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BOUNDEDNESS IN NONLINEAR PERTURBED FUNCTIONAL DIFFERENTIAL SYSTEMS

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Abstract. In this paper, we investigate bounds for solutions of nonlinear perturbed functional differential systems.

1. Introduction

Integral inequalities play a vital role in the study of boundedness and other qualitative properties of solutions of differential equations. The method incorporating integral inequalities takes an important place among the methods developed for the qualitative analysis of solutions to linear and nonlinear system of differential equations. As is traditional in a pertubation theory of nonlinear differential equations, the behavior of solutions of a perturbed system is determined in terms of the behavior of solutions of an unperturbed system. There are three useful methods for showing the qualitative behavior of the solutions of perturbed nonlinear system : Lyapunov's second method, the use of integral inequalities , and the method of variation of constants formula. In the presence the method of integral inequalities is as efficient as the direct Lyapunov's method.

Pinto [15,16] introduced h-stability (hS) with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called h-systems. Using this notion, Choi and Ryu[3,4] investigated bounds of solutions for nonlinear perturbed systems and nonlinear functional differential systems. Also, Goo et al.[8,11] studied the boundedness of solutions for nonlinear functional perturbed systems.

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In this paper, we obtain some results on boundedness of solutions of nonlinear perturbed functional differential systems under suitable conditions on perturbed term. To do this we need some integral inequalities.

2. Preliminaries

We consider the nonlinear functional differential equation

$$y' = f(t,y) + \int_{t_0}^{t} g(s, y(s), Ty(s))ds,\ y(t_0) = y_0, \tag{2.1}$$

where $t \in \mathbb{R}^+ = [0,\infty)$, $x \in \mathbb{R}^n$,$f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $f(t,0) = 0$, the derivative $f_x \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t,0,0) = 0$, and T is a continuous operator mapping from $C(\mathbb{R}^+, \mathbb{R}^n)$ into $C(\mathbb{R}^+, \mathbb{R}^n)$. The symbol $|\cdot|$ will be used to denote arbitrary vector norm in $\mathbb{R}^n$. We assume that for any two continuous functions $u, v \in C(I)$ where I is the closed interval and the operator T satisfies the following property:

$$u(t) \le v(t), 0 \le t \le t_1, t_1 \in I,$$

implies $Tu(t) \le Tv(t), 0 \le t \le t_1$, and $|Tu| \le T|u|$.

Equation (2.1) can be considered as the perturbed equation of

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0. \tag{2.2}$$

Let $x(t, t_0, x_0)$ be denoted by the unique solution of (2.2) passing through the point $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ such that $x(t_0, t_0, x_0) = x_0$. Also, we can consider the associated variational systems around the zero solution of (2.2) and around $x(t)$, respectively,

$$v'(t) = f_x(t,0)v(t),\ v(t_0) = v_0 \tag{2.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t),\ z(t_0) = z_0. \tag{2.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0)$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.3).

We recall some notions of h-stability [15].

Definition 2.1. The system (2.2)(the zero solution $x = 0$ of (2.2)) is called an *h-stable* if there exist a constant $c \ge 1$, $\delta > 0$, and a positive bounded continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \le c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ (here $h(t)^{-1} = \frac{1}{h(t)}$).

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [6].

DEFINITION 2.2. A matrix $A(t) \in \mathcal{M}$ is t_∞-*similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t), \dot{} = \frac{d}{dt} \tag{2.5}$$

for some $S(t) \in \mathcal{N}$.

We give some related properties that we need in the sequel.

LEMMA 2.3. [16] *The linear system*

$$x' = A(t)x, \ x(t_0) = x_0, \tag{2.6}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h-system(h-stable, respectively) if and only if there exist $c \geq 1$ and a positive continuous(bounded, repectively) function h defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{2.7}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (2.6).

We need Alekseev formula to compare between the solutions of (2.2) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \ y(t_0) = y_0, \tag{2.8}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.8) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

LEMMA 2.4. *If $y_0 \in \mathbb{R}^n$, then for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^t \Phi(t, s, y(s))g(\tau, y(\tau))d\tau\, ds.$$

THEOREM 2.5. [3] *If the zero solution of (2.2) is hS, then the zero solution of (2.3) is hS.*

THEOREM 2.6. [4] *Suppose that $f_x(t,0)$ is t_∞-similar to $f_x(t,x(t,t_0,x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (2.3) is hS, then the solution $z = 0$ of (2.4) is hS.*

LEMMA 2.7. [13] *Let $u, f, g \in C(\mathbb{R}^+)$, for which the inequality*

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)\Big\{\int_0^s g(\tau)u(\tau))d\tau\Big\}ds, \;\; t \in \mathbb{R}^+,$$

holds, where u_0 is a nonnegative constant. Then,

$$u(t) \leq u_0\Big(1 + \int_0^t f(s)\exp(\int_0^s (f(\tau) + g(\tau))d\tau)\Big)ds, \;\; t \in \mathbb{R}^+.$$

LEMMA 2.8. [5] *Let $u, \lambda_1, \lambda_2, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u, and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. If , for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s)u(s)ds + \int_{t_0}^t \lambda_1(s)\Big\{\int_{t_0}^s \lambda_2(\tau)w(u(\tau))d\tau\Big\}ds, \;\; t \geq t_0 \geq 0,$$

then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^t \lambda_2(s)ds\Big]\exp(\int_{t_0}^t \lambda_1(s)ds), \;\; t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^u \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$, and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^t \lambda_2(s)ds \in \mathrm{domW}^{-1}\Big\}.$$

LEMMA 2.9. [11] *Let $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, and $w(u)$ be nondecreasing in u, $u \leq w(u)$. Suppose that for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s)w(u(s))ds + \int_{t_0}^t \lambda_2(s)(\int_{t_0}^s \lambda_3(\tau)u(\tau)d\tau)ds, \;\; 0 \leq t_0 \leq t.$$

Then

(2.9)

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s)\int_{t_0}^s \lambda_3(\tau)d\tau)ds\Big], \;\; t_0 \leq t < b_1,$$

where W, W^{-1} are the same functions as in Lemma 2.8, and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s)\int_{t_0}^s \lambda_3(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 2.10. [2] *Let $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, and $w(u)$ be nondecreasing in u. Suppose that for some $c > 0$,*

$$u(t) \leq c+\int_{t_0}^{t} \lambda_1(s)w(u(s))ds+\int_{t_0}^{t} \lambda_2(s)(\int_{t_0}^{s} \lambda_3(\tau)w(u(\tau))d\tau)ds,\ 0 \leq t_0 \leq t.$$

Then

(2.10)

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s} \lambda_3(\tau)d\tau)ds\Big],\ t_0 \leq t < b_1,$$

where W, W^{-1} are the same functions as in Lemma 2.8, and

$$b_1 = \sup\Big\{t \geq t_0 : W(c)+\int_{t_0}^{t}(\lambda_1(s)+\lambda_2(s)\int_{t_0}^{s} \lambda_3(\tau)d\tau)ds \in \mathrm{domW^{-1}}\Big\}.$$

3. Main results

In this section, we investigate bounds for the nonlinear functional differential systems. Also, we examine the bounded property for the perturbed system of (2.2)

$$y' = f(t,y) + \int_{t_0}^{t} g(s,y(s),Ty(s))ds,\ y(t_0) = y_0, \qquad (3.1)$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t,0,0) = 0$, and T is a continuous operator mapping from $C(\mathbb{R}^+, \mathbb{R}^n)$ into $C(\mathbb{R}^+, \mathbb{R}^n)$.

The generalization of a function h's condition and the strong condition of a function g in Theorem 3.4[10] are the following result.

Theorem 3.1. *Suppose that $f_x(t,0)$ is t_∞-similar to $f_x(t,x(t,t_0,x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (2.2) is hS with a positive continuous function h, and g in (3.1) satisfies*

$$\Big|\int_{t_0}^{s} g(\tau,y(\tau),Ty(\tau))d\tau\Big| \leq a(s)(|y(s)| + |Ty|), t \geq t_0 \geq 0,$$

and

$$|Ty| \leq h(t)\int_{t_0}^{t} k(s)|y(s)|ds\,,$$

where $a, k \in C(\mathbb{R}^+)$, $\int_{t_0}^{\infty} a(s)ds < \infty$, and $\int_{t_0}^{s} k(s)ds < \infty$. Then, the solution $y = 0$ of (3.1) is hS.

Proof. Using the nonlinear variation of Alekseev[1], any solution $y(t) = y(t, t_0, y_0)$ of (3.1) passing through (t_0, y_0) is given by
(3.2)

$$y(t,t_0,y_0) = x(t,t_0,y_0) + \int_{t_0}^{t} \Phi(t,s,y(s)) \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau))\, d\tau ds.$$

By Theorem 2.5, since the solution $x = 0$ of (2.2) is hS, the solution $v = 0$ of (2.3) is hS. Therefore, by Theorem 2.6, the solution $z = 0$ of (2.4) is hS. By Lemma 2.3 and (3.2) , we obtain

$$\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \left| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau)) d\tau \right| ds \\
&\le c_1 |y_0| h(t)\, h(t_0)^{-1} \\
&\quad + \int_{t_0}^{t} c_2 h(t)\, h(s)^{-1} a(s) \Big(|y(s)| + h(s) \int_{t_0}^{s} k(\tau)|y(\tau)| d\tau \Big) ds \\
&\le c_1 |y_0| h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\, a(s) h(s)^{-1} |y(s)| ds \\
&\quad + \int_{t_0}^{t} c_2 h(t) a(s) \int_{t_0}^{s} k(\tau) h(\tau) h(\tau)^{-1} |y(\tau)| d\tau ds.
\end{aligned}$$

Set $u(t) = |y(t)| h(t)^{-1}$. Then, by Lemma 2.7, we have

$$\begin{aligned}
|y(t)| &\le c_1 |y_0| h(t)\, h(t_0)^{-1} \\
&\quad \cdot \Big(1 + c_2 \int_{t_0}^{t} a(s) \exp\Big(\int_{t_0}^{s} (c_2 a(\tau) + k(\tau) h(\tau)) d\tau \Big) ds \Big) \\
&\le c |y_0| h(t)\, h(t_0)^{-1}, \\
&\qquad c = c_1 \Big(1 + c_2 \int_{t_0}^{\infty} a(s) \exp(\int_{t_0}^{s} (c_2 a(\tau) + k(\tau) h(\tau)) d\tau) ds \Big).
\end{aligned}$$

It follows that $y = 0$ of (3.1) is hS. Hence, we obtain the result. □

Remark 3.2. In the linear case, we can obtain that if the zero solution $x = 0$ of (2.6) is hS, then the perturbed system

$$y' = A(t)y + \int_{t_0}^{t} g(s, y(s), Ty(s)) ds, y(t_0) = y_0,$$

is also hS under the same hypotheses in Theorem 3.1 except the condition of t_∞-similarity.

Remark 3.3. Letting $k(t) = 0$, $g(s, y(s), Ty(s)) = g(s, y(s))$, and adding the increasing condition of the function h in Theorem 3.1, we obtain the same result as that of Theorem 3.3 in [9].

Theorem 3.4. *Let $a, k, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u, and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (2.2) is hS with a positive continuous function h, and g in (3.1) satisfies*

$$\left| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau))d\tau \right| \le a(s)(|y(s)| + |Ty|)$$

and

$$|Ty| \le h(t) \int_{t_0}^{t} k(s)w(|y(s)|)ds,$$

where $\int_{t_0}^{\infty} a(s)ds < \infty$ and $\int_{t_0}^{\infty} k(s)ds < \infty$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (3.1) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + \int_{t_0}^{t} k(s)h(s)ds\Big] \exp(\int_{t_0}^{t} c_2 a(s)ds), t_0 \le t < b_1,$$

where $c = c_1|y_0|h(t_0)^{-1}$, W, W^{-1} are the same functions as in Lemma 2.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} k(s)h(s)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. It is known that the solution of (3.1) is represented by the integral equation(3.2). By Theorem 2.5, since the solution $x = 0$ of (2.2) is hS, the solution $v = 0$ of (2.3) is hS. Therefore, by Theorem 2.6, the solution $z = 0$ of (2.4) is hS. Using Lemma 2.3 and (3.2), we have

$$\begin{aligned} |y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \left| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau))d\tau \right| ds \\ &\le c_1|y_0|h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)a(s)\frac{|y(s)|}{h(s)}ds \\ &\quad + \int_{t_0}^{t} c_2 h(t)a(s) \int_{t_0}^{s} k(\tau)h(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau ds. \end{aligned}$$

Set $u(t) = |y(t)|h(t)^{-1}$. Now an application of Lemma 2.8 yields

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + \int_{t_0}^{t} k(s)h(s)ds\Big] \exp(\int_{t_0}^{t} c_2 a(s)ds), t_0 \le t < b_1,$$

where $c = c_1|y_0|h(t_0)^{-1}$. The above estimation implies the boundedness of $y(t)$, and the proof is complete. □

THEOREM 3.5. *Let $a, b, k, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u, $u \le w(u)$, and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (2.2) is hS with the positive continuous function h, and g in (3.1) satisfies*

$$\left| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau))d\tau \right| \le a(s)w(|y(s)|) + b(s)|Ty|$$

and

$$|Ty| \le h(t) \int_{t_0}^{t} k(s)|y(s)|ds,$$

where $\int_{t_0}^{\infty} a(s)ds < \infty$, $\int_{t_0}^{\infty} b(s)ds < \infty$, and $\int_{t_0}^{\infty} k(s)ds < \infty$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (3.1) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} (a(s) + b(s) \int_{t_0}^{s} k(\tau)h(\tau)d\tau)ds\Big],$$

where W, W^{-1} are the same functions as in Lemma 2.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c)+c_2 \int_{t_0}^{t} (a(s)+b(s) \int_{t_0}^{s} k(\tau)h(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.2) and (3.1), respectively. By Theorem 2.5, since the solution $x = 0$ of (2.2) is hS, the solution $v = 0$ of (2.3) is hS. Therefore, by Theorem 2.6, the solution $z = 0$ of (2.4) is hS. Using Lemma 2.3 and (3.2) , we obtain

$$\begin{aligned} |y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \left| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau))d\tau \right| ds \\ &\le c_1|y_0|h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2h(t)a(s)w(\frac{|y(s)|}{h(s)})ds \\ &\quad + \int_{t_0}^{t} c_2h(t)b(s) \int_{t_0}^{s} k(\tau)h(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau ds. \end{aligned}$$

Set $u(t) = |y(t)|h(t)^{-1}$. Now an application of Lemma 2.9 have

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} (a(s) + b(s) \int_{t_0}^{s} k(\tau)h(\tau)d\tau)ds\Big],$$

where $c = c_1|y_0|h(t_0)^{-1}$. The above estimation yields the desired result since the function h is bounded, and the theorem is proved. □

REMARK 3.6. Letting $k(t) = 0$, $g(s, y(s), Ty(s)) = g(s, y(s))$ in Theorem 3.5, and adding the increasing condition of the function h, we obtain the same result as that of Theorem 3.2 in [8].

THEOREM 3.7. *Let $a, b, k, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u, and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (2.2) is hS with the positive continuous function h, and g in (3.1) satisfies*

$$\left| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau))d\tau \right| \le a(s)w(|y(s)|) + b(s)|Ty|$$

and

$$|Ty| \le h(t) \int_{t_0}^{t} k(s)|w(|y(s)|)|ds,$$

where $\int_{t_0}^{\infty} a(s)ds < \infty$, $\int_{t_0}^{\infty} b(s)ds < \infty$, and $\int_{t_0}^{\infty} k(s)ds < \infty$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (3.1) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} (a(s) + b(s) \int_{t_0}^{s} k(\tau)h(\tau)d\tau)ds\Big],$$

where W, W^{-1} are the same functions as in Lemma 2.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c)+c_2 \int_{t_0}^{t} (a(s)+b(s) \int_{t_0}^{s} k(\tau)h(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.2) and (3.1), respectively. By Theorem 2.5, since the solution $x = 0$ of (2.2) is hS, the solution $v = 0$ of (2.3) is hS. Therefore, by Theorem 2.6, the solution $z = 0$ of (2.4) is hS. Using Lemma 2.3 and (3.2), we have

$$\begin{aligned} |y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \left| \int_{t_0}^{s} g(\tau, y(\tau), Ty(\tau))d\tau \right| ds \\ &\le c_1|y_0|h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2h(t)a(s)w(\frac{|y(s)|}{h(s)})ds \\ &\quad + \int_{t_0}^{t} c_2h(t)b(s) \int_{t_0}^{s} k(\tau)h(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau ds. \end{aligned}$$

Set $u(t) = |y(t)|h(t)^{-1}$. Now an application of Lemma 2.10 yields

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} (a(s) + b(s) \int_{t_0}^{s} k(\tau)h(\tau)d\tau)ds\Big],$$

where $c = c_1|y_0|h(t_0)^{-1}$. The above estimation implies the boundedness of $y(t)$, and the proof is complete. □

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