \chi_{\bar{D}}). \tag{2.2}$$

LEMMA 2.1. *Let $\chi_{\bar{D}}$ be a quadratic character, where $D \in \mathcal{H}_{2g+2}$. Then*

$$\begin{aligned}
&\mathcal{L}(q^{-\frac{1}{2}}, \chi_{\bar{D}}) \\
&= \sum_{n=0}^{g} (-1)^n q^{-\frac{n}{2}} \sum_{N \in \mathbb{A}_n^+} \chi_D(N) + (-1)^g q^{-\frac{(g+1)}{2}} \sum_{n=0}^{g} \sum_{N \in \mathbb{A}_n^+} \chi_D(N) \\
&\quad + \sum_{n=0}^{g-1} (-1)^n q^{-\frac{n}{2}} \sum_{N \in \mathbb{A}_n^+} \chi_D(N) + (-1)^{g+1} q^{-\frac{g}{2}} \sum_{n=0}^{g-1} \sum_{N \in \mathbb{A}_n^+} \chi_D(N).
\end{aligned}$$

Proof. Write $\tilde{\mathcal{L}}(u, \chi_{\bar{D}}) = \sum_{n=0}^{2g} \tilde{\sigma}_n(\bar{D}) u^n$. Since $\mathcal{L}(u, \chi_{\bar{D}}) = (1+u)$ $\tilde{\mathcal{L}}(u, \chi_{\bar{D}})$, we have $\sigma_0(\bar{D}) = \tilde{\sigma}_0(\bar{D})$, $\sigma_n(\bar{D}) = \tilde{\sigma}_{n-1}(\bar{D}) + \tilde{\sigma}_n(\bar{D})$ $(1 \leq n \leq 2g)$ and $\sigma_{2g+1}(\bar{D}) = \tilde{\sigma}_{2g}(\bar{D})$, or

$$\tilde{\sigma}_n(\bar{D}) = \sum_{i=0}^{n} (-1)^{n-i} \sigma_i(\bar{D}) \ (0 \leq n \leq 2g). \tag{2.3}$$

By substituting $\tilde{\mathcal{L}}(u, \chi_{\bar{D}}) = \sum_{n=0}^{2g} \tilde{\sigma}_n(\bar{D}) u^n$ into (2.2) and equating coefficients, we have $\tilde{\sigma}_n(\bar{D}) = \tilde{\sigma}_{2g-n}(\bar{D}) q^{-g+n}$ or $\tilde{\sigma}_{2g-n}(\bar{D}) = \tilde{\sigma}_n(\bar{D}) q^{g-n}$.

Hence,

$$\tilde{\mathcal{L}}(u, \chi_{\bar{D}}) = \sum_{n=0}^{g} \tilde{\sigma}_n(\bar{D}) u^n + q^g u^{2g} \sum_{n=0}^{g-1} \tilde{\sigma}_n(\bar{D}) q^{-n} u^{-n}.$$

In particular, we have

$$\tilde{\mathcal{L}}(q^{-\frac{1}{2}}, \chi_{\bar{D}}) = \sum_{n=0}^{g} \tilde{\sigma}_n(\bar{D}) q^{-\frac{n}{2}} + \sum_{n=0}^{g-1} \tilde{\sigma}_n(\bar{D}) q^{-\frac{n}{2}}. \tag{2.4}$$

By substituting (2.3) into (2.4) and using $\sigma_n(\bar{D}) = (-1)^n \sigma_n(D)$, we have

$$\begin{aligned}\tilde{\mathcal{L}}(q^{-\frac{1}{2}}, \chi_{\bar{D}}) = \frac{1}{1+q^{-\frac{1}{2}}} \sum_{n=0}^{g} (-1)^n q^{-\frac{n}{2}} \sigma_n(D) + \frac{(-1)^g q^{-\frac{(g+1)}{2}}}{1+q^{-\frac{1}{2}}} \sum_{n=0}^{g} \sigma_n(D) \\ + \frac{1}{1+q^{-\frac{1}{2}}} \sum_{n=0}^{g-1} (-1)^n q^{-\frac{n}{2}} \sigma_n(D) + \frac{(-1)^{g+1} q^{-\frac{g}{2}}}{1+q^{-\frac{1}{2}}} \sum_{n=0}^{g-1} \sigma_n(D).\end{aligned}$$

So we get the result since $\mathcal{L}(q^{-\frac{1}{2}}, \chi_{\bar{D}}) = (1+q^{-\frac{1}{2}}) \tilde{\mathcal{L}}(q^{-\frac{1}{2}}, \chi_{\bar{D}})$. □

2.2. Contribution of square parts

The square part contributions in the summation of $L(\frac{1}{2}, \chi_{\bar{D}})$ over $D \in \mathcal{H}_{2g+2}$ are given as follow.

PROPOSITION 2.2. *We have*

$$\begin{aligned}\sum_{m=0}^{[\frac{g}{2}]} q^{-m} \sum_{L \in \mathbb{A}_m^+} \sum_{\substack{D \in \mathcal{H}_{2g+2} \\ (L,D)=1}} 1 \\ = \frac{\mathrm{P}(1)}{\zeta_{\mathbb{A}}(2)} |D| \left\{ \left[\frac{g}{2}\right] + 1 + \frac{1}{\log q} \frac{\mathrm{P}'}{\mathrm{P}}(1) \right\} + O\left(g q^{\frac{3}{2}g+2}\right)\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}(-1)^g q^{-\frac{(g+1)}{2}} \sum_{m=0}^{[\frac{g}{2}]} \sum_{L \in \mathbb{A}_m^+} \sum_{\substack{D \in \mathcal{H}_{2g+2} \\ (L,D)=1}} 1 \\ = (-1)^g q^{-\frac{(g+1)}{2} + [\frac{g}{2}]} \mathrm{P}(1) |D| + O\left(g q^{\frac{3}{2}g+\frac{3}{2}}\right).\end{aligned} \tag{2.6}$$

Proof. See Proposition 3.5 in [2]. □

2.3. Contribution of non square parts

The non square part contributions in the summation of $L(\frac{1}{2},\chi_{\bar{D}})$ over $D\in\mathcal{H}_{2g+2}$ are given as follow.

Proposition 2.3. *We have*

$$\sum_{n=0}^{g}(-1)^n q^{-\frac{n}{2}}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{\substack{N\in\mathbb{A}_n^+\\ N\neq\square}}\chi_D(N)=O\left(2^{g+1}q^{\frac{3}{2}g+\frac{3}{2}}\right) \tag{2.7}$$

and

$$(-1)^g q^{-\frac{(g+1)}{2}}\sum_{n=0}^{g}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{\substack{N\in\mathbb{A}_n^+\\ N\neq\square}}\chi_D(N)=O\left(2^{g+1}q^{\frac{3}{2}g+\frac{3}{2}}\right). \tag{2.8}$$

Proof. As in [1, Lemma 6.4], for any non-square $N\in\mathbb{A}^+$, we have

$$\sum_{D\in\mathcal{H}_{2g+2}}\chi_D(N)\ll q^{g+1}2^{\deg N-1}.$$

Hence we have

$$\begin{aligned}&\sum_{n=0}^{g}(-1)^n q^{-\frac{n}{2}}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{\substack{N\in\mathbb{A}_n^+\\ N\neq\square}}\chi_D(N)\\ &\leq\sum_{n=0}^{g}q^{-\frac{n}{2}}\sum_{N\in\mathbb{A}_n^+}q^{g+1}2^{n-1}\leq q^{g+1}\sum_{n=0}^{g}2^n q^{\frac{n}{2}}\ll 2^{g+1}q^{\frac{3}{2}g+\frac{3}{2}}\end{aligned}$$

and

$$\begin{aligned}&(-1)^g q^{-\frac{(g+1)}{2}}\sum_{n=0}^{g}\sum_{D\in\mathcal{H}_{2g+2}}\sum_{\substack{N\in\mathbb{A}_n^+\\ N\neq\square}}\chi_D(N)\\ &\leq q^{-\frac{(g+1)}{2}}\sum_{n=0}^{g}\sum_{N\in\mathbb{A}_n^+}q^{g+1}2^{n-1}\leq q^{\frac{(g+1)}{2}}\sum_{n=0}^{g}2^n q^n\ll 2^{g+1}q^{\frac{3}{2}g+\frac{3}{2}}.\end{aligned}$$

$\square$

3. Proof of main theorem

In this section we give the proof of Theorem 1.1. By Lemma 2.1, we have

$$
\begin{aligned}
\sum_{D\in\mathcal{H}_{2g+2}} L(\tfrac{1}{2},\chi_{\bar{D}}) &= \sum_{n=0}^{g}(-1)^n q^{-\frac{n}{2}} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{N\in\mathbb{A}_n^+} \chi_D(N) \\
&\quad + (-1)^g q^{-\frac{(g+1)}{2}} \sum_{n=0}^{g} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{N\in\mathbb{A}_n^+} \chi_D(N) \\
&\quad + \sum_{n=0}^{g-1}(-1)^n q^{-\frac{n}{2}} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{N\in\mathbb{A}_n^+} \chi_D(N) \\
&\quad + (-1)^{g+1} q^{-\frac{g}{2}} \sum_{n=0}^{g-1} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{N\in\mathbb{A}_n^+} \chi_D(N).
\end{aligned} \tag{3.1}
$$

By (2.5) and (2.7), we have

$$
\begin{aligned}
&\sum_{n=0}^{g}(-1)^n q^{-\frac{n}{2}} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{N\in\mathbb{A}_n^+} \chi_D(N) \\
&= \sum_{n=0}^{g}(-1)^n q^{-\frac{n}{2}} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{\substack{N\in\mathbb{A}_n^+ \\ N=\square}} \chi_D(N) \\
&\quad + \sum_{n=0}^{g}(-1)^n q^{-\frac{n}{2}} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{\substack{N\in\mathbb{A}_n^+ \\ N\neq\square}} \chi_D(N) \\
&= \sum_{m=0}^{[\frac{g}{2}]} q^{-m} \sum_{L\in\mathbb{A}_m^+} \sum_{\substack{D\in\mathcal{H}_{2g+2} \\ (L,D)=1}} 1 + \sum_{n=0}^{g}(-1)^n q^{-\frac{n}{2}} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{\substack{N\in\mathbb{A}_n^+ \\ N\neq\square}} \chi_D(N) \\
&= \frac{\mathrm{P}(1)}{\zeta_{\mathbb{A}}(2)}|D| \left\{ \left[\frac{g}{2}\right] + 1 + \frac{1}{\log q}\frac{\mathrm{P}'}{\mathrm{P}}(1) \right\} + O\left(2^{g+1} q^{\frac{3}{2}g+\frac{3}{2}}\right)
\end{aligned} \tag{3.2}
$$

and, by (2.6) and (2.8), we have

(3.3)

$$
\begin{aligned}
&(-1)^g q^{-\frac{(g+1)}{2}} \sum_{n=0}^{g} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{N\in\mathbb{A}_n^+} \chi_D(N) \\
&= (-1)^g q^{-\frac{(g+1)}{2}} \sum_{n=0}^{g} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{\substack{N\in\mathbb{A}_n^+ \\ N=\square}} \chi_D(N) \\
&\quad + (-1)^g q^{-\frac{(g+1)}{2}} \sum_{n=0}^{g} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{\substack{N\in\mathbb{A}_n^+ \\ N\neq\square}} \chi_D(N) \\
&= (-1)^g q^{-\frac{(g+1)}{2}} \sum_{m=0}^{[\frac{g}{2}]} \sum_{L\in\mathbb{A}_m^+} \sum_{\substack{D\in\mathcal{H}_{2g+2} \\ (L,D)=1}} 1 + (-1)^g q^{-\frac{(g+1)}{2}} \sum_{n=0}^{g} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{\substack{N\in\mathbb{A}_n^+ \\ N\neq\square}} \chi_D(N) \\
&= (-1)^g q^{-\frac{(g+1)}{2}+[\frac{g}{2}]} \mathrm{P}(1)|D| + O\left(2^{g+1} q^{\frac{3}{2}g+\frac{3}{2}}\right).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
(3.4)\qquad &\sum_{n=0}^{g-1} (-1)^n q^{-\frac{n}{2}} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{N\in\mathbb{A}_n^+} \chi_D(N) \\
&= \frac{\mathrm{P}(1)}{\zeta_{\mathbb{A}}(2)} |D| \left\{ \left[\frac{g-1}{2}\right] + 1 + \frac{1}{\log q} \frac{\mathrm{P}'}{\mathrm{P}}(1) \right\} + O\left(2^g q^{\frac{3}{2}g+\frac{1}{2}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(3.5)\qquad &(-1)^{g+1} q^{-\frac{g}{2}} \sum_{n=0}^{g-1} \sum_{D\in\mathcal{H}_{2g+2}} \sum_{N\in\mathbb{A}_n^+} \chi_D(N) \\
&= (-1)^{g+1} q^{-\frac{g}{2}+[\frac{g-1}{2}]} \mathrm{P}(1)|D| + O\left(2^g q^{\frac{3}{2}g}\right).
\end{aligned}
$$

By inserting (3.2), (3.3), (3.4) and (3.5) into (3.1), we have

$$
\begin{aligned}
&\sum_{D\in\mathcal{H}_{2g+2}} L(\tfrac{1}{2}, \chi_{\bar{D}}) \\
&= \frac{\mathrm{P}(1)}{2\zeta_{\mathbb{A}}(2)} |D| \left\{ \log_q |D| + \frac{4}{\log q} \frac{\mathrm{P}'}{\mathrm{P}}(1) + 2(\sqrt{q}+1) \right\} + O\left(|D|^{\frac{3}{4}+\frac{\log_q 2}{2}} \right).
\end{aligned}
$$

The Corollary 1.2 follows immediately from Theorem 1.1 by using (2.1) and computing the limit as $g \to \infty$.

References

[1] J. C. Andrade and J. P. Keating, *The mean value of $L(\frac{1}{2},\chi)$ in the hyperelliptic ensemble.* J. Number Theory **132** (2012), no. 12, 2793-2816.

[2] H. Jung, *Note on the mean value of $L(\frac{1}{2},\chi)$ in the hyperelliptic ensemble.* To appear in Int. J. Number Theory.

[3] M. Jutila, *On the mean value of $L(\frac{1}{2},\chi)$ for real characters.* Analysis **1** (1981), no. 2, 149-161.

[4] J. P. Keating and N. C. Snaith, *Random matrix theory and L-functions at $s = 1/2$.* Comm. Math. Phys. **214** (2000), no. 1, 91-110.

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