JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **27**, No. 2, May 2014
http://dx.doi.org/10.14403/jcms.2014.27.2.297

SOME PROPERTIES OF DERIVATIONS ON CI-ALGEBRAS

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Abstract. The present paper gives the notion of a derivation on a CI-algebra X and investigates related properties. We define a set $Fix_d(X)$ by $Fix_d(X) = \{x \in X \mid d(x) = x\}$, where d is a derivation on a CI-algebra X. We show that $Fix_d(X)$ is a subalgebra of X. Also, we prove some one-to-one and onto derivation theorems. Moreover, we study a regular derivation on a CI-algebra and an isotone derivation on a transitive CI-algebra.

1. Introduction

The notion of a BCK-algebra was proposed by Y. Imai and K. Iséki in 1966 [1]. In the same year, K. Iséki introduced the notion of a BCI-algebra [2], which is a generalization of a BCK-algebra. H. S. Kim and Y. H. Kim defined a BE-algebra as a dualization of generalization of a BCK-algebra [4]. In [5], B. L. Meng introduced the notion of a CI-algebra as a generalization of a BE-algebra. Y. B. Jun and X. L. Xin applied the notion of a derivation on a BCI-algebra which is defined in a way similar to the notion of derivation in ring and near-ring theory [3]. In fact, the notion of a derivation in ring theory is quite old and plays a significant role in analysis, algebraic geometry and algebra. In this paper we introduce the notion of a derivation on a CI-algebra and obtain some properties related to it. In Section 2, we recall some definitions and some properties for a CI-algebra. In Section 3, we not only prove some one-to-one and onto derivation theorems, but also we study a regular derivation on a CI-algebra and an isotone derivation on a transitive CI-algebra.

Received March 03, 2014; Accepted April 07, 2014.
2010 Mathematics Subject Classification: Primary 06F35, 03G25, 08A30.
Key words and phrases: CI-algebra, self-distributive, derivation, transitive, isotone.
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2. Preliminaries

We review some definitions and properties that will be used later.

DEFINITION 2.1. [5] Let X be a nonempty set, and let $*$ be a binary operation on X. Then $(X, *, 1)$ is said to be a CI-algebra if the following axioms are satisfied:

(CI1) $x * x = 1$.
(CI2) $1 * x = x$.
(CI3) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

In the above definition we simply say that X is a CI-algebra. In [6], the author defined a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 1$, where $x, y \in X$.

A CI-algebra X is said to be *self-distributive* if $x*(y*z) = (x*y)*(x*z)$ for all $x, y, z \in X$ [5]. A nonempty subset S of a CI-algebra X is said to be a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. Also, we define a binary operation $\vee$ on a CI-algebra X by $x \vee y = (y * x) * x$.

PROPOSITION 2.2. [5] *Let x, y be arbitrary elements of a CI-algebra X. Then:*

(1) $y * ((y * x) * x) = 1$.
(2) $(x * 1) * (y * 1) = (x * y) * 1$.
(3) $1 \leq x$ *implies* $x = 1$ *for all* $x, y \in X$.

DEFINITION 2.3. [5] A CI-algebra X is said to be *transitive* if

$$(y * z) * ((x * y) * (x * z)) = 1, \text{ for all } x, y, z \in X.$$

EXAMPLE 2.4. Let $X = \{1, a, b, c\}$ be a CI-algebra, where $*$ is defined as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then X is a transitive CI-algebra.

Note that if X is a self-distributive CI-algebra, then X is a transitive CI-algebra.

PROPOSITION 2.5. [7] *Let X be a CI-algebra. If X is transitive, then:*

(1) $y \leq z$ *implies* $x * y \leq x * z$.
(2) $y \leq z$ *implies* $z * x \leq y * x$ *for all* $x, y, z \in X$.

Definition 2.6. [7] A nonempty subset of F of a CI-algebra X is said to be a filter of X if the following axioms are satisfied:
(F1) $1 \in F$.
(F2) $x \in F$ and $x * y \in F$ imply $y \in F$.

3. Derivations on CI-algebras

We start this section with the definition of a derivation on a CI-algebra X.

Definition 3.1. [3] Let X be a CI-algebra. A map $d : X \to X$ is said to be a *right-left derivation* (*(r, l)-derivation*) on X if it satisfies the identity

$$d(x * y) = (x * d(y)) \vee (d(x) * y)$$

for all $x, y \in X$. Similarly, a map d is said to be a *left-right derivation* (*(l, r)-derivation*) on X if it satisfies the identity

$$d(x * y) = (d(x) * y) \vee (x * d(y))$$

for all $x, y \in X$. Moreover, if d is both a (l, r)-derivation and a (r, l)-derivation, then d is called a *derivation* on X.

Example 3.2. Let $X = \{1, a, b\}$ be a CI-algebra, where $*$ is defined as follows:

$*$	1	a	b
1	1	a	b
a	1	1	b
b	1	a	1

Then the map $d : X \to X$ defined by

$$d(x) = \begin{cases} 1 & \text{if } x = 1,\, b \\ a & \text{if } x = a \end{cases}$$

is a derivation on X.

Example 3.3. Let $X = \{1, a, b, c\}$ be a CI-algebra, where $*$ is defined as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	a
b	1	a	1	c
c	1	1	b	1

Then the map $d : X \to X$ defined by

$$d(x) = \begin{cases} 1 & \text{if } x = 1,\, a \\ a & \text{if } x = c \\ b & \text{if } x = b \end{cases}$$

is a derivation on X.

A map d on a CI-algebra X is said to be *regular* if $d(1) = 1$ [3]. Then we get two propositions for a regular map.

PROPOSITION 3.4. *Let d be a regular map on a CI-algebra X. Then the following properties are satisfied:*

(1) *If d is a (l, r)-derivation of X, then $d(x) = x \vee d(x)$ for every $x \in X$.*
(2) *If d is a (r, l)-derivation of X, then $d(x) = d(x) \vee x$ for every $x \in X$.*

Proof. Note that $d(1) = 1$ and $1 * x = x$ for every $x \in X$.
(1) If d is a (l, r)-derivation on X, then by Definition 3.1, we get

$$d(x) = d(1 * x) = (d(1) * x) \vee (1 * d(x)) = x \vee d(x)$$

for every $x \in X$.
(2) If d is a (r, l)-derivation on X, then by Definition 3.1, we get

$$d(x) = d(1 * x) = (1 * d(x)) \vee (d(1) * x) = d(x) \vee x$$

for every $x \in X$. □

PROPOSITION 3.5. *Let d be a derivation on a CI-algebra X. If $x * 1 = 1$ for every $x \in X$, then $d(1) = 1$. That is, d is a regular derivation on X.*

Proof. Let d be a (r, l)-derivation on X. Then by hypothesis we have

$$\begin{aligned} d(1) &= d(x * 1) \\ &= (x * d(1)) \vee (d(x) * 1) \quad \text{(since Definition 3.1)} \\ &= (x * d(1)) \vee 1 \\ &= (1 * (x * d(1))) * (x * d(1)) \quad \text{(since } x \vee y = (y * x) * x) \\ &= (x * d(1)) * (x * d(1)) = 1 \end{aligned}$$

for every $x \in X$. Hence d is a regular (r, l)-derivation on X.
Similarly, let d be a (l, r)-derivation on X. Then by hypothesis we have

$$\begin{aligned} d(1) &= d(x * 1) \\ &= (d(x) * 1) \vee (x * d(1)) \quad \text{(since Definition 3.1)} \\ &= 1 \vee (x * d(1)) \quad \text{(since } x \vee y = (y * x) * x) \\ &= 1 \end{aligned}$$

for every $x \in X$. Hence d is a regular (l, r)-derivation on X. Therefore d is a regular derivation on X. $\square$

Now, we study some properties of a derivation on a CI-algebra X.

PROPOSITION 3.6. *Let d be a derivation on a CI-algebra X and let $x \in X$. If $x * 1 = 1$, then $x * ((d(x) * x) * x) = x * ((x * d(x)) * d(x))$.*

Proof. By Proposition 3.4, $d(x) = d(x) \vee x = x \vee d(x)$. Hence $x * d(x) = x * (d(x) \vee x) = x * ((x * d(x)) * d(x))$ and $x * d(x) = x * (x \vee d(x)) = x * ((d(x) * x) * x)$. Thus we get $x * ((d(x) * x) * x) = x * ((x * d(x)) * d(x))$. $\square$

PROPOSITION 3.7. *Let d be a (r, l)-derivation on a CI-algebra X. Then:*

(1) *$x \leq d(x)$ for every $x \in X$.*
(2) *If X is transitive, $d(x) * y \leq x * y \leq x * d(y)$ for all $x, y \in X$.*

Proof. (1) By Definition 2.1 (CI3) and Proposition 3.4 (2), we have

$$\begin{aligned} x * d(x) &= x * (d(x) \vee x) = x * ((x * d(x)) * d(x)) \\ &= (x * d(x)) * (x * d(x)) = 1 \end{aligned}$$

for every $x \in X$. Hence $x \leq d(x)$.

(2) By (1) and Proposition 2.5, we have $d(x) * y \leq x * y \leq x * d(y)$ for all $x, y \in X$. $\square$

THEOREM 3.8. *Let d be a (r, l)-derivation on a transitive CI-algebra X. Then $d(x * y) = x * d(y)$ for all $x, y \in X$.*

Proof. Let d be a (r, l)-derivation on X and let $x, y \in X$. Then by Proposition 3.7 (2) we have $d(x) * y \leq x * d(y)$. Hence we get

$$\begin{aligned} d(x * y) &= (x * d(y)) \vee (d(x) * y) \\ &= ((d(x) * y) * (x * d(y))) * (x * d(y)) \\ &= 1 * (x * d(y)) = x * d(y). \end{aligned}$$

$\square$

PROPOSITION 3.9. *Let d be a map on a CI-algebra X and let $x, y \in X$. Then:*

(1) *If d is a (l, r)-derivation on X, then $x * d(y) \leq d(x * y)$.*
(2) *If d is a (r, l)-derivation on X, then $d(x) * y \leq d(x * y)$.*
(3) *If d is a regular (r, l)-derivation on X, then $d(x * d(x)) = 1$.*

Proof. (1) Let d be a (l,r)-derivation of X. Then note that

$$\begin{aligned} d(x*y) &= (d(x)*y)\vee(x*d(y)) \\ &= ((x*d(y))*(d(x)*y))*(d(x)*y). \end{aligned}$$

Hence $(x*d(y))*d(x*y) = (x*d(y))*(((x*d(y))*(d(x)*y))*(d(x)*y)) = 1$ for all $x,y\in X$ and so we get $x*d(y)\le d(x*y)$.

(2) Let d be a (r,l)-derivation on X. Then note that

$$\begin{aligned} d(x*y) &= (x*d(y))\vee(d(x)*y) \\ &= ((d(x)*y)*(x*d(y)))*(x*d(y)). \end{aligned}$$

Hence $(d(x)*y)*d(x*y) = (d(x)*y)*(((d(x)*y)*(x*d(y)))*(x*d(y))) = 1$ for all $x,y\in X$ and so we get $d(x)*y\le d(x*y)$.

(3) If d is a (r,l)-derivation on X, then by Proposition 3.7 (1) and by hypothesis, we get $d(x*d(x)) = d(1) = 1$ for every $x\in X$. □

Proposition 3.10. *Let X be a CI-algebra. If $d_1, d_2, \cdots, d_n$ are regular (r,l)-derivations on X, then $x\le d_n(d_{n-1}(...(d_2(d_1(x)))...))$ for every $n\in\mathbb{N}$.*

Proof. For $n=1$, we get

$$d_1(x) = d_1(1*x) = (1*d_1(x))\vee(d_1(1)*x) = d_1(x)\vee x = (x*d_1(x))*d_1(x).$$

Then by Definition 2.1 we have

$$x*d_1(x) = x*((x*d_1(x))*d_1(x)) = (x*d_1(x))*(x*d_1(x)) = 1.$$

Hence $x\le d_1(x)$.

Suppose that $x\le d_n(d_{n-1}(...(d_2(d_1(x)))...))$ for any positive integer $n\ge 2$. For simplicity, let

$$D_n = d_n(d_{n-1}(...(d_2(d_1(x)))...)).$$

Then

$$\begin{aligned} d_{n+1}(D_n) &= d_{n+1}(1*D_n) = (1*d_{n+1}(D_n))\vee(d_{n+1}(1)*D_n) \\ &= d_{n+1}(D_n)\vee D_n = (D_n*d_{n+1}(D_n))*d_{n+1}(D_n). \end{aligned}$$

Hence $D_n*D_{n+1}=1$, which implies $D_n\le D_{n+1}$. By assumption, we get $x\le D_n\le D_{n+1}$. Therefore Proposition 3.10 holds. □

Let d be a map on a CI-algebra X. Define a set $Fix_d(X)$ by

$$Fix_d(X) = \{x\in X \mid d(x) = x\}.$$

Now, we write $d^2(x)$ for $(d\circ d)(x)$, where $\circ$ is a composition.

Proposition 3.11. *Let d be a derivation on a CI-algebra X and let $x,y\in Fix_d(X)$. Then:*

(1) $d(x) \in Fix_d(X)$. *Hence* $Fix_d(X) \subset Fix_{d^2}(X)$.
(2) $x * y \in Fix_d(X)$. *That is,* $Fix_d(X)$ *is a subalgebra of* X.
(3) $x \vee y \in Fix_d(X)$.

Proof. (1) Note that $d^2(x) = d(d(x)) = d(x) = x$ for every $x \in Fix_d(X)$. Hence $d(x) \in Fix_d(X)$ and $x \in Fix_{d^2}(X)$. Thus $Fix_d(X) \subset Fix_{d^2}(X)$.

(2) Let $x, y \in Fix_d(X)$. Then we get $d(x) = x$ and $d(y) = y$. Hence

$$d(x * y) = (x * d(y)) \vee (d(x) * y) = (x * y) \vee (x * y) = x * y.$$

So $x * y \in Fix_d(X)$. Therefore $Fix_d(X)$ is a subalgebra of X.

(3) Let $x, y \in Fix_d(X)$. Then we get

$$\begin{aligned} d(x \vee y) &= d((y * x) * x) \\ &= ((y * x) * d(x)) \vee (d(y * x) * y) \\ &= ((y * x) * x) \vee ((y * d(x) \vee d(y) * x) * x) \\ &= ((y * x) * x) \vee ((y * x) * x) \\ &= (y * x) * x = x \vee y \end{aligned}$$

since $x \vee x = (x * x) * x = 1 * x = x$. □

In the above proposition we say that d is idempotent if $d^2(x) = x$.

PROPOSITION 3.12. *Let* d *be a* (r, l)*-derivation of a self-distributive CI-algebra* X. *If* $y \in Fix_d(X)$, *then* $x * y \in Fix_d(X)$ *for every* $x \in X$.

Proof. Let $y \in Fix_d(X)$. Then $d(y) = y$. By Definition 2.1 (CI3) and Proposition 3.7 (1), we have

$$\begin{aligned} d(x * y) &= (x * d(y)) \vee (d(x) * y) = ((d(x) * y) * (x * y)) * (x * y) \\ &= (x * ((d(x) * y) * y)) * (x * y) \\ &= ((x * (d(x) * y)) * (x * y)) * (x * y) \\ &= (((x * d(x)) * (x * y)) * (x * y)) * (x * y) \\ &= ((1 * (x * y)) * (x * y)) * (x * y) \\ &= ((x * y) * (x * y)) * (x * y) = 1 * (x * y) = x * y. \end{aligned}$$

This completes the proof. □

By Proposition 3.11 (1) we know $Fix_d(X)$ is a subset of $Fix_{d^2}(X)$. But in general $Fix_{d^2}(X)$ is not a subset of $Fix_d(X)$. Also, if $x \in Fix_d(X)$, then $d(x) \in Fix_d(X)$. But there is an element $x \notin Fix_d(X)$ even though $d(x) \in Fix_d(X)$. In fact, $d(b) \in Fix_d(X)$ but $b \notin Fix_d(X)$ in Example 3.2.

THEOREM 3.13. *Let d be a derivation on a CI-algebra X. If $d(x) \in Fix_d(X)$ for every $x \in X$, then the following conditions are equivalent:*
(1) *$d(x) = x$ for every $x \in X$.*
(2) *d is a one-to-one derivation.*
(3) *d is a onto derivation.*

Proof. (1) $\Rightarrow$(2) and (1) $\Rightarrow$(3) are clear.
(2) $\Rightarrow$(1) Let d be a one-to-one derivation and let $x \in X$. Since $d(x) \in Fix_d(X)$, we have $d(d(x)) = d(x)$. Since d is one-to-one, we get $d(x) = x$ for every $x \in X$.
(3) $\Rightarrow$(1) Let d be a onto derivation and let $x \in X$. Then there exists $y \in X$ such that $d(y) = x$. Since $d(x) \in Fix_d(X)$, we have $d(x) = d(d(y)) = d(y) = x$. □

Let d be a map on a CI-algebra X. Define the kernel of d, denoted by $Kerd$,

$$Kerd = \{x \in X \mid d(x) = 1\}.$$

PROPOSITION 3.14. *Let d be a (r, l)-derivation on a CI-algebra X. If $x * 1 = 1$ for every $x \in X$, then $Kerd$ is a subalgebra of X.*

Proof. Let $x, y \in Kerd$. Then $d(x) = 1$ and $d(y) = 1$. Hence we have

$$d(x * y) = (x * d(y)) \vee (d(x) * y) = (x * 1) \vee (1 * y) = 1 \vee y = 1.$$

So $x * y \in Kerd$. Therefore $Kerd$ is a subalgebra of X. □

A CI-algebra X is said to be *commutative* if $(y * x) * x = (x * y) * y$ for all $x, y \in X$.

PROPOSITION 3.15. *Let d be a (r, l)-derivation on a commutative CI-algebra X and let $x \leq y$. If $x \in Kerd$ and $z * 1 = 1$ for every $z \in X$, then $y \in Kerd$.*

Proof. Let $x \in Kerd$ and let $x \leq y$. Then $d(x) = 1$ and $x * y = 1$. By hypothesis and Proposition 2.2 (2), we get

$$\begin{aligned} d(y) &= d(1 * y) = d((x * y) * y) = d((y * x) * x) \\ &= ((y * x) * d(x)) \vee (d(y * x) * x) \\ &= ((y * x) * 1) \vee (d(y * x) * x) \\ &= 1 \vee (d(y * x) * x) = 1, \end{aligned}$$

and so $y \in Kerd$. □

A map d on a CI-algebra X is said to be an *isotone* if $x \leq y$ implies $d(x) \leq d(y)$ for all $x, y \in X$ [8].

Theorem 3.16. *Let d be an isotone derivation on a transitive CI-algebra X and let $x \in Fix_d(X)$. If d is idempotent and regular on X, then $Kerd$ is a filter of X.*

Proof. Since d is regular, we get $1 \in Kerd$. Let $x \in Kerd$ and $x * y \in Kerd$. Then by Theorem 3.8, we get $1 = d(x * y) = x * d(y)$, which means $x \leq d(y)$. Since d is isotone and idempotent derivation on X, then $1 = d(x) \leq d^2(y) = d(y)$. That is, $d(y) = 1$. This implies $y \in Kerd$. Therefore $Kerd$ is a filter of X. □

Proposition 3.17. *Let d be a regular derivation on a CI-algebra X. If d is one-to-one, then $Kerd = \{1\}$.*

Proof. If $x \in Kerd$, then $d(x) = 1$. Since d is regular, $d(1) = 1$. Hence $d(x) = d(1)$. Since d is one-to-one, we get $x = 1$. Therefore $Kerd = \{1\}$. □

Proposition 3.18. *Let d be an isotone derivation on a CI-algebra X. If $x \leq y$ and $x \in Kerd$, then $y \in Kerd$ for all $x, y \in X$.*

Proof. Let $x \leq y$ and $x \in Kerd$. Then $d(x) = 1$, and so $1 = d(x) \leq d(y)$. By Proposition 2.2 (3), we get $d(y) = 1$. □

Definition 3.19. [3] Let d be a derivation on a CI-algebra X. A nonempty subset F of X is said to be *d-invariant* if $d(F) \subseteq F$, where $d(F) = \{d(x) \mid x \in F\}$.

It follows from $d(Fix_d(X)) \subset Fix_d(X)$ that $Fix_d(X)$ is d-invariant.

Proposition 3.20. *Let d be a regular derivation on a CI-algebra X. Then every filter F of X is d-invariant.*

Proof. Let F be a filter of X and let $y \in d(F)$. Then $y = d(x)$ for some $x \in F$. It follows from Proposition 3.9 that $x * y = x * d(x) = 1 \in F$. This implies $y \in F$. Thus $d(F) \subseteq F$. Therefore F is d-invariant. □

Definition 3.21. [7] A nonempty subset F of a CI-algebra X is said to be a *normal filter* of X if the following conditions are satisfied:
(NF1) $1 \in F$.
(NF2) $x \in X$ and $y \in F$ imply $x * y \in F$.

Proposition 3.22. *Let d be a (r, l)-derivation on a self-distributive CI-algebra X. Then:*

(1) *$Fix_d(X)$ is a normal filter of X.*
(2) *$Kerd$ is a normal filter of X.*

Proof. Since X is a self-distributive CI-algebra, then $x * 1 = x * (x * x) = (x * x) * (x * x) = 1$ for every $x \in X$. By Proposition 3.5 we know $d(1) = 1$. Hence $1 \in Fix_d(X)$ and so $1 \in Kerd$.

(1) Let $x \in X$ and $y \in Fix_d(X)$. Then by Theorem 3.8, we get $d(x * y) = x * d(y) = x * y$. Hence $x * y \in Fix_d(X)$ and so $Fix_d(X)$ is a normal filter of X.

(2) Similarly, let $x \in X$ and $y \in Kerd$. Then by Theorem 3.8, we get $d(x * y) = x * d(y) = x * 1 = 1$. Hence $x * y \in Kerd$. and so $Kerd$ is a normal filter of X. □

PROPOSITION 3.23. *Let d be a regular derivation on a self-distributive CI-algebra X and let $a \in X$. Then $F_a = \{x \in X \mid a \leq d(x)\}$ is a normal filter of X.*

Proof. Since $a \leq d(1) = 1$ for every $a \in X$, we get $1 \in F_a$. Let $x \in X$ and $y \in F_a$. Then $a * d(y) = 1$, and so by Theorem 3.8

$$\begin{aligned} a * d(x * y) &= a * (x * d(y)) = (a * x) * (a * d(y)) \\ &= (a * x) * 1 \\ &= 1. \end{aligned}$$

Hence $x * y \in F_a$ and so F_a is a normal filter of X. □

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