CHEMICAL HYPERSTRUCTURES FOR VANADIUM

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Abstract. In this paper, we give chemical hyperstructures of chemical reactions for a set of vanadium(V).

1. Introduction

In 1935 F. Marty [6] introduced algebraic hyperstructures which are generalized concepts of groups. Up to now, many mathematicians [1, 2] have been studying about hyperstructures. In 1994 T. Vougiouklis [7] gave about H_v-groups and in 2013 B. Davvaz [3] summarized the basic algebraic hyperstructures.

In 2012, B. Davvaz, A. D. Nezhad and A. Benvidi [5] showed chemical hyperalgebra in dismutation reactions for three sets of tin(Sn), indium(In) and vanadium(V).

In this paper, we obtain different chemical hypersructures for the set of vanadium(V) as B. Davvaz, A. D. Nezhad and A. Benvidi got in [5].

2. Hyperalgebraic structures

Let H be a non-empty set and $\cdot : H \times H \longrightarrow \wp^*(H)$ be a function, where $\wp^*(H)$ is the set of all non-empty subsets of H. Then the function $\cdot$ is called a *hyperoperation* on H and the couple $(H, \cdot)$ is called a *hypergroupoid*. For two subsets A, B of H, we define $A \cdot B = \cup_{a \in A, b \in B} a \cdot b$, and for a singleton $\{a\}$ we denote $\{a\} \cdot B = a \cdot B$ and $B \cdot \{a\} = B \cdot a$.

DEFINITION 2.1. The hypergroupoid $(H, \cdot)$ is called a *semihypergroup* if

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z, \text{ for all } x, y, z \in H.$$

In this case, the hyperoperation $(\cdot)$ is called *associative*.

The hypergroupoid $(H, \cdot)$ is called an H_v-*semigroup* if

$$x \cdot (y \cdot z) \cap (x \cdot y) \cdot z \neq \emptyset, \text{ for all } x, y, z \in H.$$

In this case, the hyperoperation $(\cdot)$ is called *weakly associative*.

The hypergroupoid $(H, \cdot)$ is called a *quasihypergroup* if

$$x \cdot H = H \cdot x = H, \text{ for all } x \in H.$$

The hyperoperation $(\cdot)$ is called *commutative* if

$$x \cdot y = y \cdot x, \text{ for all } x, y \in H.$$

The hypergroupoid $(H, \cdot)$ is called a *hypergroup* if it is a semihypergroup and a quasihypergroup.

The hypergroupoid $(H, \cdot)$ is called an H_v-*group* if it is an H_v-semigroup and a quasihypergroup.

The hypergroupoid $(H, \cdot)$ is called a *commutative hypergroup* if it is a hypergroup with a commutative hyperoperation $(\cdot)$.

The hypergroupoid $(H, \cdot)$ is called a *commutative H_v-group* if it is an H_v-group with a commutative hyperoperation $(\cdot)$.

A non-empty subset K of a hypergroup (respectively, H_v-group) H is called a *subhpergroup* (respectively, H_v-subgroup) of H if $a \cdot K = K \cdot a = K$ for all $a \in K$.

Let $(H_1, \cdot)$ and $(H_2, *)$ be two H_v-groups. A map $f : H_1 \longrightarrow H_2$ is called an H_v-*homomorphism* or *weak homomorphism* if

$$f(x \cdot y) \cap f(x) * f(y) \neq \emptyset, \text{ for all } x, y \in H_1.$$

f is called an *inclusion homomorphism* if

$$f(x \cdot y) \subset f(x) * f(y), \text{ for all } x, y \in H_1.$$

Finally, f is called a *strong homomorphism* if

$$f(x \cdot y) = f(x) * f(y), \text{ for all } x, y \in H_1.$$

If f is onto, one to one and strong homomorphism, then it is called an *isomorphism*. In this case, H_1 and H_2 are called *isomorphic* and we write $H_1 \cong H_2$.

3. Oxidation-reduction reactions and chemical hyperstructures for vanadium

Oxidation-Reduction reactions are a vital necessity for biochemical reactions and industrial processes. Oxidation-Reduction Reactions(or redox reactions) are a type of chemical reactions that involves a transfer of electrons between two species. Oxidation is the loss of electrons from the reducing agent and reduction is the acquiring of electrons by the oxidizing agent. The galvanic cell has two electrode potentials; anode potential and cathode potential. It is impossible to measure the potential of just one electrode and then it can be obtained by selecting a particular electrode refer to hydrogen electrode. All the electrode potentials are measured in relation to this standard electrode and are termed as standard potentials and are indicated by the symbol $E°$(the E stands for electromotive force or the EMF). By convention, the standard EMF of the cell is the sum of oxidation and reduction potentials. If we gained the positive sign on the potential, the free energy change($\triangle G°$) is negative value, it is spontaneous reaction, because the EMF can be related to the free energy change($\triangle G°$) by the equation

$$\triangle G° = -nFE°$$

where n is the number of equivalents oxidized or reduced, and F is the conversion between electrochemical and thermodynamic units, $96,500 J$ $V^{-1}mol^{-1}$.

DEFINITION 3.1. Let G be a set of two or more chemical species and a hyperoperation $\oplus$ on G is defined as follows;
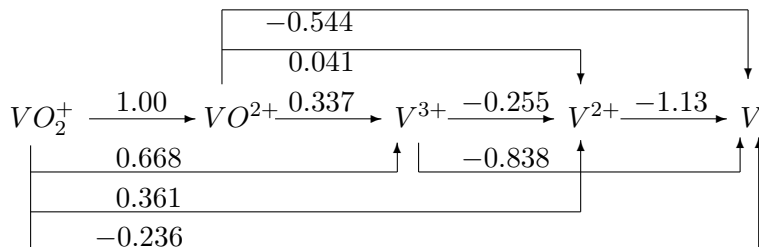
$$\oplus : G \times G \to \wp^*(G)$$

where $\wp^*(G)$ is the set of all non-empty subsets of G. For all $x, y \in G$, $x \oplus y$ is defined the *major product*, that is, it is selected from the largest combination among all spontaneous redox reactions.

The following standard reduction potentials for the set $\{V, V^{2+}, V^{3+}, VO^{2+}, VO_2^+\}$ of vanadium(V) is given in [8]. Values are not given in [8] are calculated by way of the reference [4].

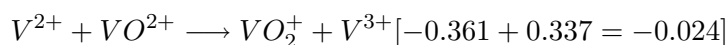EXAMPLE 3.2. For the following case

$$V^{2+} + VO^{2+} \longrightarrow V^{3+} + V^{3+}[0.255 + 0.337 = 0.632]$$
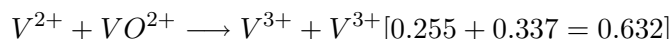
since the electromotive force($E° = 0.632$) is positive, the free energy change ($\triangle G° = -nFE°$) is negative. Hence redox reactions under the
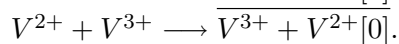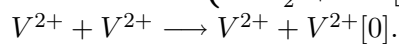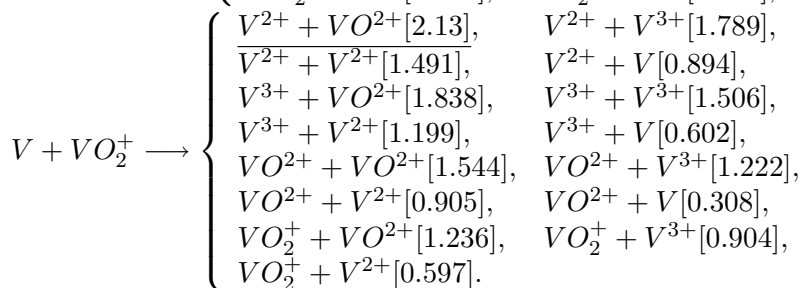
$$
\begin{array}{ccccccccc}
& & \multicolumn{3}{c}{\overline{-0.544}} & & & \\
& & \multicolumn{3}{c}{\overline{0.041}} & & \downarrow & & \downarrow\\
VO_2^+ & \xrightarrow{1.00} & VO^{2+} & \xrightarrow{0.337} & V^{3+} & \xrightarrow{-0.255} & V^{2+} & \xrightarrow{-1.13} & V \\
& \overline{0.668} & & \uparrow & & -0.838 & \uparrow & & \uparrow \\
& \overline{0.361} & & & & & & \\
& \overline{-0.236} & & & & & &
\end{array}
$$

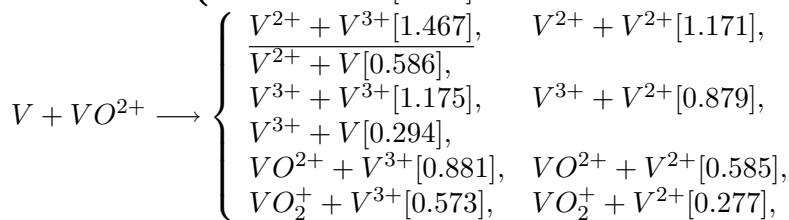standard conditions is spontaneous. But for the following case

$$V^{2+} + VO^{2+} \longrightarrow VO_2^+ + V^{3+} [-0.361 + 0.337 = -0.024]$$

since the electromotive force(-0.024) is negative, the free energy change ($\triangle G^\circ$) is positive. Hence redox reactions under the standard conditions is not spontaneous.

The followings are all possible spontaneous redox combinations for the set $\{V, V^{2+}, V^{3+}, VO^{2+}, VO_2^+\}$ of vanadium ions. From now on, we denote $a + b \longrightarrow c + d[\gamma]$ instead of $a + b \longrightarrow c + d[\alpha + \beta = \gamma]$, and the underline ___ is the major product.

$V + V \longrightarrow \underline{V + V}[0]$.

$V + V^{2+} \longrightarrow \underline{V^{2+} + V}[0]$.

$V + V^{3+} \longrightarrow \begin{cases} \underline{V^{2+} + V^{2+}}[0.875], & V^{2+} + V[0.292], \\ V^{3+} + V^{2+}[0.583], \\ VO^{2+} + V^{2+}[0.289]. \end{cases}$

$V + VO^{2+} \longrightarrow \begin{cases} \underline{V^{2+} + V^{3+}}[1.467], & V^{2+} + V^{2+}[1.171], \\ V^{2+} + V[0.586], \\ V^{3+} + V^{3+}[1.175], & V^{3+} + V^{2+}[0.879], \\ V^{3+} + V[0.294], \\ VO^{2+} + V^{3+}[0.881], & VO^{2+} + V^{2+}[0.585], \\ VO_2^+ + V^{3+}[0.573], & VO_2^+ + V^{2+}[0.277], \end{cases}$

$V + VO_2^+ \longrightarrow \begin{cases} \underline{V^{2+} + VO^{2+}}[2.13], & V^{2+} + V^{3+}[1.789], \\ V^{2+} + V^{2+}[1.491], & V^{2+} + V[0.894], \\ V^{3+} + VO^{2+}[1.838], & V^{3+} + V^{3+}[1.506], \\ V^{3+} + V^{2+}[1.199], & V^{3+} + V[0.602], \\ VO^{2+} + VO^{2+}[1.544], & VO^{2+} + V^{3+}[1.222], \\ VO^{2+} + V^{2+}[0.905], & VO^{2+} + V[0.308], \\ VO_2^+ + VO^{2+}[1.236], & VO_2^+ + V^{3+}[0.904], \\ VO_2^+ + V^{2+}[0.597]. \end{cases}$

$V^{2+} + V^{2+} \longrightarrow \underline{V^{2+} + V^{2+}}[0]$.
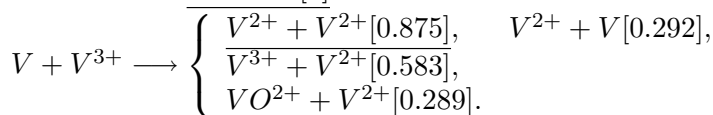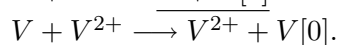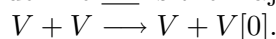
$V^{2+} + V^{3+} \longrightarrow \underline{V^{3+} + V^{2+}}[0]$.

$$V^{2+} + VO^{2+} \longrightarrow \begin{cases} \overline{V^{3+} + V^{3+}}[0.597], & V^{3+} + V^{2+}[0.266], \\ \overline{VO^{2+} + V^{3+}}[0.296]. \end{cases}$$

$$V^{2+} + VO_2^+ \longrightarrow \begin{cases} V^{3+} + VO^{2+}[1.255], & V^{3+} + V^{3+}[0.923], \\ \overline{V^{3+} + V^{2+}}[0.613], & V^{3+} + V[0.019], \\ VO^{2+} + VO^{2+}[0.959], & VO^{2+} + V^{3+}[0.627], \\ VO^{2+} + V^{2+}[0.32], \\ VO_2^+ + VO^{2+}[0.639], & VO_2^+ + V^{3+}[0.307]. \end{cases}$$

$V^{3+} + V^{3+} \longrightarrow \overline{V^{3+} + V^{3+}}[0].$

$V^{3+} + VO^{2+} \longrightarrow \overline{VO^{2+} + V^{3+}}[0].$

$$V^{3+} + VO_2^+ \longrightarrow \begin{cases} \overline{VO^{2+} + VO^{2+}}[0.663], & VO^{2+} + V^{3+}[0.331], \\ \overline{VO_2^+ + VO^{2+}}[0.332], & VO^{2+} + V^{2+}[0.024]. \end{cases}$$

$VO^{2+} + VO^{2+} \longrightarrow \overline{VO^{2+} + VO^{2+}}[0].$

$VO^{2+} + VO_2^+ \longrightarrow \overline{VO_2^+ + VO^{2+}}[0].$

$VO_2^+ + VO_2^+ \longrightarrow \overline{VO_2^+ + VO_2^+}[0].$ Then we obtain the following hyperoperation table for the set $\{V, V^{2+}, V^{3+}, VO^{2+}, VO_2^+\}$:

$\oplus$	V	V^{2+}	V^{3+}	VO^{2+}	VO_2^+
V	V	V, V^{2+}	V^{2+}	V^{2+}, V^{3+}	V^{2+}, VO^{2+}
V^{2+}	V, V^{2+}	V^{2+}	V^{2+}, V^{3+}	V^{3+}	V^{3+}, VO^{2+}
V^{3+}	V^{2+}	V^{2+}, VO^{3+}	V^{3+}	V^{3+}, VO^{2+}	VO^{2+}
VO^{2+}	V^{2+}, V^{3+}	V^{3+}	V^{3+}, VO^{2+}	VO^{2+}	VO^{2+}, VO_2^+
VO_2^+	V^{2+}, VO^{2+}	V^{3+}, VO^{2+}	VO^{2+}	VO^{2+}, VO_2^+	VO_2^+

In the above table, if we change the name from V, V^{2+}, V^{3+}, VO^{2+} and VO_2^+ to a, b, c, d and e, respectively, then the following theorem holds.

We have a different hyperoperation table for the set $\{V, V^{2+}, V^{3+}, VO^{2+}, VO_2^+\}$ of vanadium(V) ions from ([5], p. 61).

THEOREM 3.3. *Let $G = \{a, b, c, d, e\}$ be the set of vanadium ions and $\oplus$ be the hyperoperation on G. Consider the following commutative hyperoperation table:*

$\oplus$	a	b	c	d	e
a	$\{a\}$	$\{a, b\}$	$\{b\}$	$\{b, c\}$	$\{b, d\}$
b	$\{a, b\}$	$\{b\}$	$\{b, c\}$	$\{c\}$	$\{c, d\}$
c	$\{b\}$	$\{b, c\}$	$\{c\}$	$\{c, d\}$	$\{d\}$
d	$\{b, c\}$	$\{c\}$	$\{c, d\}$	$\{d\}$	$\{d, e\}$
e	$\{b, d\}$	$\{c, d\}$	$\{d\}$	$\{d, e\}$	$\{e\}$

Then we have the following.

(1) *The hyperstructures $(\{a, b\}, \oplus), (\{b, c\}, \oplus), (\{c, d\}, \oplus)$ and $(\{d, e\}, \oplus)$ are commutative subhypergroups of G and isomorphic.*

(2) *The hyperstructures $(\{a, b, c\}, \oplus), (\{b, c, d\}, \oplus)$ and $(\{c, d, e\}, \oplus)$ are commutative H_v-subsemigroups of G and isomorphic.*

(3) *The hyperstructures $(\{a, b, c, d\}, \oplus)$ and $(\{b, c, d, e\}, \oplus)$ are commutative H_v-subsemigroups of G and isomorphic.*

(4) *The hyperstructure $(G, \oplus)$ is a commutative H_v-semigroup.*

Proof. For all $x, y, z \in G$, we show that $[x \oplus (y \oplus z)] \cap [(x \oplus y) \oplus z] \neq \emptyset$.

(1) It is obvious.

(2) For all $x, y, z \in \{a, b, c\}$, we have
$$\begin{cases} [x \oplus (y \oplus z)] \cap [(x \oplus y) \oplus z] = \{a\}, & \text{if } x = a, y = a, z = a; \\ [x \oplus (y \oplus z)] \cap [(x \oplus y) \oplus z] = \{c\}, & \text{if } x = c, y = c, z = c; \\ [x \oplus (y \oplus z)] \cap [(x \oplus y) \oplus z] \cap \{b\} = \{b\}, & \text{otherwise.} \end{cases}$$

But clearly it is not a quasihypergroup. Thus $(\{a, b, c\}, \oplus)$ is a commutative H_v-semigroup,

Since $\{a, b, c\}, \{b, c, d\}$ and $\{c, d, e\}$ have the same similar hyperoperation tables, they are isomorphic.

(3) For all $x, y, z \in \{a, b, c, d\}$, we have
$$\begin{cases} [x \oplus (y \oplus z)] \cap [(x \oplus y) \oplus z] = \{a\}, & \text{if } x = a, y = a, z = a; \\ [x \oplus (y \oplus z)] \cap [(x \oplus y) \oplus z] = \{d\}, & \text{if } x = d, y = d, z = d; \\ [x \oplus (y \oplus z)] \cap [(x \oplus y) \oplus z] \cap \{b, c\} \neq \emptyset, & \text{otherwise.} \end{cases}$$

Thus $(\{a, b, c, d\}, \oplus)$ is a commutative H_v-semigroup.

Since $\{a, b, c, d\}$ and $\{b, c, d, e\}$ have the same similar hyperoperation tables, they are isomorphic.

(4) For the weak associative law, it is sufficient to show three cases $\{a, b, e\}, \{a, c, e\}$ and $\{a, d, e\}$. The other cases follow (3).

For all $x, y, z \in \{a, b, e\}$, we have the following.
$$\begin{cases} [x \oplus (y \oplus z)] \cap [(x \oplus y) \oplus z] = \{a\}, & \text{if } x = a, y = a, z = a; \\ [x \oplus (y \oplus z)] \cap [(x \oplus y) \oplus z] = \{e\}, & \text{if } x = e, y = e, z = e; \\ [x \oplus (y \oplus z)] \cap [(x \oplus y) \oplus z] \cap \{b, c, d\} \neq \emptyset, & \text{otherwise.} \end{cases}$$

For all $x, y, z \in \{a, c, e\}$, we have the following.
$$\begin{cases} [x \oplus (y \oplus z)] \cap [(x \oplus y) \oplus z] = \{a\}, & \text{if } x = a, y = a, z = a; \\ [x \oplus (y \oplus z)] \cap [(x \oplus y) \oplus z] = \{e\}, & \text{if } x = e, y = e, z = e; \\ [x \oplus (y \oplus z)] \cap [(x \oplus y) \oplus z] \cap \{b, d\} \neq \emptyset & \text{otherwise.} \end{cases}$$

For all $x, y, z \in \{a, d, e\}$, we have the following.
$$\begin{cases} [x \oplus (y \oplus z)] \cap [(x \oplus y) \oplus z] = \{a\}, & \text{if } x = a, y = a, z = a; \\ [x \oplus (y \oplus z)] \cap [(x \oplus y) \oplus z] = \{e\}, & \text{if } x = e, y = e, z = e; \\ [x \oplus (y \oplus z)] \cap [(x \oplus y) \oplus z] \cap \{b, d\} \neq \emptyset, & \text{otherwise.} \end{cases}$$

Hence $(G, \oplus)$ is a commutative H_v-semigroup. $\square$

REMARK 3.4. The hyperstructure $(\{a, b, c\}, \oplus)$ is not a semihyper-group since the hyperoperation $\oplus$ is not associative; for example, $c \oplus (a \oplus a) = c \oplus a = \{b\}$ and $(c \oplus a) \oplus a = b \oplus a = \{a, b\}$. Hence $c \oplus (a \oplus a) \neq (c \oplus a) \oplus a$.

DEFINITION 3.5. Let G be a set of two or more chemical species and a hyperoperation $\oplus'$ on G is defined as follows; for all $x, y, x', y' \in G$, consider the following all possible combinations for oxidation-reduction reactions
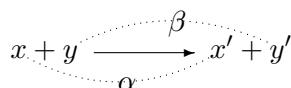
$$x + y \xrightarrow[\alpha]{\beta} x' + y'$$

where α and β are potential differences. We define $x \oplus' y$ as follows;

$$x \oplus' y = \begin{cases} \cup_{\alpha+\beta>0}\{x', y'\}, & \text{whenever } x + y \to x' + y'; \\ \{x, y\}, & \text{otherwise.} \end{cases}$$

THEOREM 3.6. *Let $G = \{a, b, c, d, e\}$ be the set of vanadium ions and $\oplus'$ be the hyperoperation on G. Then we have the following.*

(1) *The hyperstructure $(G, \oplus')$ has the following commutative hyper-operation table.*

$\oplus'$	a	b	c	d	e
a	$\{a\}$	$\{a, b\}$	$\{a, b, c, d\}$	$\{a, b, c, d, e\}$	$\{a, b, c, d, e\}$
b	$\{a, b\}$	$\{b\}$	$\{b, c\}$	$\{b, c, d\}$	$\{a, b, c, d, e\}$
c	$\{a, b, c, d\}$	$\{b, c\}$	$\{c\}$	$\{c, d\}$	$\{b, c, d, e\}$
d	$\{a, b, c, d, e\}$	$\{b, c, d\}$	$\{c, d\}$	$\{d\}$	$\{d, e\}$
e	$\{a, b, c, d, e\}$	$\{a, b, c, d, e\}$	$\{b, c, d, e\}$	$\{d, e\}$	$\{e\}$

(2) *The hyperstructures $(\{a, b\}, \oplus')$, $(\{b, c\}, \oplus')$, $(\{c, d\}, \oplus')$ and $(\{d, e\}, \oplus')$ are commutative subhypergroups of G and isomorphic.*

(3) *The hyperstructure $(\{b, c, d\}, \oplus')$ is a commutative subhypergroup of G.*

(4) *The hyperstructure $(G, \oplus')$ is a commutative H_v-group, and ob-viously the commutative H_v-semigroup $(G, \oplus)$ is a subhyperstruc-ture of $(G, \oplus')$.*

Proof. (1) Using all possible redox reactions(p. 312), we get the hy-peroperation table.

(2) It is obvious.

(3) For all $x, y, z \in \{b, c, d\}$, we have

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z = \begin{cases} \{b\}, & \text{if } \{b\} = \{x,y,z\}; \\ \{c\}, & \text{if } \{c\} = \{x,y,z\}; \\ \{d\}, & \text{if } \{d\} = \{x,y,z\}; \\ \{b,c\}, & \text{if } \{b,c\} = \{x,y,z\}; \\ \{c,d\}, & \text{if } \{c,d\} = \{x,y,z\}; \\ \{b,c,d\}, & \text{if } \{b,d\} \subset \{x,y,z\}. \end{cases}$$

Thus $(\{b,c,d\}, \oplus')$ is a commutative hypergroup since it is a quasi-hypergroup.

(4) Since $\{x,y\} \subset x \oplus' y$, $\{x,y,z\} \subset [x \oplus' (y \oplus' z)] \cap [(x \oplus' y) \oplus' z]$ for all $x,y,z \in G$.

Thus $(G, \oplus')$ is a commutative H_v-group since it is a quasihypergroup. $\qquad\square$

REMARK 3.7. The hyperstructure $(G, \oplus')$ is not a hypergroup since the hyperoperation $\oplus'$ is not associative; for example, $c \oplus' (a \oplus' a) = c \oplus' a = \{a,b,c,d\}$ and $(c \oplus' a) \oplus' a = \{a,b,c,d\} \oplus' a = \{a,b,c,d,e\}$. Hence $c \oplus' (a \oplus' a) \neq (c \oplus' a) \oplus' a$.

Acknowledgement

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