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REPRESENTATIONS OF THE AUTOMORPHISM GROUP OF A SUPERSINGULAR K3 SURFACE OF ARTIN-INVARIANT 1 OVER ODD CHARACTERISTIC

Junmyeong Jang*

Abstract. In this paper, we prove that the image of the representation of the automorphism group of a supersingular K3 surface of Artin-invariant 1 over odd characteristic p on the global two forms is a finite cyclic group of order $p+1$. Using this result, we deduce, for such a K3 surface, there exists an automorphism which cannot be lifted over a field of characteristic 0.

1. Introduction

Let k be an algebraically closed field and X be an algebraic K3 surface defined over k. $H^0(X, \Omega^2_{X/k})$ is a one dimensional k space and the canonical representation of the automorphism group of X on $H^0(X, \Omega^2_{X/k})$

$$\rho : \mathrm{Aut}\, X \to Gl(H^0(X, \Omega^2_{X/k}))$$

is a character. If the characteristic of k is not 2, the image of ρ is a finite cyclic group. ([14], [7]) We let N be the order of $\mathrm{Im}\, \rho$. If the characteristic of k is 0 or X is of finite height over odd characteristic, $\phi(N)$ is at most 20. ([13], [7]) Here ϕ is the Euler ϕ-function. If X is a supersingular K3 surface of Artin-invariant σ over odd characteristic, N divides $p^\sigma + 1$. ([14], Prop. 2.4) For the definition of height and Artin-invariant, see section 2. In this paper, we prove that for a supersingular K3 surface of Artin-invariant 1 over odd characteristic, the order of $\mathrm{Im}\, \rho$ is $p+1$.

Theorem 3.3. Let X be a supersingular K3 surface of Artin invariant 1 over an algebraically closed field k of odd characteristic p. Then

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Im ρ is a cyclic group of order $p+1$.

Assume k is an algebraically closed field of odd characteristic and (R, m) is a discrete valuation ring of characteristic 0 whose residue field R/m is isomorphic to k. In this case $p \in m$. A proper connected smooth scheme $\mathbb{X}$ over R is a lifting of X over R if the base change $\mathbb{X} \otimes k$ is isomorphic to X. Every K3 surface defined over k has a lifting over the ring of Witt vectors. ([3], [15]) Let $\alpha : X \to X$ be an automorphism of X. A lifting of (X, α) over R is a pair $(\mathbb{X}, \mathfrak{a})$ such that $\mathbb{X}$ is a lifting of X over R and $\mathfrak{a}$ is an R-isomorphism of $\mathbb{X}$ satisfying $\mathfrak{a} \otimes k$ is equal to α under the identification $\mathbb{X} \otimes k = X$. If α is of finite order and the order of α is prime to p, α has a lifting over W. ([8]) When X is a K3 surface of finite height, we let T_l be the l-adic transcendental lattice of X. In other words, $T_l(X)$ is the orthogonal complement of the embedding

$$NS(X) \otimes \mathbb{Z}_l \hookrightarrow H^2_{\acute{e}t}(X, \mathbb{Z}_l).$$

The representation

$$\chi_l : \operatorname{Aut} X \to O(T_l(X))$$

has a finite image. ([7], Prop. 2.5) If the order of $\chi_l(\alpha)$ is not divisible by p, (X, α) has a lifting over W. ([8]) Since the characteristic polynomial of $\chi_l(\alpha)$ has integer coefficients ([4], 3.7.3), if $p \geq 23$ the order of $\chi_l(\alpha)$ is not divisible by p for any $\alpha \in \operatorname{Aut} X$.

From Theorem 3.3, we obtain the following.

Corollary 3.5. If $\phi(p+1) > 20$, a supersingular K3 surface of Artin-invariant 1 over k has an automorphism which does not have a lifting over a field of characteristic 0.

2. Preliminary: supersingular K3 surfaces

In this section we review some properties of supersingular K3 surfaces. Let k be an algebraically closed field of odd characteristic p. $W = W(k)$ is the ring of Witt vectors of k and K is the fraction field of W. W and K are equipped with the natural Frobenius operators

$$\sigma : W \to W,\ \sigma : K \to K.$$

Let X be a K3 surface defined over k. The second crystalline cohomology of X/k, $H^2_{cris}(X/W)$ is a free W-module of rank 22 equipped with a Frobenius linear endomorphism

$$\mathbf{F} : H^2_{cris}(X/W) \to H^2_{cris}(X/W).$$

$H^2_{cris}(X/K) = H^2_{cris}(X/W) \otimes K$ has the induced Frobenius-linear automorphism

$$\mathbf{F} : H^2_{cris}(X/K) \to H^2_{cris}(X/K).$$

By the Dieudonné-Manin theorem, an F-isocrystal $(H^2_{cris}(X/K), \mathbf{F})$ has a decomposition

$$(H^2_{cris}(X/K), \mathbf{F}) = \oplus K[T]/(T^{r_i} - p^{s_i}).$$

Here $K[T]$ is a Frobenius semi-commutative polynomial ring satisfying $Ta = \sigma(a)T$ for any $a \in K$. Under the identification, the operator $\mathbf{F}$ corresponds to the multiplication by T on the right hand side. We say the rational number $\frac{s_i}{r_i}$ is a Newton slope of $H^2_{cris}(X/K)$. The length of Newton slope $\frac{s_i}{r_i}$ is r_i. The sum of the lengths of all the slopes is equal to the dimension of $H^2_{cris}(X/K)$. If the only slopes of $H^2_{cris}(X/K)$ is 1, we say the height of X is ∞ or X is supersingular. If X is not supersingular, there exists an integer h $(1 \le h \le 10)$ such that $1 - 1/h,\ 1,\ 1 + 1/h$ are slopes of $H^2_{cris}(X/K)$ of length $h,\ 22 - 2h,\ h$ respectively. In this case, the height of X is h.

The Neron-Severi group of X, $NS(X)$ is a finite free abelian group equipped with a lattice structure given by the intersection theory. The Picard number of X, $\rho(X)$ is the rank of $NS(X)$. The Neron-Severi lattice of a K3 surface is even by the Riemann-Roch theorem and the signature of $NS(X)$ is $(1, \rho(X) - 1)$ by the Hodge index theorem. The Picard number of X is at most the length of the Newton slope 1. ([5]) It follows that $\rho(X) \le 22 - 2h$ if the height of X is $h < \infty$. Also it is known that X is supersingular if and only if $\rho(X) = 22$. ([2], [10]) Note that over a field of characteristic 0, the Picard number of a K3 surface is at most 20 since $h^{1,1} = 20$ for a K3 surface.

For a lattice L, we denote the discriminant of L and the discriminant group L^*/L by $d(L)$ and $A(L)$ respectively. Let $d(X) = d(NS(X)) \in \mathbb{Z}$ be the discriminant of the lattice $NS(X)$ and $A(X) = (NS(X))^*/NS(X)$ be the discriminant group of $NS(X)$. When X is a supersingular K3 surface over k, $d(X) = -p^{2\sigma}$ for an integer $1 \le \sigma \le 10$. We say σ is the Artin-invariant of X. It is known that the moduli space of K3 surfaces of height $h < \infty$ is $20 - h$ dimensional and the moduli space of supersingular K3 surfaces of Artin-invariant σ is $\sigma - 1$ dimensional. ([1]) Moreover a supersingular K3 surface of Artin-invariant 1 is unique up to isomorphism and it is isomorphic to the Kummer surface of the self-product of a supersingular elliptic curve. For a supersingular K3 surface X, the lattice structure of $NS(X)$ is determined by the Artin-invariant and the base characteristic p. ([17]) Let us denote the Neron-Severi lattice of a supersingular K3 surface of Artin-invariant σ over a field of characteristic p by $\Lambda_{p,\sigma}$. After tensor product with $\mathbb{Z}_p$, we obtain a decomposition of $\mathbb{Z}_p-$lattice

$$\Lambda_{p,\sigma} \otimes \mathbb{Z}_p = E_0(p) \oplus E_1.$$

Here E_0 and E_1 are unimodular $\mathbb{Z}_p-$lattices of rank 2σ and $22 - 2\sigma$ respectively. And $d(E_0) = (-1)^\sigma \delta$ and $d(E_1) = (-1)^{\sigma+1}\delta$, where δ is a non-square unit of $\mathbb{Z}_p$. Note that a unimodular $\mathbb{Z}_p-$lattice is uniquely determined up to isomorphism by the rank and the discriminant, square or non-square. It follows that

$$A(\Lambda_{p,\sigma}) = A(\Lambda_{p,\sigma} \otimes \mathbb{Z}_p) = A(E_0(p)) = E_0(p)/pE_0(p)$$

is a $2\sigma-$dimensional quadratic $\mathbb{Z}/p$ space. Note that the discriminant of $A(\Lambda_{p,\sigma})$ is a $(-1)^\sigma$ times non-square and $A(\Lambda_{p,\sigma})$ does not contain a $\sigma-$dimensional isotropic $\mathbb{Z}/p$-subspace.

By the flat Kummer sequence

$$0 \to \mu_{p^n} \to \mathbb{G}_{m,X} \overset{p^n}{\to} \mathbb{G}_{m,X} \to 0,$$

we have a canonical inclusion

$$NS(X) \otimes \mathbb{Z}_p \hookrightarrow H^2_{fl}(X, \mathbb{Z}_p(1)).$$

Also there exists an exact sequence ([5])

$$0 \to H^2_{fl}(X, \mathbb{Z}_p(1)) \to H^2_{cris}(X/W) \overset{id-p}{\to} H^2_{cris}(X/W).$$

Composing two embeddings, we obtain the cycle map

$$NS(X) \otimes W \hookrightarrow H^2_{cris}(X/W).$$

If X is a supersingular K3 surface, the cycle map is an embedding of $W-$lattices of the same rank. Since $H^2_{cris}(X/W)$ is unimodular by the Poincaré duality,

$$NS(X) \otimes W \subset H^2_{cris}(X/W) \subset (NS(X) \otimes W)^*$$

and $K_X = H^2_{cris}(X/W)/(NS(X) \otimes W)$ is a $\sigma-$dimensional isotropic k-subspace of the discriminant group

$$A(NS(X) \otimes W) = (NS(X) \otimes W)^*/(NS(X) \otimes W) = A(NS(X)) \otimes k.$$

K_X is equipped with a Frobenius-inverse linear operator $V : K_X \to K_X$ such that $V^\sigma = 0$ and we can choose $x \in K_X$ such that $\{x, Vx, \cdots, V^{\sigma-1}x\}$ is a basis of K_X. ([14]) Also we have a canonical isomorphism ([6], Prop.2.2)

$$K_X/VK_X \simeq H^2(X, \mathcal{O}_X).$$

Let $\mathcal{P} = \{x \in NS(X) \otimes \mathbb{R} | (x,x) > 0\}$, the positive cone of X and $\Delta = \{v \in NS(X) | (v,v) = -2\}$, the set of roots of X. For any $v \in \Delta$, let s_v be the reflection with respect to v,

$$s_v : u \mapsto u + (u,v)v.$$

Let W_X be the subgroup of the orthogonal group of $NS(X)$ generated by all the refections $s_v, (v \in \Delta)$ and $-id$. Let $\mathcal{P}^0 = \{v \in \mathcal{P} | (v,w) \neq 0, \ \forall w \in \Delta\}$. It is known that the W_X acts simply transitively on the set of connected components of $\mathcal{P}^0$. Moreover, the connected component which contains an ample divisor is the ample cone of X. ([16], [11]) It follows that $v \in NS(X)$ represents an ample divisor if and only if $(v,v) > 0$ and $(v,w) > 0$ for all effective $w \in \Delta$. If $(v,v) > 0$ and $(v,w) \neq 0$ for any $w \in \Delta$, there exists a unique element γ in W_X such that $\gamma(v)$ represents an ample divisor.

Theorem 2.1 (Crystalline Torelli theorem, [16], p.371). *Let X and Y be supersingular K3 surfaces defined over k. Assume $\Psi : NS(X) \to NS(Y)$ is an isometry. If Ψ takes the ample cone of $NS(X) \otimes \mathbb{R}$ into the ample cone of $NS(Y) \otimes \mathbb{R}$ and K_X into K_Y, then there exists a unique isomorphism $\psi : Y \to X$ such that $\Psi = \psi^*$*

Remark 2.2. *Our definition of K_X is slightly different from the definition of the period space in* [16]. *However, in the statement of the crystalline Torelli theorem, we can use K_X instead of the period space.*

For a supersingular K3 surface X, let

$$\rho^{-1} : \operatorname{Aut} X \to Gl(H^2(X, \mathcal{O}_X))$$

and

$$\chi : \operatorname{Aut} X \to Gl(A(NS(X))$$

be the representation of the automorphism group of X on $H^2(X, \mathcal{O}_X)$ and $A(NS(X))$ respectively. Since ρ^{-1} is a character, ρ^{-1} is isomorphic to the representation

$$\rho : \operatorname{Aut} X \to Gl(H^0(X, \Omega^2_{X/k})).$$

Any automorphism of X preserves K_X in $A(NS(X)) \otimes k$, so there is a canonical projection

$$pr : \operatorname{Im} \chi \to \operatorname{Im} \rho^{-1} \simeq \operatorname{Im} \rho.$$

Proposition 2.3 ([7], Prop.2.1). *pr is an isomorphism. In particular, both of* $\operatorname{Im} \chi$ *and* $\operatorname{Im} \rho$ *are finite cyclic groups.*

For the order of $\operatorname{Im} \rho$, the following is known.

Proposition 2.4 ([14], Prop.2.4). *If the Artin-invariant of X is σ, the order of* $\operatorname{Im}\rho$ *divides* $p^\sigma + 1$.

3. Proof

Assume k is an algebraically closed field of odd characteristic p and X is a supersingular K3 surface of Artin-invariant 1. Then $A(X)$ is a 2 dimensional space over $\mathbb{F}_p$ equipped with a non degenerate quadratic form $q = q_{A(X)}$. Here $\mathbb{F}_p$ is a prime field of characteristic p. Also we can see $A(X)$ does not contain a non-zero isotropic vector over $\mathbb{F}_p$. Let us choose an orthogonal basis of $A(X)$, $\{x, y\}$ such that $x \cdot x = 1$, $y \cdot y = -\delta$ and $x \cdot y = 0$. The following lemma is well-known. We present a proof using the zeta function.

Lemma 3.1. *For any $\alpha \in \mathbb{F}_p^*$, the cardinality of the set $\{v = ax + by \in A(X) | a, b \in \mathbb{F}_p, q(v) = \alpha\}$ is $p + 1$.*

Proof. Let C be the smooth conic $X^2 - \delta Y^2 - \alpha Z^2$ in $\mathbb{P}_k^2$. Let $Z_C(t)$ be the zeta function of C. Since $Z_C(t) = \frac{1}{(1-t)(1-pt)}$, $|C(\mathbb{F}_p)| = p + 1$. For any $(X, Y) \neq (0, 0)$, $X^2 - \delta Y^2 \neq 0$, so each point of $C(\mathbb{F}_p)$ gives a distinct solution $(X/Z)^2 - \delta(Y/Z)^2 = \alpha$. This completes the proof. □

Lemma 3.2. *The special orthogonal group $SO(q)$ is a finite cyclic group of order $p + 1$*

Proof. Assume $\gamma \in O(q)$. We have $p + 1$ choices of $\gamma(x)$ by Lemma 3.1. Because $\gamma(x) \cdot \gamma(y) = 0$ and $\gamma(y) \cdot \gamma(y) = -\delta$, for each choice of $\gamma(x)$, there are two possibilities of $\gamma(y)$. Therefore the order of $O(q)$ is $2(p + 1)$ and the order of $SO(q)$ is $p + 1$. There are two isotropic lines in $A(X) \otimes \mathbb{F}_{p^2}$. Any $\gamma \in O(q)$ fixes or interchanges two isotropic lines and $\gamma \in O(q)$ is contained in $SO(q)$ if and only if γ fixes the isotropic lines. Let $v \in A(X) \otimes \mathbb{F}_{p^2}$ be an isotropic vector. The character $\lambda : SO(q) \to k^*$ defined by $\gamma(v) = \lambda(\gamma)v$ is an injection. It follows that $SO(q) = \mathbb{Z}/(p + 1)$. □

Theorem 3.3. *Let X be a supersingular K3 surface of Artin-invariant 1 over an algebraically closed field k of odd characteristic p. Then* $\operatorname{Im}\rho$ *is a cyclic group of order $p + 1$.*

Proof. Since $NS(X)$ is even indefinite of rank 22 and $A(NS(X)) = (\mathbb{Z}/p)^2$, the canonical map

$$\pi : O(NS(X)) \to O(q)$$

is surjective. ([12], 1.14.2) Assume $\gamma \in \pi^{-1}(SO(q))$. Since K_X is an isotropic line of $A(X) \otimes k$, γ preserves K_X. By [18], p.456 there is a decomposition

$$NS(X) = U \oplus H^{(p)} \oplus E_8^2.$$

Here U is an even unimodular hyperbolic lattice of rank 2 and E_8 is a negative definite unimodular root lattice. $H^{(p)}$ is the maximal order of the quaternion algebra over $\mathbb{Q}$ which is ramified only at p and ∞. The lattice structure of $H^{(p)}$ is induced by the trace map of the quaternion algebra. $H^{(p)}$ is an negative definite even lattice of rank 4 and

$$A(H^{(p)}) = A(NS(X)) = (\mathbb{Z}/p)^2.$$

The Weyl group $W_X \subset O(NS(X))$ is generated by $-id$ and reflections $s_v (v \in \Delta)$. For $v \in \Delta$, $v \cdot v = -2$, so $\mathbb{Z}_p v$ is a unimodular sublattice of $NS(X) \otimes \mathbb{Z}_p$ and we have a decomposition

$$NS(X) \otimes \mathbb{Z}_p = M \oplus \mathbb{Z}_p v,$$

here M is the orthogonal complement of $\mathbb{Z}_p v$ in $NS(X) \otimes \mathbb{Z}_p$. Then $s_v | NS(X) \otimes \mathbb{Z}_p = id \oplus -id$ with respect to the decomposition. Since $A(M) = A(X)$, $s_v|A(X) = id$ and $s_v|K_X = id$. The positive cone $\mathcal{P}$ has two connected components. Since s_v fixes a positive vector, s_v fixes connected components of $\mathcal{P}$. On the other hand, $-id$ interchanges the connected components of $\mathcal{P}$. Let $\iota = id \oplus -id \oplus id \in O(NS(X))$ for the decomposition

$$NS(X) = U \oplus H^{(p)} \oplus E_8^2.$$

ι preserves the connected components of $\mathcal{P}$ and $\iota|A(NS(X)) = -id$. Assume $\psi \in O(NS(X))$ and $\pi(\psi) \in SO(q)$. There exists a unique $\gamma \in W_X \cup W_X \cdot \iota$ such that $\gamma \circ \psi$ preserves the ample cone and $\pi(\gamma \circ \psi) = \pi(\psi)$. Since $\gamma \circ \psi$ preserves the ample cone and K_X, $\gamma \circ \psi \in \operatorname{Aut} X \subset O(NS(X))$ by the crystalline Torelli theorem. Therefore

$$\operatorname{Im} \chi = \pi(\operatorname{Aut} X) = \pi(\pi^{-1}(SO(q))) = SO(q)$$

and $\operatorname{Im} \rho \simeq \operatorname{Im} \chi$ is a cyclic group of order $p+1$. □

Remark 3.4. *For $\sigma > 1$, there exists a supersingular K3 surface of Artin-invariant σ over k such that the order of $\operatorname{Im} \chi$ is equal to or less than 2. ([9], Theorem 1.7)*

Corollary 3.5. *If $\phi(p+1) > 20$, a supersingular K3 surface of Artin-invariant 1 over k has an automorphism which can not be lifted over a field of characteristic 0.*

Proof. Let α be an automorphism of X such that the order of $\rho(\alpha)$ is $p+1$. Assume R is a discrete valuation ring of characteristic 0 whose residue field is isomorphic to k and $(\mathbb{X}, \mathfrak{a})$ is a lifting of (X, α) over R. Let F be the fraction field of R and $X_F = \mathbb{X} \otimes F$. X_F is a K3 surface defined over F and $H^0(\mathbb{X}, \Omega^2_{\mathbb{X}/R})$ is a free R-module of rank 1. The order of $\mathfrak{a}^*|H^0(\mathbb{X}, \Omega^2_{\mathbb{X}/R})$ is equal to the order of $\mathfrak{a}^*|H^0(X_F, \Omega^2_{X_F/F})$. Since $\mathfrak{a}^*|H^0(\mathbb{X}, \Omega^2_{\mathbb{X}/R})$ is a multiplication by a root of unity, the order of $\mathfrak{a}^*|H^0(\mathbb{X}, \Omega^2_{\mathbb{X}/R})$ is a p-power times of the order of $\alpha^*|H^0(X, \Omega^2_{X/k})$. But the ϕ value of the order of $\mathfrak{a}^*|H^0(X_F, \Omega^2_{X_F/F})$ is at most 20 and it is a contradiction. Therefore (X, α) is not liftable. □

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Department of Mathematics
University of Ulsan
Ulsan 680-749, Republic of Korea
E-mail: jmjang@ulsan.ac.kr